ARBEITSGEMEINSCHAFT ON THE BEZRUKAVNIKOV EQUIVALENCE

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1. Introduction

Let G be a pinned reductive group over $\bar{\mathbb{F}}_p$. Our coefficient ring for sheaves below will always be $\bar{\mathbb{Q}}_\ell$, so we omit it from the notation. In the eighties, Kazhdan–Lusztig [KL87] constructed an identification

$$K_0(\mathrm{Shv}_{\mathrm{coh}}(G^\vee \backslash \mathrm{St})) \simeq \mathbb{Z}[W_{\mathrm{ext}}]$$

between the group algebra of the extended affine group of G and the Grothendieck group of coherent sheaves on the Steinberg variety $\mathrm{St} = \tilde{\mathcal{N}} \times_{G^\vee} \tilde{\mathcal{N}}$ for the dual group $G^\vee$ over $\bar{\mathbb{Q}}_\ell$, where $\tilde{\mathcal{N}}$ denotes the Springer resolution of the nilpotent cone of the Lie algebra $\mathfrak{g}^\vee$ of $G^\vee$. At the same time, it is not hard to show that one also has

$$K_0(\mathrm{Shv}_{\text{ét}}(L^+I \backslash \mathrm{Fl}_I)) \simeq \mathbb{Z}[W_{\mathrm{ext}}]$$

where I is the standard Iwahori of G, Fl_I the affine flag variety over $\bar{\mathbb{F}}_p$, and we consider the Grothendieck group of I-equivariant étale $\bar{\mathbb{Q}}_\ell$-sheaves on Fl_I. This raises the natural question of whether there exists a categorical enhancement of the above combinatorial identities in the form of an equivalence of bounded derived categories

$$\Phi\colon D^b_{\mathrm{coh}}(G^\vee \backslash \mathrm{St}) \xrightarrow{\sim} D_{\mathrm{cons}}(L^+I \backslash \mathrm{Fl}_I) \subset D_{\text{ét}}(L^+I \backslash \mathrm{Fl}_I),$$

where Steinberg variety is now understood as a derived scheme. The above exists by work of Bezrukavnikov [Bez16] and is therefore called the *Bezrukavnikov equivalence*. Our goal in this AG is to sketch a proof and digest its various aspects.

One of the starting points of the equivalence is the construction of central sheaves on Fl_I due to Gaitsgory [Gai01]. The idea behind it is to use the following two ingredients. On the one hand, we've got the geometric Satake equivalence for the affine Grassmannian Gr_G whose perverse equivariant sheaves are equivalent to $G^\vee$-representations, see [MV07]. On the other hand, there is a deformation over the affine line $\mathbb{A}^1_{\bar{\mathbb{F}}_p}$ from the flag variety Fl_I at the origin to the affine Grassmannian Gr_G elsewhere, see [Zhu14]. Hence, the composition of nearby cycles with geometric Satake yields a functor

$$Z : \mathrm{Rep}(G^\vee) \to \mathrm{Perv}(L^+I \backslash \mathrm{Fl}_I).$$

A fusion-type argument constructs commutativity constraints on Z, so that its essential image consists of *central* perverse sheaves. It is known therefore as the *Gaitsgory central functor* and it comes equipped with a natural monodromy operator induced by the Galois action, which is unipotent.

Another geometric input are the so-called Wakimoto sheaves J_λ for $\lambda \in \mathbb{X}_*(T)$, see [AB09]. Discovered by Mirković, they categorify the translation elements $\mathbb{X}_*(T) \subset W_{\mathrm{ext}}$, giving rise to a functor

$$J\colon \mathrm{Rep}(T^\vee) \to \mathrm{Perv}(L^+I \backslash \mathrm{Fl}_I)$$

uniquely determined by requiring that $J_\lambda = \nabla_\lambda$ is costandard for λ dominant. The key statement for Bezrukavnikov is that the central sheaves in the essential image of Z are filtered by perverse sheaves with grading in the essential image of J. This is known as the *Wakimoto filtration*. We note also that Z and J have led to a better understanding of the geometry of the reduction of Shimura varieties, see [Zhu14, HR21, AGLR22, GL22].

These two collections of sheaves allowed Arkhipov–Bezrukavnikov [AB09] to prove a weaker version of the equivalence, using Iwahori–Whitaker averaging. Let us explain the broad strategy. Consider the Springer variety $\tilde{\mathcal{N}} := G^\vee \times^{B^\vee} \mathfrak{n}^\vee$ with its natural quasi-affine $T^\vee$-torsor pulled back from the flag variety. Taking global sections, the latter embeds in a Springer analogue of the basic affine space (i.e. the affine hull of $G^\vee/U^\vee$). This leads via formal considerations to the definition of a functor

$$F\colon D^b_{\mathrm{coh}}(G^\vee\backslash\tilde{\mathcal{N}}) \to D_{\text{ét}}(L^+I\backslash\mathrm{Fl}_I)$$

extending the previously constructed $Z \times J$. Here, F is defined at the level of (honest) complexes and one checks that the image of those whose cohomology is supported in the boundary of the affine hull is sent to an acyclic complex of perverse sheaves in $\mathrm{Perv}(L^+I\backslash\mathrm{Fl}_I)$ with a Wakimoto filtration.

Note that F is still very far from being an equivalence, but one may compose with the projection to the Iwahori–Whittaker category $D^b_{\mathrm{IW}}(\mathrm{Fl}_I)$. For this, one fixes an auxiliary Artin–Schreier local system $\mathcal{L}_{\mathrm{AS}}$ on $\mathbb{G}_{\mathrm{a},\bar{\mathbb{F}}_p}$, a homomorphism $\chi : I_0^- \to \mathbb{G}_{a,\bar{\mathbb{F}}_p}$ from the unipotent radical of the opposite Iwahori and considers the bounded derived category $D^b_{\mathrm{IW}}(\mathrm{Fl}_I)$ of $(I_0^-, \chi^*\mathcal{L}_{\mathrm{AS}})$-equivariant sheaves on Fl_I. Its advantage is that the abelian heart $\mathrm{Perv}_{\mathrm{IW}}(\mathrm{Fl}_I)$ is a *highest weight category*, so it carries a notion of tilting modules, simplifying the calculation of Ext groups (by contrast, $\mathrm{Perv}(L^+I\backslash\mathrm{Fl}_I)$ is *not* highest weight and it misses certain tilting modules). There is a unique simple object Ξ with support contained in $G/B \subset \mathrm{Fl}_I$, so we have an averaging functor $\mathrm{Av}_{\mathrm{IW}} := \Xi \star (-)$ sending $D_{\text{ét}}(L^+I\backslash\mathrm{Fl}_I)$ to $D^b_{\mathrm{IW}}(\mathrm{Fl}_I)$ as desired. The composition

$$\mathrm{Av}_{\mathrm{IW}} \circ F\colon D^b_{\mathrm{coh}}(G^\vee\backslash\tilde{\mathcal{N}}) \to D^b_{\mathrm{IW}}(\mathrm{Fl}_I)$$

turns out to be an equivalence, the so-called *AB equivalence*, because averaging maps the central sheaves Z_V to tilting modules in $\mathrm{Perv}(L^+I\backslash\mathrm{Fl}_I)$. A highly recommended source for this material is the book [AR] by Achar–Riche.

Finally, we can discuss the actual Bezrukavnikov equivalence [Bez16]. The patterns behind the ideas are quite similar to those of [AB09], but we'd like to highlight the following extra difficulties, that appear technical but might just be essential. The first issue is that St is not actually a scheme, but rather a *derived scheme*, because the higher $\mathrm{Tor}^{\mathcal{O}_{G^\vee}}_{>0}(\mathcal{O}_{\tilde{\mathcal{N}}}, \mathcal{O}_{\tilde{\mathcal{N}}})$ groups do not vanish. The correct fix for this is to enlarge $\tilde{\mathcal{N}}$ to the its torsor $\tilde{\mathcal{N}}_0 := G^\vee \times^{B^\vee} \mathfrak{b}^\vee$ for the adjoint action of $B^\vee$ on its Lie algebra, and correspondingly for the Grothendieck resolution $\mathrm{St}_0 := \tilde{\mathcal{N}}_0 \times_{G^\vee} \tilde{\mathcal{N}}_0$ of the Steinberg variety which becomes an actual scheme. A somewhat parallel issue at the étale level is that $\mathrm{Perv}(L^+I\backslash\mathrm{Fl}_I)$ is not a highest weight category, so it isn't exhausted by tilting modules. Here, the right fix consists in looking at the derived cateogory $D(L^+I_0\backslash\mathrm{Fl}_{I_0})$ of I_0-equivariant sheaves with unipotent I-monodromy, where I_0 is the pro-p-Iwahori. In both situations, we get two extra torus factors, so it is natural to expect the following upgrade of the equivalence

$$\Phi_0\colon D^b_{\mathrm{coh}}(G^\vee\backslash\mathrm{St}_0) \xrightarrow{\sim} D_{\mathrm{cons}}(L^+I_0\backslash\mathrm{Fl}_{I_0}) \subset D_{\text{ét}}(L^+I_0\backslash\mathrm{Fl}_{I_0}),$$

which is the main focus of [Bez16]. Roughly, the idea behind the construction of Φ_0 is to repeat the construction of the functor F from [AB09]. However, the left (resp. right) $\tilde{\mathcal{N}}_0$-factor of the fiber product St_0 now contributes with a left (resp. right) action on the étale side, and Φ_0 is obtained by acting on the free-monodromic object Ξ_0 lifting Ξ. For the reasons sketched above, it is also necessary to generalize the approach of [AB09] to the $T^\vee$-monodromic setting, and even to a certain completion thereof, which was discussed at length in Yun's appendix to the paper [BY13] of Bezrukavnikov–Yun.

The Bezrukavnikov equivalence has been relevant in geometric Langlands for providing the means of constructing coherent sheaves on the moduli stacks of Langlands parameters. Most recently, a paper by Hemo–Zhu [HZ20] has been announced to contain a proof of Zhu's geometric Langlands conjecture [Zhu20] for tame representations. Note that an important input here is a version of the Bezrukavnikov equivalence for the Witt flag variety announced by Yun–Zhu. At the same time, the perfectoid theory in Scholze–Weinstein and Fargues–Scholze [SW20, FS21] made possible to reproduce the geometric features of the central functor in [AGLR22]. Furthermore, Bezrukavnikov–Riche–Rider [BRR20, BR22] have an ongoing project with the goal of proving a considerably harder modular version, i.e. with $\bar{\mathbb{F}}_\ell$-coefficients, of the Bezrukavnikov equivalence. These recent developments strongly motivate our AG.

2. Description of the talks

We are now going over the details that should be covered in each of the talks. Their duration should be of approximately 90 minutes.

Session 1. In this session, we are going to discuss some background material needed for the Bezrukavnikov equivalence, such as the combinatorial identity and geometric Satake, and also the main geometric inputs of the equivalence, such as the construction of central and Wakimoto sheaves.

Talk 1: Combinatorics. Following [AR, Section 5.1], define the Iwahori–Hecke algebra of the pinned split group G, the Bernstein translation elements θ_λ and show that their W_{fin}-orbit is a central element, cf. [AR, Lemma 5.1.3]. Describe the structure of the Iwahori–Hecke algebra as in [AR, Theorem 5.1.4]. Relate the Grothendieck group of L^+I-equivariant étale sheaves on Fl_I to the group algebra $\mathbb{Z}[W_{\mathrm{ext}}]$, see [AR, Lemma 5.2.1] and compute the images of IC, cf. [AR, Theorem 5.2.3]. Next, define the Springer unipotent variety $\tilde{\mathcal{U}}$ and the Steinberg unipotent variety $\mathrm{St}^{\mathrm{u,u}}$, compare with [CG97, Sections 3.2-3.3]. Identify $\mathbb{Z}[W_{\mathrm{ext}}]$ with the Grothendieck group of $G^\vee$-equivariant sheaves on the Steinberg variety $\mathrm{St}^{\mathrm{u,u}}$ following [CG97, Section 7.3], skipping over most of the K-theoretic background.

Talk 2: Geometric Satake. Define the affine Grassmannian Gr_G over $\bar{\mathbb{F}}_p$ attached to a split group G as in [Zhu17, Section 1.2]. Following [Zhu17, Section 2.1], introduce the basics on the geometry of Schubert cells $\mathrm{Gr}_{G,\mu}$ and their closures, especially [Zhu17, Proposition 2.1.5] and [Zhu17, Theorem 2.1.21] (the hypothesis $p \nmid \pi_1(G_{\mathrm{der}})$ is sharp by [HLR18, Theorem 2.5]). Introduce the category $\mathrm{Perv}(L^+G\backslash \mathrm{Gr}_G)$ of L^+G-equivariant perverse $\bar{\mathbb{Q}}_\ell$-sheaves on Gr_G as in [MV07, Section 2]. Define the convolution product on $\mathrm{Perv}(L^+G\backslash \mathrm{Gr}_G)$ using [MV07, Proposition 4.2]. Introduce constant terms CT_B for the Borel $B \subset G$, see [MV07, Theorem 3.5] and [FS21, Corollary VI.3.5], and the monoidal fiber functor, see [MV07, Theorem 3.6, Lemma 6.1]. Construct the symmetry constraint via the fusion interpretation as in [MV07, Section 5]. Sketch an identification of the abstract dual group with $G^\vee$, see [MV07, Theorem 7.3] and [FS21, Section VI.11, pp. 231-232].

Talk 3: Central sheaves. Recall the flag variety Fl_I with its ind-scheme structure, L^+I-action and Bruhat stratification into Schubert cells, see [Ric13]. Introduce the affine Grassmannian Gr_I over $\mathbb{A}^1_{\bar{\mathbb{F}}_p}$ deforming Fl_I to Gr_G, see [Zhu14, Section 3.1]. Define the functor $Z\colon \mathrm{Rep}(G^\vee) \to \mathrm{Perv}(L^+I\backslash \mathrm{Fl}_I)$ via nearby cycles composed with geometric Satake, see [Zhu14, Section 7.2]. Show that it is a central functor verifying the several compatibilities as in [AR, Chapter 3]. Mention the local model $\mathrm{M}_{I,\mu}$ as the closure of $\mathrm{Gr}_{G,\mu}$ as in [HR21, Definition 6.11] and compute its reduced special fiber in Fl_I with the help of central sheaves, see [HR21, Theorem 6.12] or [AGLR22, Theorem 6.16].

Talk 4: Wakimoto sheaves. Define the standard Δ_w and costandard sheaves ∇_w on Fl_I and study their behavior under convolution, cf. [AR, Section 4.1]. Next, define Wakimoto sheaves J_λ

and prove that they are perverse, see [AR, Lemma 4.1.7]. Upgrade these to a monoidal functor $J\colon \mathrm{Rep}(T^\vee) \to \mathrm{Perv}(L^+I\backslash \mathrm{Fl}_I)$ as in [AR, Section 4.2]. Show that Z_V admits a filtration by Wakimoto sheaves, see [AR, Theorem 4.4.5], by proving [AR, Lemma 4.3.2] and the strategy in [AR, Subsection 4.4.1]. Compute CT_{B^-} of Wakimoto sheaves as in [AR, Lemma 4.5.8] and deduce that monodromy is unipotent, see [AR, Proposition 4.6.9]. Construct the highest weight arrows as in [AR, Subsection 4.6.3] skipping most of the verifications. Briefly discuss the monoidal grading functor, cf. [AR, Proposition 4.7.5] and compare it to the Satake fiber functor, see [AR, Proposition 4.8.2].

Session 2. In this session, we are going to study the AB equivalence from [AB09] following the very detailed account of [AR, Chapter 6].

Talk 5: Coherent sheaves on the unipotent Springer variety. Compute $\mathcal{O}(G^\vee/U^\vee)$ as a $G^\vee \times T^\vee$-module, see [AR, Equation (6.2.3)], and show it is a finitely generated $\bar{\mathbb{Q}}_\ell$-algebra, see [AR, Lemma 6.2.1]. Mention that $G^\vee/U^\vee \to \mathrm{Spec}\,\mathcal{O}(G^\vee/U^\vee)$ is a dense open immersion and describe the complement via [AR, Lemma 6.2.2]. Define the Springer variety $\tilde{\mathcal{N}}$, its canonical $\mathfrak{t}^\vee$-torsor $\tilde{\mathcal{N}}_0$ and the affine completion $\tilde{\mathcal{N}}_{0,\mathrm{af}}$ induced by $\mathcal{O}(G^\vee/U^\vee)$, compare with [AR, Section 6.2]. Prove that $D^b_{\mathrm{coh}}(G^\vee\backslash\tilde{\mathcal{N}})$ is spanned by $\mathrm{Rep}(G^\vee \times T^\vee)$ as a tensor derived category, see [AR, Lemma 6.2.7]. Use it to show that $D^b_{\mathrm{coh}}(G^\vee\backslash\tilde{\mathcal{N}}_0)$ is the quotient of $K^b_{\mathrm{coh}}(G^\vee\backslash\tilde{\mathcal{N}}_{0,\mathrm{af}})$ by the full subcategory whose objects have boundary supports as in [AR, Proposition 6.2.8].

Talk 6: Construction of the functor F. Recall the functor $Z \times J\colon \mathrm{Rep}(G^\vee \times T^\vee) \to \mathrm{Perv}(L^+I\backslash \mathrm{Fl}_I)$ and its compatibility structures, see [AR, Subsection 6.3.3]. Factor $Z\times J$ through the non-full symmetric monoidal subcategory $\mathscr{C} = \mathrm{Mod}(G^\vee \times T^\vee\backslash A)$ of $\mathrm{Perv}(L^+I\backslash \mathrm{Fl}_I)$ as in [AR, Proposition 6.3.5]. Construct the preliminary $\tilde{F}\colon K^b_{\mathrm{coh}}(G^\vee\backslash\tilde{\mathcal{N}}_{0,\mathrm{af}}) \to D_{\mathrm{\acute{e}t}}(L^+I\backslash \mathrm{Fl}_I)$ by defining a $G^\vee \times T^\vee$-equivariant homomorphism $\mathcal{O}(\tilde{\mathcal{N}}_{0,\mathrm{af}}) \to A$ as in [AR, Lemma 6.3.7]. Prove following [AR, Proposition 6.3.9] that $\tilde{F}$ descends to the sought for monoidal functor $F\colon D^b_{\mathrm{coh}}(G^\vee\backslash\tilde{\mathcal{N}}) \to D_{\mathrm{\acute{e}t}}(L^+I\backslash \mathrm{Fl}_I)$.

Talk 7: Iwahori–Whittaker averaging and tilting objects. Define the Artin–Schreier sheaf $\mathcal{L}_{\mathrm{AS}}$ on $\mathbb{G}_{a,\bar{\mathbb{F}}_p}$, the additive character $\chi\colon I_0^- \to \mathbb{G}_{a,\bar{\mathbb{F}}_p}$, and the bounded derived category $D^b_{\mathrm{IW}}(\mathrm{Fl}_I)$ in terms of $\chi^*\mathcal{L}_{\mathrm{AS}}$ following [AR, Subsection 6.4.2] and note it carries a right action of $D^b_{\mathrm{cons}}(L^+I\backslash \mathrm{Fl}_I)$. Define standard $\Delta^{\mathrm{IW}}_\lambda$, costandard $\nabla^{\mathrm{IW}}_\lambda$, and intersection complexes $\mathrm{IC}^{\mathrm{IW}}_\lambda$ for $\lambda \in \mathbb{X}_*(T)$ and note that $\Delta^{\mathrm{IW}}_0 = \nabla^{\mathrm{IW}}_0$ cf. [AR, Subsection 6.4.3]. Show that $\mathrm{Av}_{\mathrm{IW}} := \Delta^{\mathrm{IW}}_0 \star - :$ $D^b_{\mathrm{cons}}(L^+I\backslash \mathrm{Fl}_I) \to D^b_{\mathrm{IW}}(\mathrm{Fl}_I)$ is perverse t-exact as in [AR, Subsection 6.4.4] and verify that it is fully faithful after passing to the anti-spherical Serre quotient following [AR, Subsection 6.4.5]. Recall the notion of highest weight category, tilting objects, see [RW21, Section 20.1], and state that $Z^{\mathrm{IW}}_V := \mathrm{Av}_{\mathrm{IW}}(Z_V)$ is tilting, cf. [AR, Theorem 6.5.2]. Show that it propagates via tensor products as in [AR, Proposition 6.5.7] so it suffices to check minuscule and quasi-minuscule V, see [AR, Proposition 6.5.9].

Talk 8: The regular quotient and proof of the equivalence. Verify that Z^{IW}_V is tilting for minuscule V following [AR, Subsection 6.5.5]. Define the regular quotient of $\mathrm{Perv}(L^+I\backslash \mathrm{Fl}_I)$ as the Serre quotient by the positive dimensional IC sheaves, and show it respects various structures introduced so far, compare with [AR, Subsection 6.5.6]. Sketch the proof that the regular quotient is isomorphic to $\mathrm{Rep}(Z_{G^\vee}(u_0))$ where $u_0 \in G^\vee(\bar{\mathbb{Q}}_\ell)$ is regular unipotent, see [AR, Propositions 6.5.18, 7.2.8, 8.5.5]. Finish the proof that Z^{IW}_V is tilting for V quasi-minuscule, see [AR, Subsection 6.5.10], and mention the relation between the regular quotient of $\mathrm{Perv}(L^+I\backslash \mathrm{Fl}_I)$ and the regular orbit of $\tilde{\mathcal{N}}$, see [AR, Subsection 6.5.11]. Explain the preparations of [AR, Subsection 6.6.2] for proving the AB equivalence. Show $\mathrm{Av}_{\mathrm{IW}} \circ F$ is fully faithful via the dimension counting of [AR, Subsection 6.6.3].

Session 3. In this session, we are finally going to discuss the paper [Bez16].

Talk 9: Strategy outline and monodromic sheaves Sketch the proof of Bezrukavnikov's equivalence following [Bez16, Section 2], stressing the similarities to [AB09]. Briefly recall the notions on free-monodromic sheaves of [BY13, Appendix A], and show that our previous functor $Z \times J$ upgrades to a free-monodromic version following either [Bez16, Section 3] or preferably [BR22, Sections 7] for the construction of free-monodromic central sheaves. Cover at least [Bez16, Proposition 7, Corollary 12, Propositions 14 and 15, and Lemma 16].

Talk 10: Spectral action Construct a functor $\Phi^{\mathrm{Ho}}_{\mathrm{diag}}$ using Wakimoto sheaves, the central functor, the lowest weight arrow and the monodromy following [Bez16, Section 4] up to [Bez16, Proposition 20]. Discuss the torus monodromy and how to use it to construct an action of $\mathrm{Perf}(\widehat{\mathrm{St}}_0/G^\vee)$ on $\widehat{D}$. State at least [Bez16, Lemma 22] (and prove it if time permits). Discuss how convolving with Ξ_0 behaves following [Bez16, Section 5]. Discuss the properties of Φ_{Perf} following [Bez16, Section 6].

Talk 11: Extending from perfect complexes to coherent sheaves Explain the fully faithful embedding $D_{\mathrm{coh}}(X) \subset \mathrm{Fun}(\mathrm{Perf}(X)^{\mathrm{op}}, \mathrm{Vect})$ following [Bez16, Section 7] (If you wish, point out how this is similar to functions embedding into distributions, leading to the philosophy that perfect complexes are functions, and coherent sheaves are distributions). Check that the spectral actions constructed in previous talk satisfy the properties from [Bez16, Section 7] following [Bez16, Section 8].

Talk 12: Monoidality and DG categories Discuss the equivalences Φ_0 and Φ following [Bez16, Section 9]. Discuss the monoidal structure on Φ_0, Φ as well as the other compatibilites following [Bez16, Section 10]. Skip [Bez16, Subsection 10.1] if there is not enough time.

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