Geometric representation theory and p-adic geometry

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INTRODUCTION

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ABSTRACT. My work is focused on the geometry of affine Grassmannians in positive and mixed characteristic with a view toward applications to the arithmetic of Shimura varieties and the categoral local Langlands program. We introduce some of the motivation behind these questions, describe our own contributions, and finally explain some future directions. In positive characteristic, we study singularities of Schubert varieties in the wildly ramified case [FHLR22, CL25] and when the prime p is torsion [HLR24]. In mixed characteristic, we have constructed an analogue of Gaitsgory's central functor in [AGLR22, ALWY23] and applied it to the geometry of local models as in [AGLR22, GL24]. On the one hand, this geometry led us to studying p-adic period domains in [GL22]. On the other hand, central sheaves lead us to constructing an Arkhipov–Bezrukavnikov equivalence in mixed characteristic [ALWY23]. Besides that, we have proved a version of ramified Satake with mod ℓ coefficients [ALRR22, ALRR24].

CONTENTS

1. BACKGROUND

1.1. **Origins of Langlands.** One of the most celebrated theorems of Gauss [Gau86] is the quadratic reciprocity law. It states that for two distinct odd primes $p \neq q$, there is an equality

$$\left(\frac{p}{q}\right)\left(\frac{q}{p}\right) = (-1)^{\frac{p-1}{2}\frac{q-1}{2}}, \tag{1.1}$$

describing the signed anticommutativity of the Legendre symbol. This law is striking, precisely because it relates the existence of solutions to quadratic equations in two distinct finite fields. Conceptually, this is best understood as the specialization of the Artin reciprocity law for a quadratic extension, by rewriting the Legendre symbols in terms of the value of some p-Frobenius lift φ_p in $\mathrm{Gal}_{\mathbb{Q}(\sqrt{\pm q})/\mathbb{Q}} \simeq \{\pm 1\}$. Emil Artin [Art27] reformulated all previously known examples of higher degree reciprocity laws via the following statement: for any abelian Galois extension $F/\mathbb{Q}$ and every character $\rho\colon \mathrm{Gal}_{F/\mathbb{Q}} \to \mathbb{C}^\times$, there exists a Dirichlet character $\chi_\rho\colon (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$ for some N such that $\rho(\varphi_p) = \chi_\rho(p)$ for all unramified $p \nmid N$ (equivalently, one can ask for an equality of the associated L-functions). There is an adèlic formulation of the Artin law due to Chevalley [Che40] via an isomorphism

$$\mathbb{Q}^\times \backslash \mathbb{I}_\mathbb{Q} / \mathbb{R}_{>0} \simeq \mathrm{Gal}_\mathbb{Q}^{\mathrm{ab}} \tag{1.2}$$

where $\mathbb{I}_\mathbb{Q} := \mathbb{A}_\mathbb{Q}^\times$ is the group of rational idèles and $\mathbb{A}_\mathbb{Q} := \mathbb{R} \times (\hat{\mathbb{Z}} \otimes_\mathbb{Z} \mathbb{Q})$ is the ring of rational adèles. In particular, Artin reciprocity is of local-global nature and the local maps are dense injections $\mathbb{Q}_p^\times \to \mathrm{Gal}_{\mathbb{Q}_p}^{\mathrm{ab}}$ that admit explicit constructions via Lubin–Tate theory [LT65].

The full Galois group $\mathrm{Gal}_\mathbb{Q}$ being highly non-abelian, one would like to grasp its full structure as well, and the best mathematical tool to achieve this comes in the form of representations. So far, we have looked into characters, so the next natural step are 2-dimensional representations. The automorphic representations of $\mathrm{GL}_{2,\mathbb{Q}}$ can be more simply understood in terms of modular forms as investigated by Hecke [Hec27]. These are holomorphic functions f on the upper half space $\mathscr{H} = \{\tau \in \mathbb{C} : \mathrm{im}(\tau) > 0\}$ which transform equivariantly up to an automorphy factor under the natural action of some level subgroup $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ via Möbius transformations, and converge at the cusps in $\mathbb{Q} \cup \infty$. Famous examples of modular forms that one encounters in complex analysis are Eisenstein series, Jacobi theta functions, Ramanujan's Δ function, etc. Looking at the cusps of the smooth manifold $\Gamma \backslash \mathscr{H}$, we obtain a Fourier expansion for f. If f is a cusp form, we can attach to it an L-series $L(s, f)$. Deligne [Del71a] associated 2-dimensional Galois representations ρ_f with normalized cusp forms of weight ≥ 2 with the same L-series, but this goes very deep already and requires the Weil conjectures proved in [Del74]. In a converse direction, one would like to understand which 2-dimensional Galois representations arise from modular forms, a natural class being those ρ_E associated with elliptic curves E over $\mathbb{Q}$ via the natural $\mathrm{Gal}_\mathbb{Q}$-action on its Tate module $T_\ell(E) = \lim_n E[\ell^n]$ (where independence of the prime $\ell \neq p$ holds). The modularity theorem of Breuil–Conrad–Diamond–Taylor [BCDT01] states that every Galois representation ρ_E attached to an elliptic $\mathbb{Q}$-curve comes from a unique normalized cusp form. This is an even deeper theorem originally conjectured by Taniyama–Shimura [ST61], and proved in the semi-stable case by Taylor–Wiles [Wil95, TW95], famously implying Fermat's last theorem.

Now, let us try to formulate Langlands in the general setting. Let G be a connected reductive $\mathbb{Q}$-group. Automorphic representations of $G(\mathbb{A}_\mathbb{Q})$ appear as irreducible unitary subrepresentations of $L^2(G(\mathbb{Q})\backslash G(\mathbb{A}_\mathbb{Q}))$. Its corresponding L-function was defined by Langlands [Lan70] while studying constant terms of Eisenstein series: at an unramified prime p, the corresponding local factor relates to the Satake isomorphism, to be discussed next. For $G = \mathrm{GL}_{n,\mathbb{Q}}$, Langlands proposed that the Artin L-function $L(s, \rho)$ of an n-dimensional irreducible complex Galois representation ρ of $\mathrm{Gal}_\mathbb{Q}$ coincides with the L-function $L(s, \pi_\rho)$ of an automorphic representation π_ρ of $\mathrm{GL}_n(\mathbb{A})$. This is a crude version of the global Langlands correspondence that does not pin down the automorphic representation π_ρ. Its deficiencies lie in our lacking understanding of the Langlands dual group over number fields. Advances in the theory of Shimura varieties and p-adic Hodge theory have led to improved predictions, see the modern treatment by Buzzard–Gee [BG14].

Over $\mathbb{Q}_p$, we get the local Langlands correspondence (LLC) with a much clearer formulation. The LLC predicts a bijection.between the set $\Pi(\mathrm{GL}_{n,\mathbb{Q}_p})$ of isomorphism classes of smooth irreducible complex representations of $\mathrm{GL}_n(\mathbb{Q}_p)$ on the automorphic side and the set $\Phi(\mathrm{GL}_{n,\mathbb{Q}_p})$ of conjugacy classes of L-parameters of $\mathrm{GL}_n(\mathbb{C})$, where $W_{\mathbb{Q}_p}$ denotes the Weil group of $\mathbb{Q}_p$. This conjecture was proved by Harris–Taylor [HT01] using global ingredients coming from the cohomology of Shimura varieties, whose geometry will be described in the next section. For general reductive groups other than GL_n, one expects to have a map $\Pi(G) \to \Phi(G)$ with finite non-empty fibers Π_φ called L-packets, where $\Pi(G)$ denotes again the set of isomorphism classes of smooth irreducible complex representations of $G(\mathbb{Q}_p)$ and $\Phi(G)$ the set of conjugacy classes of L-parameters of the Langlands dual group $G^\vee(\mathbb{C})$. An exhaustive list of desiderata is due to Kaletha [Kal16] and we note that the structure of L-packets is still actively researched.

1.2. **Arithmetic of Shimura varieties.** In the previous section, we highlighted some of the main topics surrounding the origins of the Langlands program, a conjectural bridge relating automorphic to Galois representations. Lurking in the dark, there were some significant geometric spaces and cohomology theories from which one is supposed to extract the link between the representations. Over $\mathbb{Q}$ and for the group $\mathrm{GL}_{2,\mathbb{Q}}$, these spaces are the (non-compact) modular curves X_Γ, whose complex analytifications arise as quotients of the upper half space $\mathscr{H} = \{\tau \in \mathbb{C} : \mathrm{im}(\tau) > 0\}$ by arithmetic subgroups $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$. There are two more useful ways of perceiving this object, the first of which is via adèlic uniformization (see the general case below) as follows by writing $\mathscr{H} = \mathrm{SL}_2(\mathbb{R})/\mathrm{SO}_2(\mathbb{R})$ as an Hermitian symmetric space, and then running over compact open subgroups $K \subset \mathrm{SL}_2(\mathbb{A}_\mathbb{Q})$. More importantly, X_Γ is a moduli space of elliptic curves with level structure that is naturally defined over a finite extension $E/\mathbb{Q}$. At the level of $\mathbb{C}$-valued points, this relates to the fact that the complex tori $\mathbb{C}/\Lambda_\tau$ with $\Lambda_\tau := (\mathbb{Z} + \tau\mathbb{Z})$ is isomorphic to elliptic $\mathbb{C}$-curves as follows by considering the Weierstrass function $\wp_\tau(z)$ and its derivative. One can also study integral models of the modular curves X_Γ over (some appropriate localization of) the ring of integers O_E and their corresponding reductions at each prime. Deligne–Rapoport [DR73] provided a description of the reduction at Iwahori level for some prime p, with the famous picture of two intertwinning lines at the supersingular points. This geometry of the reduction modulo p is a crucial input to understand congruences of modular forms via the Eichler–Shimura relations.

In higher dimensions, one considers instead moduli spaces of principally polarized abelian varieties with level structure (also known as PEL type Shimura varieties), an explicit example being quotients of the Siegel upper half-space by arithmetic subgroups of $\mathrm{Sp}_{2n}(\mathbb{Z})$. It is not a coincidence that the symplectic group $\mathrm{Sp}_{2n,\mathbb{Q}}$ appears instead of the special $\mathrm{SL}_{n,\mathbb{Q}}$ or general linear groups $\mathrm{GL}_{n,\mathbb{Q}}$, as the automorphic quotients the latter for $n > 2$ are never complex manifolds. Indeed, consider the Riemannian symmetric space $\mathcal{D} = G(\mathbb{R})/Z_G(\mathbb{R})^+K_\infty$ obtained by quotienting out the connected component of the center $Z_G(\mathbb{R})$, and a compact real Lie subgroup $K_\infty \subset G(\mathbb{R})$ with maximal compact Lie subalgebra. We get a tower of real orbifolds

$$X_K := G(\mathbb{Q})\backslash(\mathcal{D} \times G(\mathbb{A}_\mathbb{Q}^\infty))/K \tag{1.3}$$

for varying compact open subgroups $K \subset G(\mathbb{A}_\mathbb{Q}^\infty)$, identifying with the disjoint union of quotients $\Gamma\backslash\mathcal{D}$ by arithmetic subgroups $\Gamma \subset G(\mathbb{Q})$. The more restrictive notion of a Shimura variety was introduced by Deligne [Del71b] building on numerous examples worked out by Shimura [Shi63]: the symmetric space $\mathcal{D}$ arises as the conjugacy class of homomorphisms $\mathbb{S} := \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m \to G_\mathbb{R}$, and satisfies a series of axioms to ensure that $\mathcal{D}$ is Hermitian and classifies variations of complex Hodge structures. These turn out to be quasi-projective smooth DM stacks defined over a number field E, a non-trivial result achieved across many decades by combining work of Baily–Borel [BB66], Deligne [Del79], Borovoi [Bor83], and Milne [Mil83]. Shimura varieties play a distinguished role in the Langlands program, because their étale cohomology relates to both automorphic and Galois representations.

There is now a consolidated industry of studying integral models $\mathscr{S}_\mathcal{G}$ of Shimura varieties at parahoric level $\mathcal{G}$, whose origins go back to work of Chai–Norman [CN90], de Jong [dJ93], and Deligne–Pappas [DP94]. Indeed, for the Siegel case, one defines parahoric integral models of the Shimura variety by classifying chains of principally polarized abelian varieties. A more systematic treatment in the PEL case was given by Rapoport–Zink [RZ96], where they also study p-adic uniformization of these integral models via moduli spaces of deformations of p-divisible groups. An important technique used to comprehend the singularities of integral modules of Shimura varieties is the local model, i.e., a smooth map

$$\mathscr{S}_\mathcal{G} \to [\mathcal{G}_{O_E}\backslash M_{\mathcal{G},\mu}] \tag{1.4}$$

of the same dimension as G, where $E/\mathbb{Q}_p$ is the reflex field of a certain minuscule coweight μ coming from the Shimura datum and $M_{\mathcal{G},\mu}$ is the so-called local model. Usually, e.g., if G is of Hodge type, then local models appear as flat closed subschemes of a parabolic quotient of GL_n. How to actually define them correctly and understand them better is one of our main motivations and will be discussed further below.

1.3. **Geometric Langlands.** Let us now consider the analogous situation over function fields. Consider a geometrically connected proper smooth curve X over $\mathbb{F}_p$; it carries a canonical Frobenius homomorphism φ. Let G be a connected reductive group over the curve X and consider the moduli stack Bun_G classifying G-bundles on the curve X. This is a smooth Artin $\mathbb{F}_p$-stack and we can write

$$\mathrm{Bun}_G(\mathbb{F}_q) = G(\mathbb{F}_p)\backslash G(\mathbb{A}_X)/G(\mathbb{O}_X) \tag{1.5}$$

This automorphic quotient allows us to define automorphic forms by looking at the ℓ-adic constant sheaf for some prime $\ell \neq p$. Besides Bun_G, we also have the Hecke stack Hk_G classifying modifications $\mathcal{E}_0 \dashrightarrow \mathcal{E}_1$ of G-bundles along a point of the curve $x \in X$: it comes equipped with a Hecke correspondence towards Bun_G, geometrizing the classical Hecke operators on modular forms. Moreover, there is the stack of G-shtukas Sht_G given as the pullback of the Frobenius graph of Bun_G along the natural map $\mathrm{Hk}_G \to \mathrm{Bun}_G$. Shtuka stacks are the function field analogue of Shimura varieties over number fields, but they are no longer bounded by minuscule coweights. Their cohomology has been exploited by many authors, most notably Drinfeld [Dri80], Laumon–Rapoport–Stuhler [LRS93], L. Lafforgue [Laf02], V. Lafforgue [Laf18], and Genestier–Lafforgue [GL17] to prove various instances of the local and global Langlands correspondence. These last two works in particular construct a map $\Pi_G \to \Phi_G^{\mathrm{ss}}$ from either automorphic irreducible representations of G in the global setting or smooth admissible irreducible representations of G in the local setting to semisimple L-parameters of the dual group $G^\vee$.

Several ingredients go into the proof of this theorem, the most original one being the construction of excursion operators. Another fundamental ingredient that occurs frequently in our work is the geometric Satake equivalence. Let $O_{\bar{x}}$ be the complete local ring of X at the geometric point $\bar{x}$ and $F_{\bar{x}}$ be its fraction field. The classical Satake isomorphism following [Sat63] identifies the spherical Hecke algebra $\mathcal{H}_G := \mathbb{C}[G(O_{\bar{x}})\backslash G(F_{\bar{x}})/G(O_{\bar{x}})]$ of G with the Weyl invariants $\mathcal{H}_T^W = \mathbb{C}[X_*(T)]^W$ of the Hecke algebra of a maximal torus $T \subset G$. Upon passing to ℓ-adic coefficients, we can identify $\mathcal{H}_G$ with the Grothendieck group K_0 of étale ℓ-adic sheaves on the fiber $\mathrm{Hk}_{G,\bar{x}}$ of the Hecke stack at the geometric point x and $\mathcal{H}_T^W$ with that of representations of the dual group $G^\vee$. This observation can be upgraded to an equivalence of categories by work of Mirković–Vilonen [MV07] in the form of a symmetric monoidal equivalence of abelian categories between the category $\mathcal{P}(\mathrm{Hk}_{G,\bar{x}})$ of perverse ℓ-adic sheaves on the Hecke stack in the sense of Beilinson–Bernstein–Deligne–Gabber [BBDG18] and the category $\mathrm{Rep}(G^\vee)$ of representations of the dual group of G. In particular, the geometric Satake equivalence furnishes a plethora of perverse sheaves (also known as Satake sheaves) on the Hecke stack, which are used as the convolution kernels of geometric Hecke operators. The proof of the equivalence exploits the geometry of the affine Grassmannian Gr_G, an ind-scheme living as a pro-smooth torsor over Hk_G, which turns out to be the natural space within which local models of Shimura varieties also live.

1.4. **Perfectoids.** Recently, many concepts from geometric Langlands have been adapted to $\mathbb{Q}_p$, most notably by Fargues–Scholze [FS21], building on the theory of perfectoid spaces. The first difficulty is the absence of a decent curve over $\mathbb{F}_p$ with an absolute Frobenius and the solution comes from looking at Witt vectors. Recall that perfectoid rings (R, R^+) are infinitely ramified ϖ-adically complete rings with surjective Frobenius modulo p. The tilting functor of Scholze [Sch12] takes mixed characteristic perfectoid rings to perfect $\mathbb{F}_p$-algebras and Kedlaya–Liu [KL15] proved that untilts are parametrized by special principal ideals in $W(R^+)$. One can then consider the non-vanishing locus Y_R of $p[\varpi]$ inside the adic space of $W(R^+)$ in the sense of Huber [Hub96], and define the Fargues–Fontaine curve X_R following Fargues–Fontaine [FF18] as the quotient by Frobenius. In particular, Bun_G is now the stack of G-torsors on X. Its geometric points are in bijection with Kottwitz's set $B(G)$ classifying φ-conjugacy classes in $G(\breve{\mathbb{Q}}_p)$ by a theorem of Fargues [Far20], with topology explicitly described via the combinatorics of Newton polygons as shown by Viehmann [Vie24]. For a certain formalism of ℓ-adic sheaves, the derived category $\mathcal{D}(\mathrm{Bun}_G)$ captures smooth representations of the inner forms J_b of the Levi subgroups of G attached to $b \in B(G)$, glued in a yet mysterious way. One can also define the Hecke stacks Hk_G, affine Grassmannians Gr_G and shtuka stacks Sht_G as v-stacks over the mirror curve Div^1_X. Using the concept of universally locally acyclic sheaves, [FS21] proved the geometric Satake equivalence for Hk_G. Most of the formal arguments in [Laf18] concerning excursion operators can be repeated to yield the local Langlands map $\Pi(G) \to \Phi^{\mathrm{ss}}(G)$ over $\mathbb{Q}_p$ or other p-adic fields.

Besides, the Galois side of the LLC can be geometrized via the stack Par_G of L-parameters studied by Zhu [Zhu20] and Dat–Helm–Kurinczuk–Moss [DHKM20], which classifies conjugacy classes of continuous 1-cocycles $\varphi\colon W_{\mathbb{Q}_p} \to G^\vee(\Lambda)$ for any $\mathbb{Z}_\ell$-algebra Λ. The categorical analogue of the classical Langlands map is a correspondence between $\mathcal{D}(\mathrm{Bun}_G)$ and the category of ind-coherent sheaves on Par_G. Similar versions of this conjecture are due to Hellmann [Hel23] and Zhu [Zhu20]. The functor itself is uniquely characterized by the choice of a Whittaker sheaf $\mathcal{W}_\psi$, i.e., obtained via compact induction from a Whittaker datum ψ on a maximal unipotent subgroup $U(\mathbb{Q}_p)$, and the spectral action in [FS21]. The latter extends the Hecke operators on $\mathcal{D}(\mathrm{Bun}_G)$ given by convolving with Satake sheaves to an action of the category $\mathrm{Perf}(\mathrm{Par}_G)$ of perfect complexes on the stack of L-parameters.

2. Research description

Our work has focused on studying the geometry and cohomology of affine Grassmannians in both the function field and p-adic setting. We use Bruhat–Tits theory developed in [BT72, BT84] and the parahoric integral models $\mathcal{G}$ over the ring of integers O of connected reductive groups G over a local field F. There is a moduli space $\mathrm{Gr}_{\mathcal{G}}$ over O due to Beilinson–Drinfeld [BD91] that interpolates between a parahoric affine flag variety $\mathrm{Fl}_{\mathcal{G}}$ over the residue field k and the affine Grassmannian Gr_G over F. This also fits into the previous discussion on Shimura varieties, as local models typically embed into Beilinson–Drinfeld Grassmannians. Some of our geometric results carry mostly topological information, especially in the p-adic setting, and seem to relate more specifically to local Langlands with $\ell \neq p$ coefficients; whereas in equicharacteristic we tend to deal with non-reduced structures, hence linked more tightly to the emerging local Langlands with $\ell = p$ coefficients.

2.1. **Splinters.** This section is concerned with [FHLR22, CL25]. Let k be an algebraically closed field, $O = k[\![t]\!]$ its power series ring, and $F = k(\!(t)\!)$ its Laurent series field. Let G be a connected reductive F-group and $\mathcal{G}$ a parahoric O-model of G in the sense of Bruhat–Tits [BT72, BT84]. Following [PR08], we denote by $\mathrm{Fl}_{\mathcal{G}}$ the affine flag variety attached to $\mathcal{G}$. The Bruhat decomposition describes the $\mathcal{G}(O)$-orbits of $\mathrm{Fl}_{\mathcal{G}}$ as double cosets of the Iwahori–Weyl group W and let $\mathrm{Fl}_{\mathcal{G},\leq w}$ denote the corresponding reduced orbit closure for $w \in W$. Let $\mathrm{Fl}^{\mathrm{sn}}_{\mathcal{G},\leq w}$ be their seminormalization, i.e., the largest finite cover with the same topological space and residue fields.

Theorem 2.1 ([FHLR22, CL25])**.** *The variety $\mathrm{Fl}^{\mathrm{sn}}_{\mathcal{G},\leq w}$ is globally $+$-regular, i.e., the inclusion of the structure sheaf into its absolute integral closure splits as modules.*

We follow here the terminology from [BMP$^+$23], but this notion is equivalent to the derived splinters in the sense of [Bha12]. In particular, it follows that seminormalized Schubert varieties are normal, φ-split, and even Cohen–Macaulay by a result of Hochster–Huneke [HH92], see also [Bha20] for a modern reference in mixed characteristic. In [FHLR22], we proved it by invoking a Mehta–Ramanathan criterion [MR85], which requires a theta divisor on the full flag variety as an input: this step is difficult and could only be done on a case-by-case basis. In [CL25], we employ a criterion due to Bhatt [Bha12] or rather its refinement in [BMP$^+$23] to dispense with the theta divisor. The most striking feature of [CL25] is that it applies in the p-adic case: more precisely, we prove global $+$-regularity of sufficiently small Schubert varieties in the Witt flag variety (e.g. contained in minuscule μ-admissible loci).

Next, we look at the Beilinson–Drinfeld Grassmannian $\mathrm{Gr}_{\mathcal{G}}$ over the ring of integers O. Its generic fiber is the classical affine Grassmannian Gr_G over F and its special fiber is none other than $\mathrm{Fl}_{\mathcal{G}}$. We define the local model $M_{\mathcal{G},\mu}$ as the orbit closure of the Schubert variety $\mathrm{Gr}_{G,\leq\mu}$ inside $\mathrm{Gr}_{\mathcal{G},O_E}$, where μ is a geometric conjugacy class of coweights of G and E is its reflex field.

Theorem 2.2 ([FHLR22, CL25])**.** *The scheme $M^{\mathrm{sn}}_{\mathcal{G},\mu}$ is normal, Cohen–Macaulay, and its special fiber equals $A^{\mathrm{sn}}_{\mathcal{G},\mu}$.*

Here, $A_{\mathcal{G},\mu}$ is the union of the Schubert varieties indexed by the μ-admissible set of Kottwitz–Rapoport [KR00]. This was originally the coherence conjecture of [PR08] and it was proved for tame G in [Zhu14]. In [FHLR22] we prove this in the presence of certain Witt lifts, excluding odd unitary groups when $p = 2$. In [CL25], we give a different argument via a combinatorial trick. Again, this new approach has implications to the p-adic setting, yielding a key ingredient in [AGLR22], originally proved in the presence of Witt lifts: an equality of Hilbert polynomials between a deperfection of the μ-admissible locus and the classical flag variety G/P_μ.

2.2. **Normality.** This section is focused on [HLR24, Lou23b]. We work with affine flag varieties in equicharacteristic. We say that a prime p is torsion for G if the fundamental group $\pi_1(G)$ has non-trivial p-torsion. We have the following general result.

Theorem 2.3 ([Lou23b, HLR24])**.** *If p is not torsion, then $\mathrm{Fl}_{\mathcal{G},\leq w}$ is normal. Otherwise, it is not normal for $w \gg 0$.*

The normality assertion for non-torsion p was proved by Faltings [Fal03] for split G, then Pappas–Rapoport [PR08] for tame G, and finally [FHLR22] in the presence of Witt lifts. Our proof in [Lou23b] is the first to take place entirely in positive characteristic by replacing Lie-theoretic arguments with distributions, i.e., higher order differential operators. We also make use of condensed mathematics by Clausen–Scholze [CS19] in order to streamline certain arguments involving topological associative k-algebras.

If p is torsion, the existence of many non-normal Schubert varieties was established in [HLR24]. This discovery came as a surprise to the experts, even though the argument is remarkably simple. It was also the first time that the normalization functor appeared as a necessity in the theory of local models, vindicating the conjecture in [SW20]. Besides this, we gave an explicit criterion for normality via a tangent space obstruction that can be calculated by a formula of Kumar–Polo [Pol94, Kum95], and a complete classification was then described in [BR23].

2.3. **p-adic local models.** This section is focused on [AGLR22]. This paper was motivated by the developments in [Sch17, SW20] on the theory of diamonds allowing for the definition of affine Grassmannians and local models over p-adic rings. Let k again be algebraically closed, O be a discrete valuation ring finite over $W(k)$, and F its fraction field. We pick a connected reductive group G over F and a parahoric O-model $\mathcal{G}$ in the sense of [BT72, BT84]. We have now a v-sheaf analogue $\mathrm{Gr}_{\mathcal{G}}$ of the Beilinson–Drinfeld Grassmannian equipped with a structure map towards $\mathrm{Spd}(O)$. The local model $M_{\mathcal{G},\mu}$ is again the closure as a v-sheaf of a Schubert diamond $\mathrm{Gr}_{G,\leq\mu}$ inside $\mathrm{Gr}_{\mathcal{G},O_E}$.

Theorem 2.4 ([AGLR22])**.** *If μ is minuscule, then $M_{\mathcal{G},\mu}$ is representable by a unique normal, projective, flat O_E-scheme with φ-split special fiber and, if G has no p-divisible roots, even Cohen–Macaulay.*

This proves a conjecture by Scholze–Weinstein [SW20] and shows that previous ad hoc constructions from [PZ13, Lev16, FHLR22] in mixed characteristic are actually canonical when μ is minuscule. The statements on the geometry of local models appearing in [AGLR22] can now be mostly imported from [GL24, CL25], so the crux of the paper is proving representability. Along the way, we also compute the special fiber via nearby cycles even for non-minuscule μ. Note that the non-normality of [HLR24] ceases to be an issue, because perfectoids only capture the underlying topology.

While representability was known for groups of abelian type, we provide a uniform proof for arbitrary reductive groups based on a specialization principle going back to [Lou17] in the case of formal schemes. This became possible via work of Gleason [Gle24] on a notion of kimberlites, a class of v-sheaves relating to diamonds in the same manner formal schemes relate to rigid-analytic spaces. Therefore, one gets a fully faithful triple for the v-sheaf $M_{\mathcal{G},\mu}$ consisting of: the generic fiber, in this case a diamond paved by Banach–Colmez spaces; a special fiber representable by a perfect scheme in light of [BS17]; and a specialization map first constructed in abstract terms by Anschütz [Ans22].

The first demanding task is to compute the special fiber, so we consider the ∞-category $\mathcal{D}(\mathrm{Hk}_{\mathcal{G}})$ of étale $\bar{\mathbb{Q}}_\ell$-sheaves on the Hecke stack $\mathrm{Hk}_{\mathcal{G}}$ in the sense of [Sch17, FS21]. In the generic fiber, one has the geometric Satake equivalence, and integrally we exploit the constant term functors by taking cohomology of Borel semi-infinite orbits in the affine Grassmannian. We deduce that universally locally acyclic sheaves on the Hecke stack extend uniquely from the algebraic closure C to its ring of integers O_C and reduce in the special fiber to the usual constructible sheaves. Following [HR21] in a related setting, we calculate the support of nearby cycles $Z(V) := \Psi(S(V))$ of Satake sheaves and deduce:

Theorem 2.5 ([AGLR22])**.** *The special fiber of $M_{\mathcal{G},\mu}$ is representable by $A_{\mathcal{G},\mu}$.*

In [AGLR22], we conjectured the perversity of the $Z(V)$: this was answered positively in [ALWY23]. Meanwhile we proved that the sheaves $Z(V)$ are central for convolution. Finally, we also prove the Haines–Kottwitz conjecture [Hai14] for $M_{\mathcal{G},\mu}$ in all cases, compare with [HR21] in a slightly different setting. The conjecture predicts the existence of certain test functions in the Hecke algebra of $\mathcal{G}$ over a p-adic field, which for minuscule μ play a role in describing the cohomology of Shimura varieties within the Langlands–Kottwitz method [Kot92]. These arise via trace of Frobenius applied to the stalks of $Z(V)$ and yield the central function of [Hai15] defined via Satake parameters.

The final ingredient in [AGLR22] is characterizing the specialization map, and for this μ has to be minuscule again. We were inspired by a conjecture of He–Pappas–Rapoport [HPR20] proposing that knowing the fibers together with their $\mathcal{G}$-action should determine the specialization. Indeed, we show a weaker variant of this by working with sequences $\mu_\bullet$ of jointly minuscule coweights and fixing the specialization of points in the dense open integral $\mathcal{G}$-orbits. The proof boils down to some ramified analogue of the Iwasawa decomposition. Looking closely at the candidates for scheme-theoretic local models found in [PZ13, Lev16, FHLR22], we can now conclude that they have the same specialization triple as $M_{\mathcal{G},\mu}$, so this yields representability.

2.4. **Unibranchness.** In this section, we describe [GL24]. The normality of local models was discussed in the previous section by embedding them into actual ind-schemes, which is not available in the p-adic setting. Besides, having a reduced special fiber is usually not a topological statement. However, we observed in [AGLR22] that this reducedness would be a consequence of the local model being geometrically unibranch, i.e., the normalization map is a universal homeomorphism. This amounts to connectedness of the rigid-analytic tubes of the local model, and makes sense even for non-minuscule μ.

Theorem 2.6 ([GL24]). *The local models $M_{\mathcal{G},\mu}$ are geometrically unibranch.*

In equicharacteristic, this unibranchness was proved by Le–Le Hung–Levin–Morra [LHLM22] for tame G using dynamical methods and the rotation action on the parameter t. Over the p-adics, the rotation action does not work, so instead we consider the nearby cycles $Z(V_\mu)$.

One crucial ingredient for this is a comparison of formal nearby cycles defined via the site of kimberlites in the sense of [Gle24] and analytic nearby cycles in the sense of [Sch17] using v-descent from spatial diamonds (the latter are the ones used in [AGLR22]). This builds upon a vanishing of partially compactly supported cohomology in [FS21] and it is a quite versatile result that has found applications in subsequent work, see [GIZ23]. In particular, the central sheaves $Z(V)$ relate to the geometry of the local models $M_{\mathcal{G},\mu}$.

Another key ingredient of our proof is the perversity of the sheaves $Z(V_\mu)$ due to [ALWY23]. Moreover, it turns out that at Iwahori level $Z(V_\mu)$ is filtered by perverse sheaves with grading isomorphic to $J(V_\mu)$, where J is the Wakimoto functor defined as in [AB09]. Then, the stalks of $Z(V_\mu)$ away from codimension 2 involve a two-term extension by counting irreducible components via [Hai05], implying unibranchness. As for the codimension 2 locus, we reduce it via the kimberlite étale site to checking that $A_{\mathcal{G},\mu}$ is S_2 and then perform a combinatorial trick.

2.5. **Connectedness of period domains.** This section describes [GL22]. The moduli space Bun_G of G-torsors on the Fargues–Fontaine curve X carries the Newton stratification into locally closed strata Bun_G^b for each $b \in B(G)$. We get a Beauville–Laszlo uniformization map $\mathrm{BL}_b\colon \mathrm{Gr}_G \to \mathrm{Bun}_G$ by modifying the canonical G-torsor $\mathcal{E}_b$ over the curve. Now, we pullback the Newton stratification along BL_b and intersect it with Schubert diamonds to get Newton strata $\mathrm{Gr}_{G,\leq\mu}^{b'\mapsto b} \subset \mathrm{Gr}_{G,\leq\mu}$. If μ is minuscule, these period domains first appeared in [RZ96] and continued to be studied for many decades.

Theorem 2.7 ([GL22]). *The period domain $\mathrm{Gr}_{G,\leq\mu}^{1\mapsto b}$ is geometrically connected.*

This confirms a conjecture of Hartl [Har13] and plays a role, along with unibranchness of local models, in calculating the π_0 of affine Deligne–Lusztig varieties, see [GLX22]. The proof consists in showing that non-basic Newton strata in $\mathrm{Gr}_{G,\leq\mu}$ have cohomological dimension strictly below $\langle 2\rho, \mu\rangle$ and perform an inductive argument by passing to appropriate Levis. For this, we study the geometry of parabolic induction, i.e., the diagram $\mathrm{Bun}_M \leftarrow \mathrm{Bun}_P \to \mathrm{Bun}_G$. The left map is smooth of local dimension $\langle 2\rho_G - 2\rho_M, \nu_m\rangle$ by a result of Hamann [Ham22], where ν_m is the Newton polygon of a basic element $m \in B(M)$. The right map is rarely smooth, but it becomes so after restricting to an open subset whose complement has strictly smaller dimension by bootstrapping dominant basic elements in [FS21] to merely positive basic.

2.6. **Arkhipov–Bezrukavnikov for p-adic groups.** This section is focused on [ALWY23]. Gaitsgory [Gai01] constructed central sheaves on the equicharacteristic Hecke stack $\mathrm{Hk}_{\mathcal{I}}$ at Iwahori level by taking nearby cycles. This same functor was considered in [AGLR22] and in [ALWY23] we could prove its centrality:

Theorem 2.8 ([ALWY23]). *The functor $Z\colon \mathcal{D}(\mathrm{Hk}_{G,C}) \to \mathcal{D}(\mathrm{Hk}_{\mathcal{G},k})$ is central, i.e., it lifts to a monoidal functor towards the Drinfeld center of the right side.*

A significant novelty of our proof is that it takes place in the setting of ∞-categories, yielding a much stronger statement than its usual triangulated counterpart. If we restrict it to Satake sheaves, the functor also respects braidings, so it is $\mathbb{E}_2$-monoidal. In characteristic p algebraic geometry, nearby cycles preserve perversity by [BBDG18], thanks to Artin vanishing. A similar vanishing is outright false for general v-sheaves, but we were able to prove the following:

Theorem 2.9 ([ALWY23])**.** *The central sheaf $Z(V)$ is perverse. At Iwahori level, it admits a Wakimoto filtration with graded isomorphic to $J(V)$.*

This is not immediately clear from looking at constant terms and we complement it via the Wakimoto functor $J\colon \mathrm{Rep}(\hat{T}) \to \mathcal{P}(\mathrm{Hk}_{\mathcal{I}})$ defined in [AB09]. It depends on the choice of a Borel $B \subset G$ and arises by linearly extending costandard sheaves $\nabla_\nu = j_{\bar{\nu},*}\Lambda[\langle 2\rho, \nu\rangle]$ from the monoid of B-dominant weights ν. The central sheaves $Z(V)$ are in the stable full subcategory spanned by J, and taking constant terms yields perversity for each piece.

The natural continuation of this story is reproving the Arkhipov–Bezrukavnikov equivalence [AB09] in the p-adic setting. Assume G is split from now on and let $\mathrm{Spr}_{\hat{G}}$ be the stack quotient of the dual Springer variety over $\bar{\mathbb{Q}}_\ell$ by its $\hat{G}$-action. In [AB09, ALWY23] a functor $F\colon \mathrm{Coh}(\mathrm{Spr}_{\hat{G}}) \to \mathcal{D}(\mathrm{Hk}_{\mathcal{IW}})$ is defined extending Z, J and the nilpotent logarithm of the unipotent monodromy n_V on $Z(V)$. Here, $\mathcal{D}(\mathrm{Hk}_{\mathcal{IW}})$ is the Iwahori–Whittaker quotient of the Hecke category, whose perverse heart $\mathcal{P}(\mathrm{Hk}_{\mathcal{IW}})$ is spanned by tilting objects in the sense of Beilinson–Ginzburg–Soergel [BGS96].

Theorem 2.10 ([ALWY23])**.** *If G has enough minuscules, then F is an equivalence.*

The crux of the argument is proving that central sheaves have tilting images in the Iwahori–Whittaker category. Unfortunately, we are missing some of the ingredients of [AB09], namely Gabber's local weight-monodromy theorem proved in [BB93], and the loop rotation action to bound $\mathrm{Hom}(Z(1), Z(V))$ with V quasi-minuscule. Thanks to their representability, minuscule local models satisfy the local weight-monodromy theorem by Hansen–Zavyalov [HZ23]. We thus have to exclude those groups whose simple adjoint factors have no minuscule coweight, i.e., of type E_8, F_4 and G_2.

2.7. **Ramified Satake mod ℓ.** In this section, we discuss [ALRR22, ALRR24]. We work again in equicharacteristic and $\mathcal{G}$ is assumed to be special parahoric. We let Λ be either $\bar{\mathbb{F}}_\ell$, $\bar{\mathbb{Z}}_\ell$ or $\bar{\mathbb{Q}}_\ell$. We prove ramified geometric Satake describing the category of perverse Λ-sheaves on the Hecke stack $\mathrm{Hk}_{\mathcal{G}}$, generalizing [Zhu15, Ric16] when $\Lambda = \bar{\mathbb{Q}}_\ell$.

Theorem 2.11 ([ALRR24])**.** *There is a monoidal equivalence $\mathcal{P}(\mathrm{Hk}_{\mathcal{G}}, \Lambda) \simeq \mathrm{Rep}_\Lambda(G^\vee)^\Gamma$ compatible with unramified geometric Satake via nearby cycles.*

Here, $(G^\vee)^\Gamma$ denotes the scheme-theoretic fixed points. We studied them in [ALRR22], proving their flatness over Λ by a big cell analysis. It is reductive iff $X_*(T)_\Gamma$ is ℓ-torsion free and Φ_G has no ℓ-divisible root: indeed, if $\ell = 2$ and $G = \mathrm{SU}_{2n+1}$, then $(G^\vee)^\Gamma$ is not smooth and coincides with the counterexample of Prasad–Yu [PY06] to a conjecture in [MV07] on reductive group schemes. Our proof of the equivalence follows closely that of [MV07], but we have to circumvent the lack of a fusion product, see [BD91], because G is ramified. Instead, we decompose it via the cohomology of Mirković–Vilonen cycles. Another difference lies in determining the dual group, as some work is required to overcome the counterexample of [PY06].

3. Future projects

We describe our research projects in an initial to middle development phase. The first project is concerned with the Bezrukavnikov equivalence [Bez16] for p-adic groups, building on [AGLR22, ALWY23]. The second project amounts to comparing the derived category of étale sheaves on the classifying stack Isoc_G of G-isocrystals defined by Hemo–Zhu with the derived category of lisse sheaves on Bun_G from [FS21]. The third project is a geometrization of the Donkin conjecture on tilting modules of reductive groups, building on the mod ℓ Satake equivalence of [ALRR22, ALRR24]. The fourth project is related to the coherent and mod p cohomology of affine flag varieties for p-adic groups. The last project concerns a proposed Tannakian moduli description of local models.

3.1. **Bezrukavnikov for p-adic groups.** In this subsection, we fix a prime $\ell \neq p$ and work with $\bar{\mathbb{Q}}_\ell$-coefficients. Let G be a split connected reductive group over F with Iwahori O-model $\mathcal{I}$. Kazhdan–Lusztig [KL87] identified the K_0 of $G^\vee$-equivariant coherent sheaves on the dual derived Steinberg $\bar{\mathbb{Q}}_\ell$-stack $\mathrm{St}_{G^\vee}$ (i.e., the derived fiber product of the dual Springer stack with itself over the quotient stack $G^\vee \backslash \mathfrak{g}^\vee$) with the group ring $\mathbb{Z}[W]$ of the Iwahori–Weyl group W of G. A similar calculation holds for the K_0 of étale $\bar{\mathbb{Q}}_\ell$-sheaves on $\mathrm{Hk}_\mathcal{I}$. It is natural to wonder if there is a categorical enhancement of these identification:

Conjecture 3.1. *There is an equivalence $\Phi\colon \mathrm{IndCoh}(\mathrm{St}_{G^\vee}) \to \mathcal{D}(\mathrm{Hk}_\mathcal{I})$ of ∞-categories extending the functor F of* [ALWY23].

In equicharacteristic, Bezrukavnikov [Bez16] construct such an equivalence extending his functor with Arkhipov [AB09] (originally working at the triangulated level, but it is well understood by the experts how to upgrade this to the ∞-categorical setting). We note that Yun–Zhu had announced an equivalence between the Hecke categories $\mathcal{D}(\mathrm{Hk}_\mathcal{I})$ in equal or mixed characteristic, derived from a presentation due to Tao–Travkin [TT20]. In particular, they do not use p-adic geometry as in [SW20, FS21] and hence the equivalence is not clearly compatible with the functors in [ALWY23]. At the same time, Bando [Ban23] identified the Hecke categories for different local fields by constructing the reduction modulo p of a surface analogue of the Fargues–Fontaine curve, and then connecting the different fibers by taking nearby cycles along valuation rings. Note that these are not the same nearby cycles arising in the p-adic context of [AGLR22] but an auxiliary construction that interpolates between the Hecke stack in equal and mixed characteristic. In a joint project with Bando–Gleason–Yu, we plan on exploiting the full Fargues–Fontaine surface to compare the p-adic and t-adic nearby cycles in [Gai01] and [AGLR22] along the equivalence of [Ban23]. This will also extend the main result of [ALWY23] to all groups G, even those of type E_8, F_4 and G_2 which lack minuscule coweights. For the comparison to work, it will require a good behavior of higher dimensional nearby cycles with respect to composition and products, that is very particular to the parahoric Hecke stacks and appeared already in [ALWY23] to some extent. This usually fails in classical algebraic geometry, as Orgogozo [Org06] shows one usually has to blow up the base, and it would also fail for deeper levels than Iwahori or its pro-p unipotent radical, as we will explain in the next section.

Another goal of our project is to construct the equivalence Φ from scratch in the p-adic context. In particular, we will essentially prove that the equivalence Φ is uniquely determined by the central functor Z (together with its monodromy) and the Wakimoto functor J. This will imply that the p-adic and t-adic Bezrukavnikov equivalences coincide along the one in [Ban23]. We had been following [Bez16] and the undergraduate thesis of James Tao supervised by Gaitsgory to perform the construction of the equivalence building on [ALWY23]. However, very recent work of Dhillon–Taylor [DT25b, DT25a] provides a quicker and more pleasant avenue from F to Φ in equicharacteristic for tame monodromic Betti sheaves, and we now aim to adapt it to our context. In fact, we should mention that there are several important variants of the equivalence, obtained by enlarging the category of sheaves under consideration. For instance, one can relax left and right $\mathcal{I}$-equivariance to only equivariance for the pro-p Iwahori and a monodromic condition for the torus, and this leads to the category of free-monodromic sheaves $\mathcal{D}(\mathrm{Hk}_{\mathcal{I}_u})^\wedge$ in the sense of Yun [BY13]. Correspondingly, there's some coherent replacement for the Steinberg stack allowing one to get another equivalence of stable ∞-categories in the pro-p Iwahori setting. Recently, Étève [Ete23] has provided another perspective on these monodromic sheaves that will likely be useful, especially if we want to go beyong unipotent monodromy. From the point of view of categorical local Langlands, the upgrade to the monodromic category corresponds to passing from unipotent to tame smooth representations of the p-adic groups $J_b(\mathbb{Q}_p)$.

Hemo–Zhu crucially exploit the Bezrukavnikov equivalence together with the categorical trace of Frobenius to construct coherent sheaves on the stack of Langlands parameters as conjectured in [Zhu20]. So this equivalence plays a distinguished role in categorical local Langlands. Although one gets an ad hoc equivalence by combining [Bez16] and [Ban23], we are deeply convinced that compatibility with the p-adic nearby cycles in [AGLR22] and the Arkhipov–Bezrukavnikov functor F from [ALWY23] will be important in many future applications. In the subsection below on comparing Isoc_G and Bun_G, we provide an example by discussing the compatibility of the spectral action of [FS21] restricted to tame representations with the one imported by Hemo–Zhu from the Bezrukavnikov equivalence.

Beyond this, giving a rigorous construction of the equivalence in the ∞-categorical realm and in the p-adic context will be a valuable contribution to the literature that will open doors to subsequent developments. For instance, recently a t-adic Bezrukavnikov equivalence with $\bar{\mathbb{F}}_\ell$-coefficients was constructed by Bezrukavnikov–Riche in a series of papers [BRR20, BR22], so our methods will be flexible enough to also prove a p-adic Bezrukavnikov equivalence enjoying multiple compatibilities. In the future, we might want to work as well with coefficients in either $\mathbb{Z}[1/p]$ or $\mathbb{Q}$, whose formalism of Berkovich motives for arc-stacks has recently been written up by Scholze [Sch24], compare with [CvdHS22, CvdHS24] in equicharacteristic. Another potential avenue would be to provide an equivalence for all parahoric groups, rather than simply at Iwahori level.

3.2. **Comparing sheaves on Isoc_G and Bun_G.** In his conjecture paper [Zhu20], Zhu proposes that one finds certain coherent sheaves on the stack Par_G of Langlands parameters. His avenue for getting there is however quite different. He considers the stack Isoc_G on perfect k-algebras that classifies G-isocrystals over Witt rings. From a geometric point of view, this is a really badly behaved object. It arises as the étale quotient of the (scheme-theoretic) Witt loop group LG by its σ-conjugated adjoint action, which is not a very helpful presentation, as LG only represents an ind-pro-object in the category of perfectly finitely presented perfect schemes, a structure not retained by the σ-adjoint quotient Isoc_G. A more helpful presentation comes from realizing it as the étale quotient of the Iwahori shtuka stack $\mathrm{Sht}_{\mathcal{I}}$ by the scheme-theoretic Iwahori–Hecke stack $\mathrm{Hk}_{\mathcal{I}}$ regarded as a groupoid via its convolution structure (one could certainly replace the Iwahori by any other parahoric, that is of no consequence). More explicitly, there is a sifted presentation by convolution shtuka stacks $\mathrm{Sht}_{\mathcal{I},\bullet}$, yielding Isoc_G as its colimit. The shtuka stacks carry the structure of an ind-placid stacks, i.e., they are ind-pro-Artin. Hence, the stack of isocrystals is of quite infinite nature in both directions (i.e., points and automorphisms). Nonetheless, Hemo–Zhu have managed to define a category of sheaves $\mathrm{Shv}(\mathrm{Isoc}_G)$ by Kan extending from the well-understood case of perfectly finitely presented morphisms of perfect schemes, and verifying various descent properties. We should note that there are subtleties here having to do with the fact that !-descent works better than the $*$-descent used by Scholze [Sch17], so there are some nasty dualities coming into play that we will ignore here for the sake of convenience. The following conjectured was stated already by [FS21].

Conjecture 3.2. *There exists an equivalence* $\psi\colon \mathrm{Shv}(\mathrm{Isoc}_G) \simeq \mathcal{D}(\mathrm{Bun}_G)$ *of ∞-categories.*

In joint work with Gleason–Ivanov–Hamann–Zou, we plan on establishing this conjecture. The basic idea is that one should look at the dagger v-stack $\mathrm{Isoc}_G^\dagger$ (which is an overconvergent and thus partially proper and quasi-compact object) on perfectoids and consider the analytification functor $b^*\colon \mathrm{Shv}(\mathrm{Isoc}_G^\dagger) \to \mathcal{D}(\mathrm{Isoc}_G^\dagger)$. There is a natural map $\pi\colon \mathrm{Isoc}_G^\dagger \to \mathrm{Bun}_G$ constructed first by Gleason–Ivanov [GIZ23] (we remark that this map also perfectly explains why the closure relations for Isoc_G and Bun_G get reversed), so one can !-push sheaves forward along π and this should be our equivalence $\psi := \pi_! \circ b^*$. Now, we are being again quite sketchy in the sense that π is not really a fine map in the six functor formalism of Mann [Man22b] or analogues, so one cannot really take the !-pushforward. We actually proceed by using the stacky presentation via convolution shtukas and then have to prove some descent/gluing for the resulting maps. In any case, one should get a functor ψ and it is not hard to check that it respects the natural filtrations coming from the geometric stratifications of Bun_G and Isoc_G. In order to get an equivalence, one has to check compatibility with the gluing functors between different strata, and this will rely on a Henselian property for $\mathrm{Sht}_{\mathcal{I}} \to \mathrm{Bun}_G$ like the one that appeared in [FS21] for the local charts $\mathcal{M}_b$. Unfortunately, much of our work will go into making the categorical machinery of the paper actually working, like the analytification or the sifted ind-proper descent.

We should also say a word about the coefficients. For a while, there has been only a good étale theory of six functor formalisms with torsion coefficients. For integral or rational ℓ-adic coefficients, the first proposed solution by [FS21] was to consider a full subcategory of lisse sheaves inside solid sheaves, which then carries no six functor formalism, but has only five operations. A second solution was proposed in [Man22b] via so-called nuclear ℓ-adic sheaves. We tried for a while to make sense of the equivalence ψ using these formalisms, but it is too difficult to check that certain pushforwards land in the right categories, because we are lacking smooth charts. Recently, Scholze [Sch24] wrote up a paper on Berkovich motives over arc-stacks, and we are confident that this will allow us to finally construct the functor ψ for any ℓ-adic coefficients and prove that it is an equivalence. Even more exciting would be the prospect of actually defining $\mathrm{Shv}(\mathrm{Isoc}_G)$ with motivic integral or rational coefficients and then again construct an equivalence ψ in that setting, making the entirety of our comparison work independent of ℓ. We have already had some fruitful discussions with Richarz and van den Hove on such an undertaking.

Let us now go back to ignoring coefficient issues. It is still somewhat unclear for us whether we can prove symmetric monoidality of ψ (even though it should definitely hold!), and it seems that this is related to actual smoothness of the map $\mathrm{Sht}^{\dagger}_{\mathcal{I}} \to \mathrm{Hk}^{\diamondsuit}_{\mathcal{I}}$ which is tricky because of the infinite nature of this map (e.g., if one were to replace full shtukas by truncated ones, it would be trivial). This is a non-trivial question that we have considered with Gleason in the past and it is made considerably harder by the lack of deformation theory, e.g., in the function field setting Lafforgue [Laf18] gives an argument for a global analogue via differential calculations and exploiting the fact that the Frobenius kills these, but this makes no sense without a deperfection. Our goal here is to instead find enough charts covering the shtuka space so that we can verify a Jacobian criterion as the one in [FS21] for their local charts $\mathcal{M}_b$.

An important development in equicharacteristic has been made by Étève–Gaitsgory–Genestier–Lafforgue by providing an alternative construction of the spectral action of $\mathrm{Perf}(\mathrm{Par}_G)$ on $\mathrm{Shv}(\mathrm{Isoc}_G)$ by taking nearby cycles of Satake sheaves on (products indexed by a finite set I of) the generic fiber. Note that because Isoc_G has no inherent level structure, this captures representations of arbitrarily deep level, so one may no longer expect nearby cycles to behave well on the nose, and the Gabber–Orgogozo [Org06] theory of blow-ups to fix this becomes suddenly necessary. This is a tool that we not yet know how to perform on the p-adic side. In any case, in equicharacteristic a stronger version of the equivalence ψ arises if we ask that it is linear over the spectral action by $\mathrm{Perf}(\mathrm{Par}_G)$, i.e., it identifies the two left modules. In the p-adic setting, however, once we restrict to tame representations, then we can ask again about spectral linearity, and the key for answering this necessarily lies within the previous Bezrukavnikov project, as we need an equivalence respecting p-adic nearby cycles. Then, one can take categorical traces of Frobenius and actually expect the Hecke operators to match on both sides.

3.3. **Geometrization of the Donkin conjecture.** In this subsection we work with $\bar{\mathbb{F}}_\ell$-coefficients throughout, for some prime $\ell \neq p$. In [ALRR22, ALRR24] we construct a ramified Satake equivalence compatible with the unramified geometric Satake in [MV07] in the sense that nearby cycles get intertwined with the forgetful functor $\mathrm{Rep}(G^\vee) \to \mathrm{Rep}(G^{\vee,\Gamma})$. One of our motivations to consider geometric Satake was a theorem of Brundan [Bru98] and van der Kallen [vdK01] stating that the previous restriction to inertia fixed points preserves tilting modules. In fact, their theorem applies to any finite group action that does not preserve pinnings, whereas the geometric Satake formalism ensures that the inertia action is pinning-preserving. As we have mentioned multiple times in previous sections, tilting modules are extremely useful gadgets in modular representation theory, so this permanence result is important. However, the proof in [Bru98, vdK01] is disappointing because it is based on brute force calculations. Recently, [FS21] gave a uniform proof when ℓ does not divide the order of the finite group, but they use coherent sheaves on analytic stacks in the sense of [CS19]. Our goal in this joint project with Achar, Richarz, and Riche building on [ALRR22, ALRR24] is to instead provide another proof for finite group actions that preserve pinnings and are not necessarily prime to ℓ.

One of the main geometric tools for this project are so-called parity sheaves on stratified schemes. These were defined by Juteau–Mautner–Williamson [JMW14] by analogy with tilting modules and satisfy the decomposition theorem modulo ℓ. A parity sheaf $\mathcal{E}_\lambda$ on a stratified $X = \sqcup_\lambda X_\lambda$ splits as a direct sum of complexes whose cohomology sheaves are constant on strata and vanish according to a parity condition. Their existence for the affine flag varieties $\mathrm{Fl}_{\mathcal{G}}$ is proved via Demazure resolutions. Under some mild conditions, the same authors [JMW16] showed by explicit calculations that parity sheaves of Gr_G are exactly the tilting modules in the category of equivariant perverse sheaves. In particular, they deduced stability of tilting modules under restriction to Levis thanks to parity preservation of constant terms.

The link between parity sheaves and tilting modules can be conceptually interpreted in terms of Iwahori–Whittaker averaging as in [AB09]. Indeed, we have a category $\mathcal{D}(\mathrm{Hk}_{\mathcal{IW},\mathcal{G}})$ endowed with a distinguished object Ξ up to a central extension of G. Here, the index indicates that the sheaves are Iwahori–Whittaker equivariant on the left and $\mathcal{G}$-equivariant on the right.

Conjecture 3.3. *Convolution with Ξ induces an equivalence between $\mathcal{D}(\mathrm{Hk}_{\mathcal{I},\mathcal{G}})$ and $\mathcal{D}(\mathrm{Hk}_{\mathcal{IW},\mathcal{G}})$.*

For unramified groups, this is due to Bezrukavnikov–Gaitsgory–Mirković–Riche–Rider [BGM+19]. The proof relies on the geometric Casselman–Shalika formula that generalizes a similar concept in the representation theory of p-adic groups. We expect this to generalize to ramified groups, but as it stands, one still needs loop rotation. Consequently, we might have to find a new strategy for this in order to treat wildly ramified groups and/or the mixed characteristic case. The usefulness of Conjecture 3.3 is that tilting modules becomes more explicit on the Iwahori–Whittaker side, so they can be identified with parity sheaves. Proving this identification between tilting objects and parity sheaves (up to perverse truncation) is certainly a step in the right direction of our project. However, it is not clear yet if this will prove decisive, because nearby cycles do not preserve parity in general and the Iwahori–Whittaker realization does not behave well at the integral level because of the ramification. This will most likely forces us to define a full Whittaker realization in this ramified setting. Another interesting direction is to connect Iwahori–Whittaker averaging for $\mathrm{Fl}_{\mathcal{I}}$ and $\mathrm{Fl}_{\mathcal{G}}$, and compare with the tilting property in [AB09, ALWY23]. The non-triviality of monodromy at Iwahori level could help bound the Ext groups, but we run the risk with this approach of essentially categorifying the proof strategy found in [Bru98, vdK01].

3.4. **Geometric Satake and central sheaves when $\ell = p$.** In this section, we work with $\bar{\mathbb{F}}_p$-coefficients for our étale sheaves, so in this case $\ell = p$. Consider the derived category $\mathcal{D}(\mathrm{Hk}_{\mathcal{G}})$ of the scheme-theoretic Hecke stack. It behaves very differently from the $\ell \neq p$ case, because the functors $f^!$, f_* and Hom no longer preserve constructibility. It admits a perverse t-structure due to [Gab04] that has recently been reinterpreted in [BBL+23], so we consider the perverse heart $\mathcal{P}(\mathrm{Hk}_{\mathcal{G}})$, and the corresponding IC sheaves.

Conjecture 3.4. *The IC sheaf of* $\mathrm{Hk}_{\mathcal{G},\leq w}$ *is constant up to shift.*

If F has characteristic p, this was proved by Cass [Cas22] as an application of strong φ-regularity of $\mathrm{Fl}_{\mathcal{G},\leq w}$ via the Artin–Schreier sequence. At hyperspecial level, this yields a modulo p geometric Satake with dual monoid instead of a dual group. In [CL25] we found a new proof of this result that also applies in the p-adic setting if w is sufficiently small (this is dependent on the ramification degree of $F/\mathbb{Q}_p$). For any fixed group however, the conjecture remains unknown for infinitely many w. In a joint project with Robert Cass and Austyn Simpson, we aim to prove that the conjecture holds at least for GL_n at hyperspecial level by verifying that certain deperfections of Zhu [Zhu17] are φ-nilpotent. Note also that the notion of φ-split schemes make no sense in the perfect setting, and even if globally $+$-regular perfect varieties could be defined, they seem very hard to control because the canonical sheaf is no longer invertible.

Before we continue, let us mention some of the underlying commutative algebra. Given a local noetherian ring $(R, \mathfrak{m})$ and a finitely presented R-module M, we get the local cohomology groups $H^i_{\mathfrak{m}}(M)$ which are ind-artinian and measure the difference between M and the global sections of the associated coherent sheaf over the punctured spectrum $\mathrm{Spec}(R) \setminus \{\mathfrak{m}\}$. The R-module M is Cohen–Macaulay if $H^i_{\mathfrak{m}}(M)$ vanish below $\dim(R)$. We can extend these notions to schemes and even perfect schemes by taking φ-colimits. In characteristic p, [HH94] prove that lower local cohomology can be killed along finite extension, i.e., the absolute integral closure R^+ is Cohen–Macaulay over R. Recently, Bhatt [Bha20] proved the main theorem of [HH94] by virtue of the Riemann–Hilbert equivalence of [BL19], identifying étale $\mathbb{F}_p$-sheaves with quasi-coherent φ-sheaves with finiteness conditions. In this vein, it was proved in [BBL+23] and noticed independently during the writing of [CL25], that the constant sheaf on a perfect scheme X is perverse up to shift if and only if X is Cohen–Macaulay. Similarly, the sheaf IC_X is constant up to shift if and only if X is φ-rational, i.e., Cohen–Macaulay with Frobenius-simple top local cohomology.

There is also a link between Conjecture 3.4 and a certain kind of affine representations defined by Bhatt–Scholze [BS17]. Let $\mathcal{L}$ be an ample line bundle on $\mathrm{Fl}_{\mathcal{I}}$ and consider the associated affine cone ring $R(w, \mathcal{L})$ given by the sum of coherent cohomologies of its tensor powers indexed by $\mathbb{Z}_{\geq 0}[1/p]$. These rings $R(w, \mathcal{L})$ are φ-rational if and only if $\mathrm{Fl}_{\mathcal{I},\leq w}$ is, and it translates to a Kawamata–Viehweg vanishing result via the cohomology of negative powers of $\mathcal{L}$. In representation theory, the graded pieces are up to duality the Demazure submodules of the Weyl modules and there is an explicit character formula due to Demazure [Dem74], further studied by [Mat88] via φ-splittings and Littelmann [Lit98] through standard monomial theory and paths. In [CL25] we are able to obtain this for some globally $+$-regular deperfections when w is sufficiently small, but otherwise it is unclear. It is plausible that we may apply path combinatorics from [Lit98] to prove φ-rationality. The PhD thesis of Devansh Sehta will focus on studying a generalization of the Demazure character formula to this perfect setting.

Over p-adic fields, we also plan to study the B^+_{dR}-affine Grassmannian Gr_G. In this case, perverse $\bar{\mathbb{F}}_p$-sheaves should no longer have the pathological behavior of being constant up to shift, because the spaces lives in characteristic 0. However, it is extremely difficult to define and then control a formalism of étale sheaves with coefficients in $\bar{\mathbb{F}}_p$, $\bar{\mathbb{Z}}_p$, or $\bar{\mathbb{Q}}_p$, as the naive guess behaves badly. Instead one has to take inspiration from the Riemann–Hilbert equivalence and consider almost $\mathcal{O}^+_X/\varpi$-modules with invertible φ-structure: this was done by Mann [Man22a] over $\bar{\mathbb{F}}_p$ and more recently by Anschütz–Le Bras–Mann [ABM24] over $\bar{\mathbb{Z}}_p$ and $\bar{\mathbb{Q}}_p$ using quasi-coherent sheaves on the Fargues–Fontaine curve. Together with Mann and van den Hove, we aim at proving geometric Satake with p-adic coefficients for the Hecke stack Hk_G. Unfortunately, one cannot follow the approach of [FS21] word by word, as it makes use of a reduction to the Witt Grassmannian (which lives in characteristic p) on two occasions and the decomposition theorem for $\bar{\mathbb{Q}}_\ell$-sheaves when $\ell \neq p$ once. Inspired by the theory of parity sheaves from [JMW14] and Iwahori–Whittaker averaging from [BGM+19], we should replace the reduction to the special fiber by a finer study of the geometry of Gr_G. For instance, it seems possible to generalize work of Haines in equicharacteristic so as to pave the fibers of the Demazure resolution by successive pro-étale fibrations with fiber $\mathbb{A}^1_{\mathbb{C}_p}$ or $\mathbb{G}_{m,\mathbb{C}_p}$.

The next step in this series would be to define the central functor Z going from the generic to the special fiber. In the function field case, this appears in [Cas21], but its behavior must differ strongly in mixed characteristic, as is the case for the Satake sheaves. There is reason to believe in the existence of both Z and the Wakimoto functor J due to work of Vignéras at the Hecke algebra level. It is very much unclear how much of the Bezrukanikov equivalence actually makes sense in the $\ell = p$ case. However, by studying p-adic étale sheaves of local models, we hope to be able to find a better and uniform way of proving theorems on their singularities, such as Cohen–Macaulayness or an analogue thereof that works even for non-minuscule μ. Another motivation for us is a conjecture in [FHLR22] on the pseudo-rationality of local models. We were able to achieve this in characteristic p thanks to φ-splittings, but more sophisticated methods are needed in mixed characteristic. This also relates to the recent work [BMP+24] on perfectoid pure singularities.

3.5. **Moduli descriptions of local models.** For a very long time, local models have been approached from the perspective of certain linear algebraic moduli spaces. This was certainly the original approach by Deligne–Pappas [DP94] and Rapoport–Zink [RZ96]. One could say that a divide was first created when Görtz [Gör01] observed that the local models $M_{\mathcal{G},\mu}$ can be embedded in a Beilinson–Drinfeld Grassmannian, placing local models within geometric representation theory. In the last few years, we have had a renewel of the moduli theory of local models, especially in works of Bijakowski–Hernández [BH23], Zachos–Zhao [ZZ23], etc. Jointly with Richarz–Viehmann–Wedhorn, we formulated the following conjecture that partially unifies these approaches:

Conjecture 3.5. *Let G be unramified and simply connected. Then, $M_{\mathcal{G},\mu}$ is the subfunctor of $\mathrm{Gr}_{\mathcal{G},O_E}$ consisting of those modifications $(\mathcal{E},\alpha)$ of the trivial G-torsor such that, for every representation $\mathcal{V}$ of $\mathcal{G}$, the corresponding lattice $\Lambda_{\mathcal{V}} \subset \mathcal{V} \otimes B_{\mathrm{dR}}$ contains $\xi^{d_\mu(\mathcal{V})}\mathcal{V} \otimes B^+_{\mathrm{dR}}$, where $d_\mu(\mathcal{V})$ is the highest weight of the $\mathbb{G}_m$-representation $\mathcal{V}|_\mu$.*

Our conjecture is entirely a topological question, because we have formulated it in the perfectoid realm. In equicharacteristic or possibly even in mixed characteristic using [PZ13], it would be possible to formulate a purely geometric conjecture, that captures an actual scheme, but we postpone this discussion for later. It is not too difficult to see that the conjecture is equivalent to demanding that the diagram

$$\begin{tikzcd} M_{\mathcal{G},\mu} \arrow[r] \arrow[d] & \prod_{i=1}^n M_{\mathrm{GL}(\mathcal{V}_i),\rho_i\circ\mu} \arrow[d] \\ \mathrm{Gr}_{\mathcal{G}} \arrow[r] & \prod_{i=1}^n \mathrm{Gr}_{\mathrm{GL}(\mathcal{V}_i)} \end{tikzcd} \tag{3.1}$$

is cartesian for a finite collection of representations $\mathcal{V}_i$ of $\mathcal{G}$. The conjecture is well known in the generic fiber, but it becomes quite tricky in the special fiber. There are a few helpful observations that one can make. First of all, the problem reduces to the case of maximal parahorics by the vertexwise description of Haines–He [HH17] (in particular, the conjecture holds for type A groups and one can deduce type C via a fixed point argument in Haines–Ngô [HN02]). Secondly, one can check that Schubert cells in the proposed moduli space are μ-permissible in the sense of Kottwitz–Rapoport [KR00]. Unfortunately, there are μ-permissible elements that are not μ-admissible due to [HN02] based on certain counterexamples of Deodhar. Our first goal is to exclude these bad μ-permissible strata from our moduli space, and since they are in the W_0-double cosets of t_μ, the question should roughly reduce to recovering the Bruhat order of W_0 from that of the Weyl groups of the highest weight representations of G. This property is a consequence of the behavior of Plücker coordinates as in Fomin–Zelevinsky [FZ00]. In particular, we are very optimistic about the chances of proving the previous conjecture for μ minuscule. If μ is not minuscule, one must consider W_0-double cosets of t_λ for λ strictly below μ, but it becomes harder to have some control over the maximal μ-admissible element in this set, although there exist complicated algorithms due to He–Yu [HY24].

Let us finally discuss the much stronger version of this conjecture taking non-reduced structures into account. Then, already for the generic fiber (i.e., the corresponding Schubert varieties), the result is only known for type A and it is a conjecture of Finkelberg–Mirković [FM99] in general. Our attention was recently drawn to this by Kisin–Pappas–Zhou [KPZ24] as they provide some evidence in the form of tangent space calculations. Nonetheless, as the moduli space is not known to be reduced, there are many higher order differentials that are not captured by tangent spaces. Our hope is that one could still calculate distribution modules as used in [Lou23a] and mimic portions of [KPZ24]. We are confident that solving the Finkelberg–Mirković conjecture (so moduli descriptions only for Schubert varieties) together with our topological conjecture above would imply the stronger version of the conjecture. In any case, it seems like this program would keep us occupied for some time.

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ON THE p-ADIC THEORY OF LOCAL MODELS

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Abstract. We prove the Scholze–Weinstein conjecture on the existence and uniqueness of local models of local Shimura varieties and the test function conjecture of Haines–Kottwitz in this setting. In order to achieve this, we establish the specialization principle for well-behaved p-adic kimberlites, show that these include the v-sheaf local models, determine their special fibers using hyperbolic localization for the étale cohomology of small v-stacks and analyze the resulting specialization morphism using convolution.

Contents

1. Introduction

The general theory of Shimura varieties has been developed by Deligne [Del71, Del79] in the seventies. It generalizes classical objects such as modular curves, moduli spaces of principally polarized abelian varieties or Hilbert modular varieties. Shimura varieties naturally occur in the search for higher reciprocity laws within the Langlands program [Lan79]. Their arithmetic properties are encoded in the reduction to positive characteristic $p > 0$ and have contributed to spectacular developments in number theory and arithmetic geometry in the past decades.

Local models are flat projective schemes over complete discrete valuation rings of characteristic $(0, p)$ that are designed to model the singularities in the reduction modulo p of Shimura varieties with parahoric level structure. Starting from the pioneering works [DR73, Rap90, CN90, dJ93, DP94], the theory is formalized to some extent in the book of Rapoport–Zink [RZ96] for those Shimura varieties arising as moduli problems of abelian varieties with extra structures. The recent works of Kisin–Pappas [KP18] and Kisin–Pappas–Zhou [KPZ24] construct natural integral models for all Shimura varieties of abelian type with parahoric level structure when $p > 2$. The geometric properties of the corresponding local models are studied in a series of works notably by Faltings, Görtz, Pappas and Rapoport [Fal97, Fal01, Pap00, Gör01, Gör03, Fal03, PR03, PR05, PR08, PR09], see the survey article [PRS13] for details and further references. A breakthrough

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is due to Pappas and Zhu [PZ13, Zhu14], who gave a purely group-theoretic construction of (flat) local models, later refined by Levin [Lev16] and the third named author [Lou23a]. Roughly, the local models in this approach are constructed as flat, closed subschemes in a power series affine Grassmannian, which depends on certain auxiliary choices, see Section 7.3. A more functorial approach (without ad hoc choices) was initiated by Scholze–Weinstein [SW20], using p-adic geometry. Unfortunately, the approach has the drawback of not producing schemes, at least a priori.

The aim of the present manuscript is to connect the scheme-theoretic local models to Scholze–Weinstein's p-adic approach. More precisely, we prove the Scholze–Weinstein conjecture [SW20, Conjecture 21.4.1] on the existence and uniqueness of (weakly normal) local models, representing minuscule portions of the parahoric Beilinson–Drinfeld Grassmannian. Our methods allow us to prove, furthermore, the test function conjecture of Haines–Kottwitz [Hai14, Conjecture 6.1.1] for these local models in full generality, expressing the trace of Frobenius function on the nearby cycles sheaf in terms of spectral data.

These local models are intimately related with moduli spaces of p-adic shtukas by [SW20, Lecture XXV]. Recently, Pappas–Rapoport [PR24] made progress in the study of moduli spaces of p-adic shtukas and their relation to Shimura varieties, partially relying on a positive solution of the Scholze–Weinstein conjecture as given here. Let us now discuss our results in more detail.

1.1. **Main results.** Fix a prime number p. Denote by $\mathbb{Q}_p$ the field of p-adic numbers. (In the main body of the text, we allow more general p-adic base fields.) Let G be a connected reductive $\mathbb{Q}_p$-group and $\mathcal{G}$ a parahoric $\mathbb{Z}_p$-model in the sense of Bruhat–Tits [BT84]. Let μ be a conjugacy class of geometric cocharacters in G. Denote by $E/\mathbb{Q}_p$ its reflex field with ring of integers O_E and finite residue field k.

For a brief explanation on the following terminology the reader is referred to Section 1.2. Scholze–Weinstein [SW20, Section 20.3] introduce the v-sheaf Beilinson–Drinfeld Grassmannian $\mathrm{Gr}_{\mathcal{G}} \to \mathrm{Spd}\,\mathbb{Z}_p$. Its generic fiber is the B_{dR}^+-affine Grassmannian Gr_G and its special fiber is the v-sheaf $\mathcal{F}\ell_{\mathcal{G}}^{\diamondsuit}$ associated to the Witt vector partial affine flag variety defined by Zhu [Zhu17a] and further studied by Bhatt–Scholze [BS17]. Attached to the pair $(\mathcal{G}, \mu)$ is the v-sheaf local model

$$\mathcal{M}_{\mathcal{G},\mu} \ \subset \ \mathrm{Gr}_{\mathcal{G}}|_{\mathrm{Spd}\, O_E} \tag{1.1}$$

defined as the v-closure (Definition 2.2) of the affine Schubert variety $\mathrm{Gr}_{G,\mu}$, in analogy to the local models from [PZ13]. If μ is minuscule (that is, $\langle \mu, a\rangle \in \{0, \pm 1\}$ for every root a of $G_{\mathbb{C}_p}$), then the generic fiber $\mathcal{M}_{\mathcal{G},\mu}|_{\mathrm{Spd}\, E} = \mathrm{Gr}_{G,\mu}$ is canonically isomorphic to $\mathcal{F}_{G,\mu}^{\diamondsuit}$, the v-sheaf associated with the homogenous space $\mathcal{F}_{G,\mu}$ of parabolic subgroups of type μ. Scholze–Weinstein state the following conjecture [SW20, Conjecture 21.4.1] on the existence of local models:

Conjecture 1.1 (Scholze–Weinstein)**.** *There is a flat projective O_E-scheme $\mathcal{M}_{\mathcal{G},\mu}^{\mathrm{sch}}$ with reduced special fiber and with a closed immersion*

$$\mathcal{M}_{\mathcal{G},\mu}^{\mathrm{sch},\diamondsuit} \hookrightarrow \mathrm{Gr}_{\mathcal{G}}|_{\mathrm{Spd}\, O_E} \tag{1.2}$$

whose generic fiber is $\mathcal{F}_{G,\mu}^{\diamondsuit} \cong \mathrm{Gr}_{G,\mu}$.

Note that the schematic local model $\mathcal{M}_{\mathcal{G},\mu}^{\mathrm{sch}}$, whenever it exists, is normal and uniquely characterized by $\mathcal{M}_{\mathcal{G},\mu}^{\mathrm{sch},\diamondsuit} \cong \mathcal{M}_{\mathcal{G},\mu}$ under the closed immersion (1.2), see [SW20, Remark 21.4.2] and the text thereafter. In the thesis of the third named author, Conjecture 1.1 is separated in a "representability part" and a "geometry part". More precisely, [Lou20, Conjecture IV.4.18] states the existence of a unique flat, projective and weakly normal (Definition 2.12) O_E-scheme $\mathcal{M}_{\mathcal{G},\mu}^{\mathrm{sch}}$ together with a closed immersion as in (1.2) whose generic fiber is $\mathcal{F}_{G,\mu}^{\diamondsuit} \cong \mathrm{Gr}_{G,\mu}$ whereas [Lou20, Conjecture IV.4.19] states that the special fiber of $\mathcal{M}_{\mathcal{G},\mu}^{\mathrm{sch}}$ is reduced. The main result of the present work proves the representability part in full generality and the geometry part for all G with the exception of two non-split groups for $p = 2, 3$:

Theorem 1.2 (Theorem 7.21, Theorem 7.23)**.** *Let μ be minuscule. Then, the following hold:*

(1) *Representability: There is a unique (up to unique isomorphism) flat, projective and weakly normal O_E-scheme $\mathcal{M}^{\mathrm{sch}}_{\mathcal{G},\mu}$ with a closed immersion*
$$\mathcal{M}^{\mathrm{sch},\diamondsuit}_{\mathcal{G},\mu} \hookrightarrow \mathrm{Gr}_{\mathcal{G}}|_{\mathrm{Spd}\, O_E} \tag{1.3}$$
whose generic fiber is $\mathcal{F}^{\diamondsuit}_{G,\mu} \cong \mathrm{Gr}_{G,\mu}$. In addition, $\mathcal{M}^{\mathrm{sch}}_{\mathcal{G},\mu}$ admits a unique $\mathcal{G}_{O_E}$-action such that the isomorphism $\mathcal{M}^{\mathrm{sch},\diamondsuit}_{\mathcal{G},\mu} \cong \mathcal{M}_{\mathcal{G},\mu}$ induced from (1.3) is $\mathcal{G}^{\diamondsuit}_{O_E}$-equivariant.

(2) *Geometry: Assume that the adjoint group G_{ad} satisfies the following conditions over the completion of the maximal unramified extension $\breve{\mathbb{Q}}_p/\mathbb{Q}_p$:*
   (a) *If $p = 2$, then G_{ad} has no odd unitary $\breve{\mathbb{Q}}_2$-factors.*
   (b) *If $p = 3$, then G_{ad} has no triality $\breve{\mathbb{Q}}_3$-factors.*

   Then, the special fiber of $\mathcal{M}^{\mathrm{sch}}_{\mathcal{G},\mu}$ is reduced and equal to $\mathcal{A}^{\mathrm{can}}_{\mathcal{G},\mu}$, the canonical deperfection of the μ-admissible locus inside $\mathcal{F}\ell_{\mathcal{G}}$. Moreover, it is Cohen-Macaulay, weakly normal and Frobenius split.

Our result provides a functorial and group-theoretic framework for the theory of local models that goes beyond the previous results [SW20, HPR20, Lou20] for certain pairs (G, μ) of abelian type using Hodge embeddings. The scheme $\mathcal{M}^{\mathrm{sch}}_{\mathcal{G},\mu}$ agrees with the local models defined in [PZ13, Lev16, Lou23a] whenever $p \nmid |\pi_1(G_{\mathrm{der}})|$ and otherwise with their weak normalization (weak normalization is necessary, in general, due to the existence of non-normal Schubert varieties [HLR24]). The works [PZ13, Lev16, Lou23a] are complemented by the results of Fakhruddin–Haines–L.–R. [FHLR22] which handles new cases for wildly ramified groups and leads to the conditions in Theorem 1.2(2). Invoking the results from [Lou20], the conditions can be relaxed with our methods: for $p = 2$ one excludes odd unitary groups defined by a ramified, quadratic root-of-unit extension; for $p = 3$ one excludes trialities defined by a ramified, cubic non-(root-of-a-prime) extension. In particular, our result is complete for all primes $p \geq 5$ and, up to the two non-split examples, also for $p = 2, 3$. We remark that, after the first version of this paper was written, the remaining cases were handled in [GL24]. Hence, Conjecture 1.1 is proven in all cases, but besides reducedness of the special fiber the additional geometric properties stated in Theorem 1.2(2) are not addressed in the odd unitary case in [GL24]. This will be done in upcoming work of Cass and the third named author [CL25] which implies Theorem 1.2(2) without restrictions except that Cohen–Macaulayness is not proven for $p = 2$ and odd unitary groups defined by ramified, quadratic root-of-unit extensions, see Remarks 3.17 and 7.24. In particular, these works settle our Conjecture 7.25 stating that the special fiber of $\mathcal{M}^{\mathrm{sch}}_{\mathcal{G},\mu}$ is equal to $\mathcal{A}^{\mathrm{can}}_{\mathcal{G},\mu}$ in all cases. Furthermore, we conjecture that the singularities of $\mathcal{M}^{\mathrm{sch}}_{\mathcal{G},\mu}$ are pseudo-rational (hence, Cohen–Macaulay) in all cases, see Conjecture 7.27. In addition, let us remark that finding useful moduli interpretations of $\mathcal{M}^{\mathrm{sch}}_{\mathcal{G},\mu}$ is an interesting and difficult problem that maybe does not have a uniform solution. There are however many interesting special cases [Pap00, Gör01, Gör03, PR03, PR05, PR09].

As a cohomological application, we prove the test function conjecture for $\mathcal{M}_{\mathcal{G},\mu}$, all primes p and all pairs $(\mathcal{G}, \mu)$ as above (for general μ which are not necessarily minuscule), see Section 8. Namely, fix an auxiliary prime $\ell \neq p$, a square root $\sqrt{q}$ of the residue cardinality of E and an embedding $E \hookrightarrow \bar{\mathbb{Q}}_p$. Put $\Lambda = \mathbb{Q}_\ell(\sqrt{q})$ which we will use as sheaf coefficients. Let Γ be the absolute Galois group of E with inertia subgroup I, and fix a lift $\Phi \in \Gamma$ of geometric Frobenius. Let $E_0 = W(k)[\frac{1}{p}]$ be the maximal unramified subextension of $E/\mathbb{Q}_p$. Let IC_μ be the intersection complex on $\mathrm{Gr}_{G,\mu}$ with Λ-coefficients constructed in [FS21, Chapter VI] and normalized to be of weight zero as in (8.1). (We remark that IC_μ exists due to the orbit stratification on $\mathrm{Gr}_{G,\mu}$. So, its existence is a feature of the particular geometry at hand, not a general sheaf-theoretic construction for diamonds.) The function $\tau^{\Phi}_{\mathcal{G},\mu}\colon G(E_0)/\mathcal{G}(O_{E_0}) \to \Lambda$ is defined, up to sign, by the alternating trace of Φ on the nearby cycle stalks

$$\tau^{\Phi}_{\mathcal{G},\mu}(x) = (-1)^{\langle 2\rho,\mu\rangle}\sum_{i\in\mathbb{Z}}(-1)^i \operatorname{trace}\big(\Phi \mid \mathrm{R}^i\Psi_{\mathcal{M}_{\mathcal{G},\mu}}(\mathrm{IC}_\mu)_{\bar{x}}\big), \tag{1.4}$$

if $x \in \mathcal{M}_{\mathcal{G},\mu}(\operatorname{Spd} k)$ and 0 else. Here $\mathcal{M}_{\mathcal{G},\mu}(\operatorname{Spd} k)$ is viewed as a subset of $\mathcal{F}\ell_{\mathcal{G}}(k) = G(E_0)/\mathcal{G}(O_{E_0})$, and the trace is well-defined by Theorem 1.8, proving that the nearby cycles are constructible in this setting. Also, the function $\tau^{\Phi}_{\mathcal{G},\mu}$ depends on the choice of the geometric Frobenius lift Φ, that is, we do not take semi-simplified traces by passing to inertia invariants, compare with [HR20, Section 6.2]. However, one recovers the semi-simple trace function as in [HR20, Appendix] by "averaging over geometric Frobenius lifts Φ". The function $\tau^{\Phi}_{\mathcal{G},\mu}$ is left-$\mathcal{G}(O_{E_0})$-invariant and supported on finitely many orbits, hence $\tau^{\Phi}_{\mathcal{G},\mu} \in \mathcal{H}(G(E_0), \mathcal{G}(O_{E_0}))_\Lambda$ naturally lies in the parahoric Hecke algebra of Λ-valued functions.

Theorem 1.3 (Lemma 8.4)**.** *The function $\tau^{\Phi}_{\mathcal{G},\mu}$ lies in the center of $\mathcal{H}(G(E_0), \mathcal{G}(O_{E_0}))_\Lambda$. It is characterized as the unique function in the center that acts on any $\mathcal{G}(\mathcal{O}_{E_0})$-spherical smooth irreducible Λ-representation π by the scalar*

$$\operatorname{trace}\Big(s^{\Phi}(\pi) \;\Big|\; V_\mu\Big), \tag{1.5}$$

where $s^{\Phi}(\pi) \in [\widehat{G}^I \rtimes \Phi]_{\mathrm{ss}}/\widehat{G}^I$ is the Satake parameter for π and V_μ the representation of the L-group $\widehat{G}\rtimes\Gamma$ of highest weight μ. Moreover, $(\sqrt{q})^{\langle 2\rho,\mu\rangle}\tau^{\Phi}_{\mathcal{G},\mu}$ takes values in $\mathbb{Z}$ and is independent of $\ell \neq p$, $\sqrt{q}$ and $E \hookrightarrow \bar{\mathbb{Q}}_p$.

The theorem is a solution to the test function conjecture of Haines–Kottwitz [Hai14, Conjecture 6.1.1] for v-sheaf local models. This is new when μ is non-minuscule: then $\mathcal{M}_{\mathcal{G},\mu}$ is not representable by a scheme due to the theory of Banach–Colmez spaces and, hence, not related to their schematic counterparts defined in [PZ13, Lev16, Lou23a, FHLR22]. If μ is minuscule, then the analogue of Theorem 1.3 holds for $\mathcal{M}^{\mathrm{sch}}_{\mathcal{G},\mu}$ as well, using Theorem 1.2. In this case, we can work purely algebraically and replace IC_μ by the constant sheaf $\mathbb{Q}_\ell$ on the smooth E-scheme $\mathcal{F}_{G,\mu}$. Here, our result is new for the wildly ramified groups that were excluded in previous work [PZ13, HR20, HR21]. With a view towards applications, say, to point counting formulas in the context of the Langlands–Kottwitz method, we have an analogous result where E is replaced by a finite unramified extension, and correspondingly E_0 by its unramified subextension. Also, Theorem 1.3 implies the analogous version for the semi-simple traces following [HR20, Appendix].

1.2. **Explanation on perfectoid spaces, diamonds and *v*-sheaves.** The present manuscript relies on the p-adic geometric methods notably developed by Scholze [Sch12, SW20, Sch17]. For the reader's convenience we give a brief overview of some basic terminology including references.

Perfectoid spaces are the full subcategory of Huber's adic spaces X which can be covered by affinoid adic spaces $\operatorname{Spa}(R, R^+)$ with R a perfectoid Tate ring, see [Sch17, Definitions 3.1, 3.19]. Let $\mathrm{Perf}_{\mathbb{F}_p}$ be the category of perfectoid spaces in characteristic p equipped with the v-topology generated by surjective maps subject to some finiteness conditions analogous to the definition of the fpqc topology for schemes, see [Sch17, Definition 8.1(iii)] (here, the "v" in v-topology stands for valuation). So, $\mathrm{Perf}_{\mathbb{F}_p}$ is a site (here, we choose some cut-off cardinal) which is subcanonical by [Sch17, Theorem 8.7]. We have the following inclusions of full subcategories of the category of v-sheaves on $\mathrm{Perf}_{\mathbb{F}_p}$:

$$\{\text{perfectoid spaces}\} \subset \{\text{diamonds}\} \subset \{\text{small v-sheaves}\}$$

By definition, a diamond (respectively, small v-sheaf) admits a pro-étale cover (respectively, v-cover) by a perfectoid space, see [Sch17, Definition 11.1, Proposition 11.9, Definition 12.1]. In particular, every small v-sheaf X has an underlying topological space $|X|$ given by the topological space of a perfectoid cover modulo the equivalence relation, see [Sch17, Definition 12.8]. The notion of small v-sheaves can be generalized to stacks leading to the notion of small v-stacks. It comes with a theory of derived categories of étale sheaves that satisfy a 6-functor formalism, see [Sch17, Introduction].

One has a functor $X \mapsto X^\diamond$ from pre-adic spaces (see [SW20, Appendix to Lecture 3]) over $\mathbb{Z}_p$ to small v-sheaves on $\mathrm{Perf}_{\mathbb{F}_p}$, see [SW20, Lemma 18.1.1] and [Gle24, Proposition 1.20]. There is a surjective map $|X^\diamond| \to |X|$ of topological spaces, see [SW20, Proposition 18.2.2]. Every pre-adic space X admits a decomposition $X^{\mathrm{an}} \sqcup X^{\mathrm{na}}$ into an open locus of analytic points and a closed locus of non-analytic points, see [Gle24, Proposition 1.19, Proposition 1.21(6)] for precise statements.

If $X = X^{\mathrm{an}}$ is analytic, then $X^\diamond$ is a (locally spatial) diamond and $|X^\diamond| \cong |X|$, see [Sch17, Lemma 15.6]. Schemes X in characteristic p can be considered as discrete (hence, non-analytic) adic spaces via $\mathrm{Spec}R \mapsto \mathrm{Spa}(R, \widetilde{\mathbb{F}}_p)$ (here, $\widetilde{\mathbb{F}}_p$ is the integral closure of $\mathbb{F}_p$ in R), and we get an associated small v-sheaf $X^\diamond$ characterized as the sheafification of the functor $\mathrm{Spa}(R, R^+) \mapsto X(\mathrm{Spec}R)$ on affinoid perfectoid spaces, see Section 2.2. More generally, one can extend this construction for schemes X over $\mathbb{Z}_p$. Moreover, there is a second way of associating small v-sheaves to schemes. The second construction is characterized as the sheafification of the functor $\mathrm{Spa}(R, R^+) \mapsto X(\mathrm{Spec}R^+)$, for proper schemes over $\mathbb{Z}_p$ the two constructions agree. The reader is referred to Section 2.2 for a more detailed discussion.

For an affinoid adic space $\mathrm{Spa}(R, R^+)$ we denote by $\mathrm{Spd}(R, R^+)$, or simply by $\mathrm{Spd}\, R$ whenever R^+ is understood, the associated small v-sheaf $\mathrm{Spa}(R, R^+)^\diamond$. A basic example is $\mathrm{Spd}\, \mathbb{Z}_p = \mathrm{Spa}(\mathbb{Z}_p, \mathbb{Z}_p)^\diamond$ which is stratified by $\mathrm{Spd}\, \mathbb{Q}_p = \mathrm{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)^\diamond$ and $\mathrm{Spd}\, \mathbb{F}_p = \mathrm{Spa}(\mathbb{F}_p, \mathbb{F}_p)^\diamond$, corresponding to the decomposition into the analytic and non-analytic locus. The diamond $\mathrm{Spd}\, \mathbb{Q}_p$ is studied in [SW20, Section 8.4]. Note that $\mathrm{Spd}\, \mathbb{F}_p$ is the terminal object in the category of v-sheaves on $\mathrm{Perf}_{\mathbb{F}_p}$ (that is, the functor sending any S to a point), which however is not a diamond.

Our main example is the v-sheaf Beilinson–Drinfeld Grassmannian $\mathrm{Gr}_{\mathcal{G}} \to \mathrm{Spd}\, \mathbb{Z}_p$ associated with $\mathcal{G}$ over $\mathbb{Z}_p$ as in Section 1.1, see [SW20, Definition 20.3.1]. The map $\mathrm{Gr}_{\mathcal{G}} \to \mathrm{Spd}\, \mathbb{Z}_p$ is relatively representable by an ind-(proper, spatial diamond), that is, for every spatial diamond mapping to $\mathrm{Spd}\, \mathbb{Z}_p$ the base change is an increasing union of proper, spatial diamonds with closed immersions as transition maps, see Theorem 4.9 and the references cited there. Whereas its generic fiber $\mathrm{Gr}_G := \mathrm{Gr}_{\mathcal{G}}|_{\mathrm{Spd}\, \mathbb{Q}_p}$ is itself an ind-(proper spatial diamond), its special fiber $\mathrm{Gr}_{\mathcal{G}}|_{\mathrm{Spd}\, \mathbb{F}_p} = \mathcal{F}\ell_{\mathcal{G}}^\diamond$ is the v-sheaf associated with an ind-(perfectly proper $\mathbb{F}_p$-scheme), see [SW20, Lectures 19 and 21]. The manuscript at hand studies $\mathrm{Gr}_{\mathcal{G}}$ with the geometric and cohomological techniques outlined in the present subsection.

1.3. **Strategy of proof.** The proofs of Theorem 1.2 and Theorem 1.3 are purely group-theoretic and do not rely on the classification of reductive groups over local fields. Our method is inspired by a conjecture of He–Pappas–Rapoport [HPR20, Conjecture 2.13] of which we prove a variant, see Theorem 7.12. Following a suggestion in [Lou20, Introduction], we approach $\mathcal{M}_{\mathcal{G},\mu}$ by studying its associated *specialization triple*, that is, its generic fiber, its special fiber and the specialization map between them. The basic idea is that $\mathcal{M}_{\mathcal{G},\mu}$ should be uniquely determined by its specialization triple, making the comparison with the to-be-constructed $\mathcal{M}^{\mathrm{sch}}_{\mathcal{G},\mu}$ possible. Once there is enough progress towards Theorem 1.2, the proof of Theorem 1.3 roughly follows the method from [HR21], and the reader is referred to Section 8 for details. Along the way, we address further questions and conjectures in the field, notably Zhu's conjecture [Zhu17a, Appendix B, Conjecture III] on the geometry of deperfections of affine Schubert varieties in the Witt vector affine flag variety.

The proof of Theorem 1.2 proceeds in the following three steps outlined in the next subsections:

(1) Establish a theory of specialization triples for suitable v-sheaves including local models, see Section 1.3.1.
(2) Determine the special fibers of v-sheaf and schematic local models, see Sections 1.3.2 and 1.3.3.
(3) Study the specialization map for local models, see Section 1.3.4.

Section 1.3.5 concludes the outline of the proof with a more detailed version of Theorem 1.2, see Theorem 1.14.

1.3.1. *Kimberlites and specialization triples.* Recall from (1.1) that the v-sheaf local model $\mathcal{M}_{\mathcal{G},\mu} \subset \mathrm{Gr}_{\mathcal{G}}|_{\mathrm{Spd}\, O_E}$ is defined as the v-closure of $\mathrm{Gr}_{G,\mu}$, that is, the smallest closed sub-v-sheaf in $\mathrm{Gr}_{\mathcal{G}}|_{\mathrm{Spd}\, O_E}$ containing $\mathrm{Gr}_{G,\mu}$, see Definition 2.2. A basic difficulty is the determination of the underlying topological space $|\mathcal{M}_{\mathcal{G},\mu}|$, that is, to show the density of $|\mathrm{Gr}_{G,\mu}|$ inside $|\mathcal{M}_{\mathcal{G},\mu}|$. (We warn the reader that such density results fail in general, see Example 2.5.) The following theorem, in particular, shows that the density still holds true in the case of v-sheaf local models:

Theorem 1.4 (Theorem 2.37, Proposition 4.14)**.** *The v-sheaf local model $\mathcal{M}_{\mathcal{G},\mu}$ is a quasi-compact, separated, flat p-adic kimberlite over $\mathrm{Spd}\, O_E$ that satisfies the properties of Definition 2.36. In addition, the functor sending such kimberlites over $\mathrm{Spd}\, O_E$ to their specialization triples*

$$X \mapsto \left(X_\eta, X^{\mathrm{red}}, \mathrm{sp}_{\breve{X}}\right) \tag{1.6}$$

is fully faithful.

Kimberlites are introduced by the second named author in [Gle24] and are a certain class of v-sheaves (containing the v-sheaves associated to flat, projective O_E-schemes) that admit a well-behaved theory of specialization maps, see Section 2.3 for details. In the above result, $X_\eta = X|_{\mathrm{Spd}\, E}$ is the generic fiber, X^{red} is a perfect k-scheme that plays the role of the special fiber (see Definitions 2.19 and 2.25) and $\mathrm{sp}_{\breve{X}} \colon |X_{\breve{\eta}}| \to |X^{\mathrm{red}}_{\bar{k}}|$ is a continuous map on the underlying topological spaces after base change to $\mathrm{Spd}\, O_{\breve{E}}$. For v-sheaf local models, $\mathcal{M}^{\mathrm{red}}_{\mathcal{G},\mu}$ is a union of Schubert perfect k-varieties in $\mathcal{F}\ell_{\mathcal{G}}$ such that its associated v-sheaf is the special fiber $\mathcal{M}_{\mathcal{G},\mu}|_{\mathrm{Spd}\, k}$. Since the generic fiber of $\mathcal{M}_{\mathcal{G},\mu}$ is $\mathrm{Gr}_{G,\mu}$ by definition, it remains to determine the special fiber and the specialization map which we outline in Sections 1.3.2 and 1.3.4.

Moreover, a similar fully faithfulness result for the functor $X \mapsto X^{\diamondsuit}$ from weakly normal flat projective O_E-schemes to v-sheaves over $\mathrm{Spd}\, O_E$ is shown by the third named author in [Lou20]. So, Theorem 1.4 shows that the to-be-constructed schematic local model $\mathcal{M}^{\mathrm{sch}}_{\mathcal{G},\mu}$ is uniquely determined by the specialization triple associated to $\mathcal{M}^{\mathrm{sch},\diamondsuit}_{\mathcal{G},\mu}$. Again, its generic fiber is $\mathrm{Gr}_{G,\mu}$ by definition and it remains to determine the special fiber and the specialization map which we outline in Sections 1.3.3 and 1.3.4.

1.3.2. *Special fibers of v-sheaf local models.* The Witt vector partial affine flag variety $\mathcal{F}\ell_{\mathcal{G}}$ is the increasing union of perfect projective varieties $\mathcal{F}\ell_{\mathcal{G},w}$ indexed by the double coset of the Iwahori–Weyl group, see [Zhu17a, BS17]. The μ-admissible locus $\mathcal{A}_{\mathcal{G},\mu}$ in the sense of Kottwitz–Rapoport is defined as the k-descent of the $\bar{\mathbb{F}}_p$-union

$$\mathcal{A}_{\mathcal{G},\mu,\bar{\mathbb{F}}_p} = \bigcup_{\lambda} \mathcal{F}\ell_{\mathcal{G},\bar{\mathbb{F}}_p,\lambda_I(p)}, \tag{1.7}$$

where λ runs over all absolute Weyl conjugates of μ and $\lambda_I(p)$ denotes the associated translation in the Iwahori–Weyl group of $G_{\breve{\mathbb{Q}}_p}$, see Definition 3.11.

Theorem 1.5 (Theorem 6.16). *The special fiber of $\mathcal{M}_{\mathcal{G},\mu}$ is the v-sheaf $\mathcal{A}^{\diamondsuit}_{\mathcal{G},\mu}$ associated with the μ-admissible locus inside $\mathcal{F}\ell^{\diamondsuit}_{\mathcal{G}}$.*

The difficulty in determining the special fiber lies in the rather abstract definition of $\mathcal{M}_{\mathcal{G},\mu}$ via closure operations inside $\mathrm{Gr}_{\mathcal{G}}$. In a different context, [HR21, Theorem 6.12] determines such special fibers based on cohomological considerations by calculating the support of the nearby cycles $\Psi_{\mathcal{M}_{\mathcal{G},\mu}}\mathrm{IC}_\mu$ appearing in (1.4). A key input is the study of $\mathbb{G}_m$-actions on $\mathrm{Gr}_{\mathcal{G}}$ coming from the choice of a cocharacter $\lambda\colon \mathbb{G}_m \to \mathcal{G}$. Let $\mathcal{M}$, respectively $\mathcal{P}$ the closed subgroup scheme of $\mathcal{G}$ with Lie algebra the subspace of $\operatorname{Lie}\mathcal{G}$ with weights $\lambda = 0$, respectively $\lambda \geq 0$. This induces maps $\mathcal{M} \leftarrow \mathcal{P} \to \mathcal{G}$ of $\mathbb{Z}_p$-group schemes and, by functoriality, also of the associated Beilinson–Drinfeld Grassmannians. The geometric input towards Theorem 1.5 is the following result which extends [FS21, Chapter VI.3] beyond the case where $\mathcal{G}$ is reductive:

Theorem 1.6 (Theorem 5.2). *The $\mathbb{G}_m^{\diamondsuit}$-action on $\mathrm{Gr}_{\mathcal{G}}$ via λ induces a commutative diagram of v-sheaves*

$$\begin{tikzcd} \mathrm{Gr}_{\mathcal{M}} \arrow[d] & \mathrm{Gr}_{\mathcal{P}} \arrow[l] \arrow[r] \arrow[d] & \mathrm{Gr}_{\mathcal{G}} \arrow[d, "\mathrm{id}"'] \\ (\mathrm{Gr}_{\mathcal{G}})^0 & (\mathrm{Gr}_{\mathcal{G}})^+ \arrow[l] \arrow[r] & \mathrm{Gr}_{\mathcal{G}}, \end{tikzcd} \tag{1.8}$$

such that the vertical arrows are open and closed immersions that induce isomorphisms over $\operatorname{Spd}\mathbb{Q}_p$. If, in addition, $\mathcal{G}$ is special parahoric (for example, reductive) over $\breve{\mathbb{Z}}_p$, then the vertical arrows are also isomorphisms over $\operatorname{Spd}\mathbb{F}_p$.

Here, $(\mathrm{Gr}_{\mathcal{G}})^0$ is the fixed points locus for the $\mathbb{G}_m^{\diamondsuit}$-action and $(\mathrm{Gr}_{\mathcal{G}})^+$ its attractor locus, which is loosely speaking the union of the strata of points flowing into the fixed points, see also [FS21, Discussion above Theorem I.6.5]. Both v-sheaves are representable by increasing unions of spatial diamonds relative over $\operatorname{Spd}\mathbb{Z}_p$ in the sense of [Sch17, Definition 13.3], see Lemma 5.1.

If λ is regular, then the special fiber of $(\mathrm{Gr}_{\mathcal{G}})^+$ consists of the v-sheaves associated to the semi-infinite orbits $\mathcal{S}_w$ inside $\mathcal{F}\ell_{\mathcal{G}}$ indexed by certain cosets of the Iwahori–Weyl group, see (5.12). By comparison, $\mathrm{Gr}_{\mathcal{P}}$ corresponds to those semi-infinite orbits attached to translation elements. The following proposition generalizes the closure relation of semi-infinite orbits and the equidimensionality of Mirković–Vilonen cycles [MV07, Theorem 3.2] from split groups to twisted groups, questions left open in the context of the ramified Satake equivalence [Zhu15, Ric16]:

Proposition 1.7 (Proposition 5.4, Lemma 5.5). *For a regular coweight λ and the induced semi-infinite orbits, there is an equality inside $\mathcal{F}\ell_{\mathcal{G}}$:*

$$\overline{\mathcal{S}_w} = \bigcup_v \mathcal{S}_v, \tag{1.9}$$

where v runs through all elements less than or equal to w in Lusztig's semi-infinite Bruhat order. If $\mathcal{G}$ is special parahoric (for example, reductive) over $\breve{\mathbb{Z}}_p$, then the non-empty intersections $\mathcal{S}_{\nu_I} \cap \mathcal{F}\ell_{\mathcal{G},\mu_I}$ are equi-dimensional of dimension $\langle \rho, \nu + \mu \rangle$.

Here, $v \leq^{\frac{\infty}{2}} w$ holds in Lusztig's semi-infinite Bruhat if and only if $v \leq w$ holds in the Bruhat order after translation by a sufficiently dominant cocharacter, see (5.13).

Let us now explain the cohomological results going into Theorem 1.5. As there is no general theory of nearby cycles for v-sheaves, we develop the foundational results for the Hecke stack. Fix a prime number $\ell \neq p$ and let Λ be one of the rings $\mathbb{Z}/\ell^n$, $\mathbb{Z}_\ell$ or $\mathbb{Q}_\ell$ for some $n \geq 0$. For a small v-stack X, let $\mathrm{D}(X,\Lambda)$ be the derived category of étale Λ-sheaves on X as defined in [Sch17], where for $\Lambda = \mathbb{Q}_\ell$ we additionally invert ℓ in the adic formalism developed in [Sch17, Section 26]. We consider the inclusion of the geometric generic, respectively geometric special fiber into the integral Hecke stack

$$\mathrm{Hk}_G|_{\mathrm{Spd}\,\mathbb{C}_p} \xrightarrow{j} \mathrm{Hk}_{\mathcal{G}}|_{\mathrm{Spd}\,O_{\mathbb{C}_p}} \xleftarrow{i} \mathrm{Hk}_{\mathcal{G}}|_{\mathrm{Spd}\,\bar{\mathbb{F}}_p}, \tag{1.10}$$

see Section 4 for the definition of Hecke stacks. The following result is inspired by work of Hansen–Scholze [HS21] for schemes and proves constructibility of nearby cycles in our context:

Theorem 1.8 (Proposition 6.7, Proposition 6.12)**.** *The pullback functor*

$$j^*\colon \mathrm{D}\big(\mathrm{Hk}_{\mathcal{G}}|_{\mathrm{Spd}\,O_{\mathbb{C}_p}}, \Lambda\big) \to \mathrm{D}\big(\mathrm{Hk}_G|_{\mathrm{Spd}\,\mathbb{C}_p}, \Lambda\big) \tag{1.11}$$

induces an equivalence on the full subcategories of universally locally acyclic sheaves with bounded support, with inverse given by the derived push forward Rj_*. Consequently, the nearby cycles functor $\Psi_{\mathcal{G}} := i^* \circ Rj_*$ restricts to a functor

$$\mathrm{D}\big(\mathrm{Hk}_G|_{\mathrm{Spd}\,\mathbb{C}_p}, \Lambda\big)^{\mathrm{bd,ula}} \to \mathrm{D}\big(\mathrm{Hk}_{\mathcal{G}}|_{\mathrm{Spd}\,\bar{\mathbb{F}}_p}, \Lambda\big)^{\mathrm{bd,ula}}. \tag{1.12}$$

Furthermore, the target category is equivalent to the derived category on the schematic Witt vector Hecke stack $\mathrm{D}_{\mathrm{cons}}\big(\mathrm{Hk}^{\mathrm{sch}}_{\mathcal{G}}|_{\mathrm{Spec}\,\bar{\mathbb{F}}_p}, \Lambda\big)^{\mathrm{bd}}$ of perfect-constructible sheaves with bounded support (see Section A).

The intersection complex IC_μ descends along $\mathrm{Gr}_G \to \mathrm{Hk}_G$ and defines an object in $\mathrm{D}(\mathrm{Hk}_G|_{\mathrm{Spd}\,\mathbb{C}_p}, \Lambda)^{\mathrm{bd,ula}}$. So Theorem 1.8 implies constructibility of $\Psi_{\mathcal{M}_{\mathcal{G},\mu}}\mathrm{IC}_\mu$. To compute the support, we use that nearby cycles commute with the constant term functors defined by the generic and special fibers of diagram (1.8):

$$\mathrm{CT}_{\mathcal{P}} \circ \Psi_{\mathcal{G}} \cong \Psi_{\mathcal{M}} \circ \mathrm{CT}_P, \tag{1.13}$$

see Proposition 6.13 for details. If λ is regular, then $\mathcal{M}$ is the connected Néron model of a torus so that the right hand side of (1.13) can be computed in representation theoretic terms using the geometric Satake equivalence for the B^+_{dR}-affine Grassmannian [FS21, Chapter VI]. In down to earth terms, we are able to determine the compactly supported cohomology of $\Psi_{\mathcal{M}_{\mathcal{G},\mu}}\mathrm{IC}_\mu$ on the stratification given by the semi-infinite orbits. For carefully chosen λ (see Lemma 5.3), this is enough together with the density of generic fibers (Section 1.3.1) to deduce Theorem 1.5, see Theorem 6.16.

1.3.3. *Special fibers of schematic local models.* In this subsection, we assume that μ is minuscule. The construction of schematic local models [PZ13, Lev16, Lou20, FHLR22] relies on lifting $\mathcal{G}$ to a group scheme $\underline{\mathcal{G}}$ over $\breve{\mathbb{Z}}_p[\![t]\!]$ equipped with isomorphisms

$$\underline{\mathcal{G}} \otimes_{\breve{\mathbb{Z}}_p[\![t]\!], t\mapsto p} \breve{\mathbb{Z}}_p \simeq \mathcal{G} \otimes \breve{\mathbb{Z}}_p, \qquad \underline{\mathcal{G}} \otimes \breve{\mathbb{Q}}_p[\![t-p]\!] \simeq G \otimes \breve{\mathbb{Q}}_p[\![t-p]\!]. \tag{1.14}$$

Let us temporarily denote by $\mathcal{N}^{\mathrm{sch}}_{\underline{\mathcal{G}},\mu}$ the weak normalization of the closure of $\mathcal{F}_{G,\mu}|_{\mathrm{Spec}\,\breve{E}}$ inside the schematic Beilinson–Drinfeld Grassmannian $\mathrm{Gr}_{\underline{\mathcal{G}}}|_{\mathrm{Spec}\,O_{\breve{E}}}$ attached to $\underline{\mathcal{G}}$, a weakly normal, flat, projective $O_{\breve{E}}$-scheme. As we explain in Section 1.3.4, this is isomorphic to the base changed local model $\mathcal{M}^{\mathrm{sch}}_{\mathcal{G},\mu}|_{\mathrm{Spec}\,O_{\breve{E}}}$ appearing in Theorem 1.2, in virtually all cases in the above sense. The most general group lifts $\underline{\mathcal{G}}$ are constructed in [FHLR22], based on [Lou20], under the following assumption:

Assumption 1.9. If $p=2$, then G_{ad} has no odd unitary $\breve{\mathbb{Q}}_2$-factors.

The reason for its appearance is the difficult structure of the integral root groups inside $\mathcal{G} \otimes \breve{\mathbb{Z}}_p$ in the wildly ramified, odd unitary case for $p=2$. More precisely, quadratic field extensions of $\breve{\mathbb{Q}}_2$ fall into two classes: square roots of uniformizers and of units. The first class is handled in [Lou20], leading to the milder assumption for $p=2$ as discussed in the text after the statement of Theorem 1.2. It is the second class that appears most difficult.

The determination of the special fiber of $\mathcal{N}^{\mathrm{sch}}_{\underline{\mathcal{G}},\mu}$ relies on the coherence conjecture of Pappas–Rapoport proved by Zhu [Zhu14]. The following result is the version in our context, and moreover, confirms [Zhu17a, Appendix B, Conjecture III] for the Schubert varieties in the μ-admissible locus:

Theorem 1.10 (Theorem 3.16). *Under Assumption 1.9, the canonical deperfection $\mathcal{A}^{\mathrm{can}}_{\mathcal{G},\mu}$ of the μ-admissible locus is Cohen–Macaulay and its components are compatibly Frobenius-split. Moreover, for every ample line bundle $\mathcal{L}$ on $\mathcal{A}^{\mathrm{can}}_{\mathcal{G},\mu}$, there is an equality*

$$\dim_k \mathrm{H}^0(\mathcal{A}^{\mathrm{can}}_{\mathcal{G},\mu}, \mathcal{L}) = \dim_E \mathrm{H}^0(\mathcal{F}_{G,\mu}, \mathcal{O}(c)), \tag{1.15}$$

where $c \in \mathbb{Z}$ denotes the central charge of $\mathcal{L}$ and $\mathcal{O}(c)$ the c-th multiple of the ample generator of $\mathrm{Pic}(\mathcal{F}_{G,\mu})$.

The canonical deperfection of $\mathcal{A}_{\mathcal{G},\mu}$ is induced from the Greenberg realization of the Witt vector loop groups, see Section 3.3. The proof of Theorem 1.10 proceeds by comparing the p-adic admissible loci to their analogues in the equicharacteristic situation, and ultimately relies on the normality of Schubert varieties [Fal03, PR08] where we use [FHLR22] for wildly ramified groups. We first compare the perfect(ed) Demazure resolutions and then apply Bhatt–Scholze's h-descent results [BS17] to the ample line bundles on the resolutions. A key ingredient is He–Zhou's calculation [HZ20] of the Picard group of $\mathcal{F}\ell_{\mathcal{G}}$ as the free $\mathbb{Z}[p^{-1}]$-module dual to the lines stable under a fixed Iwahori dilated from $\mathcal{G}$.

Theorem 1.11 (Lemma 3.15, [FHLR22]). *Under Assumption 1.9, there is an isomorphism of $\bar{\mathbb{F}}_p$-schemes*

$$\mathcal{N}^{\mathrm{sch}}_{\underline{\mathcal{G}},\mu}|_{\mathrm{Spec}\,\bar{\mathbb{F}}_p} \simeq \mathcal{A}^{\mathrm{can}}_{\mathcal{G},\mu}|_{\mathrm{Spec}\,\bar{\mathbb{F}}_p}. \tag{1.16}$$

Hence, $\mathcal{N}^{\mathrm{sch}}_{\underline{\mathcal{G}},\mu}$ is normal, Cohen–Macaulay and has reduced special fiber.

The theorem holds, more generally, under the milder assumption explained above when $p = 2$ by [Lou20]. The reader is referred to [FHLR22] for a finer study of the singularities of the local models.

1.3.4. *Specialization maps.* We continue to assume that μ is minuscule and focus on the v-sheaf local models $\mathcal{M}_{\mathcal{G},\mu}$. The study of specialization maps for $\mathcal{M}_{\mathcal{G},\mu}$ is challenging. A basic problem is that, beyond rare exceptions, the set of $\mathcal{G}(O_{\mathbb{C}_p})$-orbits in $\mathcal{F}_{G,\mu}(\mathbb{C}_p)$ is infinite. However, we understand relatively well the reduction of $\mathrm{Spd}\, O_{\mathbb{C}_p}$-valued points lying in a certain cohomologically smooth sub-v-sheaf

$$\mathcal{M}^{\circ}_{\mathcal{G},\mu} \subset \mathcal{M}_{\mathcal{G},\mu}, \tag{1.17}$$

defined in Definition 7.5. Unfortunately, $\mathcal{M}^{\circ}_{\mathcal{G},\mu}$ alone does not afford sufficiently many integral points. Here we resort to variants of the splitting models of Pappas–Rapoport [PR05] in our situation, that is, we use convolutions to partially desingularize the local models. For a sequence $\mu_\bullet = (\mu_1, \dots, \mu_n)$ of minuscule coweights with pairwise disjoint supports, we consider the sub-v-sheaf

$$\mathcal{M}_{\mathcal{G},\mu_\bullet} \subset \mathrm{Gr}_{\mathcal{G}} \tilde{\times} \dots \tilde{\times} \mathrm{Gr}_{\mathcal{G}}, \tag{1.18}$$

defined as the v-closure of $\mathcal{F}^{\diamondsuit}_{G,\mu_\bullet} = \mathcal{F}^{\diamondsuit}_{G,\mu_1} \tilde{\times} \dots \tilde{\times} \mathcal{F}^{\diamondsuit}_{G,\mu_n}$ inside the convolution Beilinson–Drinfeld Grassmannian, see Section 7.1. Most of the notions discussed before have their convolution counterparts. In particular, the convolution v-sheaf local models support specialization maps just like the v-sheaf local models do. Then, functoriality in $(\mathcal{G}, \mu_\bullet)$ is enough to control the specialization map:

Theorem 1.12 (Theorem 7.12)**.** *The specialization maps*

$$\mathrm{sp}_{\mathcal{G},\mu_\bullet} : \mathcal{F}_{G,\mu_\bullet}(\mathbb{C}_p) \to \mathcal{A}_{\mathcal{G},\mu_\bullet}(\bar{\mathbb{F}}_p) \tag{1.19}$$

for all pairs $(\mathcal{G}, \mu_\bullet)$ as above are the only functorial collection of continuous maps whose restrictions to the sets $\mathcal{M}^\circ_{\mathcal{G},\mu_\bullet}(\operatorname{Spd} O_{\mathbb{C}_p})$ agree with the natural reduction maps.

The theorem is a result of our reflections on the He–Pappas–Rapoport conjecture [HPR20, Conjecture 2.12], which states that the local model $\mathcal{M}_{\mathcal{G},\mu}$ should be uniquely recovered from its fibers equipped with the $\mathcal{G}$-action. The key calculation concerns the case where G is a restriction of scalars along $E/\mathbb{Q}_p$ of a split group and where all non-zero components of $\mu_\bullet$ have support lying in disjoint irreducible components of the Dynkin diagram of G. Applying the Iwasawa decomposition and an induction on the number of non-zero components, we see $\mathcal{F}_{G,\mu_\bullet}(E) = \mathcal{M}^\circ_{\mathcal{G},\mu_\bullet}(\operatorname{Spd} O_E)$, that is, all rational points of the flag variety extend to integral points of $\mathcal{M}^\circ_{\mathcal{G},\mu_\bullet}$. Then functoriality forces uniqueness for all remaining cases.

1.3.5. *Conclusion.* The formulation of Theorem 1.12 requires functoriality of local models in $(\mathcal{G}, \mu)$. This is clear for $\mathcal{M}_{\mathcal{G},\mu}$ but, a priori, problematic for its schematic version $\mathcal{N}^{\mathrm{sch}}_{\underline{\mathcal{G}},\mu}$. We need to impose the following assumption which relates to functoriality problems with the association $\mathcal{G} \mapsto \underline{\mathcal{G}}$ of group lifts:

Assumption 1.13. If $p = 3$, then G_{ad} has no triality $\breve{\mathbb{Q}}_3$-factors.

The following result confirms Conjecture 1.1, except for the two non-split examples for $p = 2, 3$:

Theorem 1.14. *There is a unique flat, projective and weakly normal O_E-model $\mathcal{M}^{\mathrm{sch}}_{\mathcal{G},\mu}$ of the E-scheme $\mathcal{F}_{G,\mu}$ with an isomorphism of v-sheaves*

$$(\mathcal{M}^{\mathrm{sch}}_{\mathcal{G},\mu})^\diamondsuit \cong \mathcal{M}_{\mathcal{G},\mu}, \tag{1.20}$$

prolonging $\mathcal{F}^\diamondsuit_{G,\mu} \cong \mathcal{M}_{\mathcal{G},\mu}|_{\operatorname{Spd} E}$. Under Assumption 1.9 and Assumption 1.13, there is a unique isomorphism

$$\mathcal{M}^{\mathrm{sch}}_{\mathcal{G},\mu}|_{\operatorname{Spec} O_{\breve{E}}} \cong \mathcal{N}^{\mathrm{sch}}_{\underline{\mathcal{G}},\mu}|_{\operatorname{Spec} O_{\breve{E}}} \tag{1.21}$$

inducing the identity on generic fibers. So (Theorem 1.11), $\mathcal{M}^{\mathrm{sch}}_{\mathcal{G},\mu}$ is normal, Cohen–Macaulay and has reduced special fiber equal to $\mathcal{A}^{\mathrm{can}}_{\mathcal{G},\mu}$. Furthermore, the isomorphisms (1.20) *and* (1.21) *are equivariant for $\mathcal{G}^\diamondsuit_{O_E}$ and $\mathcal{G}_{O_{\breve{E}}}$ respectively.*

The unique scheme $\mathcal{M}^{\mathrm{sch}}_{\mathcal{G},\mu}$ satisfying the Scholze–Weinstein conjecture is the weak normalization of the closure of $\mathcal{F}_{G,\mu}$ inside the Beilinson–Drinfeld Grassmannian attached to a group lift of $\operatorname{Res}_{O_K/\mathbb{Z}_p}\mathcal{H}$, where K is the splitting field of G and $\mathcal{H}$ the parahoric O_K-model of the split Chevalley form. By a careful analysis, we get an isomorphism between the specialization triples associated with $\mathcal{M}_{\mathcal{G},\mu}$ and $\mathcal{M}^{\mathrm{sch},\diamondsuit}_{\mathcal{G},\mu}$, which is enough to conclude by Theorem 1.4.

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2. v-sheaf theory

Our main reference for the theory of diamonds, v-sheaves and v-stacks is [Sch17]. Here we gather some complementary results in the geometry of these objects that we will need later on.

2.1. Closures.

In this subsection, we discuss closures of small v-stacks. Let $\mathrm{Perf}_{\mathbb{F}_p}$ be the v-site of perfectoid spaces of characteristic p. All the small v-stacks in the following will be stacks on $\mathrm{Perf}_{\mathbb{F}_p}$.

Let X be a small v-stack and let $Y \subset X$ be a sub-v-stack, by which we mean a monomorphism of small v-stacks $Y \to X$, that is, a morphism whose diagonal

$$Y \xrightarrow{\Delta} Y \times_X Y \tag{2.1}$$

is an isomorphism. Just as in [Sch17, Definition 10.7], we say that Y is a locally closed sub-v-stack if, for every totally disconnected perfectoid space $S \to X$, the pullback $Y \times_X S \to S$ is representable by an immersion of perfectoid spaces, see [Sch17, Definition 5.6]. It is called a closed, respectively open immersion, if so are the respective pullbacks.

This admits a simpler description for closed sub-v-stacks. For a small v-stack X denote by $|X|$ its underlying topological space, see [Sch17, Definition 12.8].

Lemma 2.1. *A morphism $Y \to X$ of small v-stacks is a closed immersion if and only if $Y \to X$ is a quasi-compact monomorphism and the induced map $|Y| \subset |X|$ is a closed embedding.*

Proof. Assume $Y \to X$ is a closed immersion and let $f\colon S \to X$ be a surjection from a disjoint union of totally disconnected perfectoid spaces. By assumption $Z := Y \times_X S$ is representable and $Z \to S$ is a closed immersion, in particular it is quasi-compact. By [Sch17, Proposition 10.11 (o)], the map $Y \to X$ is then quasi-compact as well. We may check that $Y \xrightarrow{\Delta} Y \times_X Y$ is an isomorphism after base change to S. This amounts to verifying that $Z \xrightarrow{\Delta} Z \times_S Z$ is an isomorphism which follows from the fact that closed immersions are monomorphisms [Sch17, Definition 5.6]. The inclusion $|Z| \subset |S|$ is a closed subset and equal to $|f|^{-1}(|Y|)$. Indeed, if $s \in |S| \setminus |Z|$, $y \in |Y|$ and we let $\tilde{s}\colon \mathrm{Spa}(C_s, C_s^+) \to S$ and $\tilde{y}\colon \mathrm{Spa}(C_y, C_y^+) \to Y$ represent s and y respectively, then $\tilde{s} \times_X \tilde{y} = \emptyset$. So s and y map to different points in $|X|$. As $|f|$ is a quotient map [Sch17, Proposition 12.9], this implies that $|Y| \subset |X|$ is a closed embedding.

Conversely, assume that $Y \subset X$ is a quasi-compact monomorphism and induces a closed embedding of underlying topological spaces. Let $f\colon S \to X$ be a map from a totally disconnected space S. The base change $Y \times_X S \to S$ is still a quasi-compact monomorphism of v-sheaves. By [Sch17, Corollary 10.6, Lemma 7.6] the v-sheaf $Y \times_X S$ is representable by a pro-constructible generalizing affinoid subset of S, and $|Y \times_X S|$ carries the subspace topology of $|S|$. Arguing as above, the image of $|Y \times_X S|$ in $|S|$ is $|f|^{-1}(|Y|)$ which is closed by assumption. This implies that the morphism $Y \times_X S \to S$ is a closed immersion in the sense of [Sch17, Definition 5.6]. ☐

We now define the v-sheaf closure, or v-closure, of a sub-v-sheaf.

Definition 2.2. Let X be a small v-stack and $Y \subset X$ a sub-v-stack $Y \subset X$. We define the v-closure Y^{cl} in X as the limit (in the 2-category of v-stacks) of all closed sub-v-stacks of X containing Y.

The sub-v-stack $Y^{\mathrm{cl}} \subset X$ is a closed sub-v-stack. Indeed, we can verify this after base change by a totally disconnected perfectoid space and use [Sch17, Proposition 6.4 (o)] to conclude.

Next, we discuss the relation between the topological space $|Y^{\mathrm{cl}}|$ of the v-closure Y^{cl} and the topological closure $|Y|^{\mathrm{cl}}$ of the image of $|Y|$ in $|X|$.

Definition 2.3. Let $S \subset |X|$ be a subset.

(1) We call S weakly generalizing if, for any perfectoid field C with open and bounded valuation subring $C^+ \subset C$, and every morphism $\mathrm{Spa}(C, C^+) \to X$, the induced morphism $|\mathrm{Spa}(C, C^+)| \to |X|$ factors over S if and only if the closed point of $|\mathrm{Spa}(C, C^+)|$ maps into S.

(2) The weakly generalizing closure S^{wgc} of S is defined as the intersection of all closed, weakly generalizing subsets $S' \subset |X|$ containing S.

We note that if X is (the v-sheaf associated to) a perfectoid space, then a subset $S \subset X$ is weakly generalizing if and only if it is generalizing. Indeed, for each analytic adic space specializations happen only at the same residue field.

The images of morphisms of small v-stacks are weakly generalizing as the next lemma shows.

Lemma 2.4. *For every morphism $f \colon X \to X'$ of small v-stacks, the image of $|f| \colon |X| \to |X'|$ is weakly generalizing in $|X'|$.*

Proof. Let C be a perfectoid field with open and bounded valuation subring $C^+ \subset C$. Assume that the morphism

$$\mathrm{Spa}(C, C^+) \to X' \tag{2.2}$$

sends the closed point of $\mathrm{Spa}(C, C^+)$ into $f(|X|)$. This means that the above morphism factors through X after possibly enlarging (C, C^+), see [Sch17, Proposition 12.7]. But then the full image of $|\mathrm{Spa}(C, C^+)|$ in $|X'|$ will factor through $f(|X|)$ and this shows that $f(|X|)$ is weakly generalizing. □

In particular, the topological space of the v-closure Y^{cl} of some sub-v-stack $Y \subset X$ is always weakly generalizing. Thus, the topological space of the v-closure does not coincide, in general, with the topological closure.

Example 2.5. As a concrete example, consider the inclusion

$$\mathbb{D}_C^{\diamondsuit} \to \mathbb{B}_C^{\diamondsuit} := \mathrm{Spd}(C\langle T \rangle, \mathcal{O}_C\langle T \rangle) \tag{2.3}$$

of the open unit ball into the closed unit ball over a perfectoid base field C. Here $(-)^{\diamondsuit}$ denotes the functor from analytic pre-adic spaces to diamonds from [SW20, Lecture 10]. Then $|\mathbb{D}_C^{\diamondsuit}|^{\mathrm{cl}}$ is the complement of the torus $|\mathbb{T}_C^{\diamondsuit}| = \mathrm{Spd}(C\langle T^{\pm 1} \rangle, \mathcal{O}_C\langle T^{\pm 1} \rangle)$, hence not weakly generalizing, as it misses the Gaußpoint but contains a rank 2 specialization thereof.

The weakly generalizing closure $|\mathbb{D}_C^{\diamondsuit}|^{\mathrm{wgc}}$ is given in turn by the complement of every open unit ball $\mathbb{D}_{x,C}^{\diamondsuit}$ centered around $x \in \mathbb{T}_C(C)$. This will give rise to the v-closure of $\mathbb{D}_C^{\diamondsuit}$ inside $\mathbb{B}_C^{\diamondsuit}$, see Proposition 2.8.

Let us recall that, for every small v-stack X, there is the canonical morphism

$$X \to \underline{|X|}, \; (f \colon T \to X) \mapsto (|f| \colon |T| \to |X|) \tag{2.4}$$

of v-stacks where $\underline{|X|}$ denotes the v-sheaf represented by the topological space $|X|$, that is, it is given by $\underline{|X|}(T) = \mathrm{Hom}_{\mathrm{cts}}(|T|, |X|)$ for each $T \in \mathrm{Perf}_{\mathbb{F}_p}$.

Remark 2.6. We warn the reader that $\underline{|X|}$ is not small whenever $|X|$ fails to satisfy the separation axiom $T1$. Indeed, if $|X|$ is a trait, then $\underline{|X|}(R, R^+)$ is the set of closed subsets of $|\operatorname{Spa}(R, R^+)|$, and for each fixed cut-off cardinal κ, there is $|\operatorname{Spa}(R, R^+)|$ large enough so that not all closed subsets come from pullback of κ-small ones.

Lemma 2.7. *Let X be a small v-stack, and $S \subset |X|$ be a weakly generalizing closed subset. Then $Y := \underline{S} \times_{\underline{|X|}} X$ is a small closed sub-v-stack satisfying $|Y| = S$ and, moreover, every closed sub-v-stack is of this form.*

Proof. By [Sch17, Proposition 10.11], we may check that $Y \subset X$ is a closed sub-v-stack after pullback along a v-cover $f : Z \to X$ with Z a disjoint union of totally disconnected perfectoid spaces. Then $Y \times_X Z = \underline{|f|^{-1}(S)} \times_{\underline{|Z|}} Z$ and note that $|f|^{-1}(S) \subset |Z|$ is closed as $|f|$ is continuous. Moreover, $|f|^{-1}(S)$ is weakly generalizing, and thus generalizing because Z is a perfectoid space. Consequently, $|f|^{-1}(S)$ is representable by a perfectoid space by [Sch17, Lemma 7.6]. Its v-sheaf coincides with $Y \times_X Z$ by [Sch17, Lemma 12.5], so $Y \subset X$ is a closed immersion by Lemma 2.1. Clearly, we also have $|Y| = S$ as $|f|$ is surjective.

Now assume that $Y \subset X$ is a closed sub-v-stack and let $Y' = X \times_{\underline{|X|}} \underline{|Y|}$. The identity $Y = Y \times_X Y'$ is easy to verify (by base change to totally disconnected S and [Sch17, Proposition 5.3.(iv)]), so the map $Y \to Y'$ is a closed sub-v-stack with the same underlying topological space. By [Sch17, Lemma 12.11], $Y \to Y'$ is a surjetive map of v-stacks and consequently an isomorphism. □

Lemma 2.4 and Lemma 2.7 characterize closed weakly generalizing subsets $S \subset |X|$ as exactly those closed subsets $S \subset |X|$ for which the inclusion

$$\left|\underline{S} \times_{\underline{|X|}} X\right| \subset S \tag{2.5}$$

is an equality. Note that the v-sheaf $Y := \underline{S} \times_{\underline{|X|}} X$ may even be empty if S is not weakly generalizing. For example, this happens if $S = \{s\}$ for $s \in \operatorname{Spa}(C, C^+)$ the closed point of a perfectoid field C with $C^+ \subsetneq O_C \subset C$ an open and bounded valuation subring of rank > 1.

Proposition 2.8. *Let X be a small v-stack, and let $Y \subset X$ be a sub-v-stack. Let $Y^{\mathrm{cl}} \subset X$ be the v-closure of Y in X. Then*

$$Y^{\mathrm{cl}} = \underline{|Y|^{\mathrm{wgc}}} \times_{\underline{|X|}} X \tag{2.6}$$

as sub-v-stacks of X. Hence, $|Y^{\mathrm{cl}}| \subset |X|$ is the weakly generalizing closure $|Y|^{\mathrm{wgc}}$ of $|Y|$ in $|X|$.

Proof. Set $Y' := \underline{|Y|^{\mathrm{wgc}}} \times_{\underline{|X|}} X$. Then Y' is a closed sub-v-stack of X containing Y and $|Y'| = |Y|^{\mathrm{wgc}}$ by Lemma 2.7. Therefore, the v-closure Y^{cl} is contained in Y'. But conversely, the topological space $|Y^{\mathrm{cl}}|$ must contain $|Y|^{\mathrm{wgc}}$ by Lemma 2.4. Since $Y^{\mathrm{cl}} = \underline{|Y^{\mathrm{cl}}|} \times_{\underline{|X|}} X$ again by Lemma 2.7, we conclude that $Y' \subset Y^{\mathrm{cl}}$ and thus they coincide as desired. □

The next result will turn out to be a useful tool later on when computing v-closures.

Corollary 2.9. *The formation of v-closures commutes with base change by partially proper morphisms that are also open maps.*

Proof. In the following, we identify open substacks of small v-stacks with open subsets of their topological space, see [Sch17, Proposition 12.9]. By Proposition 2.8, we need to verify the corresponding assertion at the topological level. Let $f : Z \to X$ be an open and partially proper morphism between small v-stacks and set $g := |f|$. Let $S \subset |X|$ be a subset, clearly $g^{-1}(S)^{\mathrm{wgc}} \subset g^{-1}(S^{\mathrm{wgc}})$. Let $T := g^{-1}(S)^{\mathrm{wgc}} \subset |Z|$. Its complement V is an open subset of $|Z|$, and the map $V \to Z$ is partially proper because T is weakly generalizing. Since the map $Z \to X$ is open, the subset $U := g(V)$ is also open. Since $V \to Z$ is partially proper, the map $U \to X$ is partially proper as well. The complement $F \subset |X|$ of U is closed and $g^{-1}(F) \subset T$. Also, F is weakly generalizing since U is partially proper. This implies $S^{\mathrm{wgc}} \subset F$ and consequently $g^{-1}(S^{\mathrm{wgc}}) \subset T$, as we wanted to show. □

2.2. **The two different diamond functors.** Let O be a complete discrete valuation ring with perfect residue field k of characteristic p, and assume that O is flat over $\mathbb{Z}_p$, that is, p-torsion free. Let $\pi \in O$ be a uniformizer.

If X is a pre-adic space over O, we can attach to it a v-sheaf $X^{\diamondsuit}$ over $\mathrm{Spd}\, O$ as in [SW20, Lecture 18]. (Whenever $R^+ = R$, we simply write $\mathrm{Spd}(R)$ for $\mathrm{Spd}(R, R)$ following [SW20].) Namely, if $S \in \mathrm{Perf}_{\mathbb{F}_p}$, then

$$X^{\diamondsuit}(S) = \{(S^\sharp, \iota, f)\}/\mathrm{isom}. \tag{2.7}$$

with $(S^\sharp, \iota)$ an untilt of S and $f\colon S^\sharp \to X$ a morphism of pre-adic spaces (and the obvious notion of isomorphism between these triples). On the other hand, given an algebra A over O there are two different ways to associate a v-sheaf to $\mathrm{Spec}(A)$.

Definition 2.10. Let A be an O-algebra.

(1) We let $\mathrm{Spec}(A)^{\diamond}$ denote the functor

$$(R, R^+) \mapsto \{(R^\sharp, \iota, f)\}/\mathrm{isom}. \tag{2.8}$$

where $(R^\sharp, \iota)$ is an untilt over O and $f\colon A \to R^{\sharp,+}$ is an O-algebra homomorphism.

(2) We let $\mathrm{Spec}(A)^{\diamondsuit}$ denote the functor

$$(R, R^+) \mapsto \{(R^\sharp, \iota, f)\}/\mathrm{isom}. \tag{2.9}$$

where $(R^\sharp, \iota)$ is an untilt over O and $f\colon A \to R^\sharp$ is an O-algebra homomorphism.

Both of these constructions are compatible with localization and glue to functors from the category of schemes over O to the category of v-sheaves over $\mathrm{Spd}\, O$. Indeed, given $g \in A$, the open subscheme $\mathrm{Spec}(A[1/g])$ is sent to the open subfunctors of $\mathrm{Spec}(A)^{\diamond}$, respectively $\mathrm{Spec}(A)^{\diamondsuit}$ defined by the conditions $f(g) \in (R^\sharp)^\times$, respectively $f(g) \in (R^{\sharp,+})^\times$, that is, by the open loci $\{|g| \neq 0\} \subset \mathrm{Spa}(R^\sharp, R^{\sharp,+})$, respectively $\{|g| = 1\} \subset \mathrm{Spa}(R^\sharp, R^{\sharp,+})$. We still denote these functors on Sch_O by $\diamond$ and $\diamondsuit$.

For schemes locally of finite type over O both functors admit a two step construction following [Gle24, Section 5.1]. Let O_{disc} denote the ring O equipped with the discrete topology. The forgetful functor from the category of discrete O_{disc}-adic spaces $\mathrm{DiscAdSp}_{O_{\mathrm{disc}}}$ towards the category of O-schemes Sch_O admits a left and right adjoint:

$$\begin{tikzcd} \mathrm{Sch}_O \arrow[r, shift left=2, "(\text{-})^{\mathrm{ad}}"] \arrow[r, shift right=2, "(\text{-})^{\mathrm{ad}/O}"'] & \mathrm{DiscAdSp}_{O_{\mathrm{disc}}} \arrow[l] \end{tikzcd}$$

On affinoids, respectively affines the functors are characterized by the formulas

$$\mathrm{Spa}(A, A^+) \mapsto \mathrm{Spec}(A), \quad \mathrm{Spec}(A)^{\mathrm{ad}} = \mathrm{Spa}(A, A), \quad \mathrm{Spec}(A)^{\mathrm{ad}/O} = \mathrm{Spa}(A, \widetilde{O}),$$

where $\widetilde{O}$ is the integral closure of O in A. If X is a scheme locally of finite type over O, then the fiber products

$$\widehat{X} := X^{\mathrm{ad}} \times_{\mathrm{Spa}\, O_{\mathrm{disc}}} \mathrm{Spa}\, O \quad \text{and} \quad X^{\mathrm{an}} := X^{\mathrm{ad}/O} \times_{\mathrm{Spa}\, O_{\mathrm{disc}}} \mathrm{Spa}\, O$$

along the identity $O_{\mathrm{disc}} \to O$ exist in the category of adic spaces by Huber [Hub94, Proposition 3.7] and the structure map to $\mathrm{Spa}\, O$ is adic. In fact, if $X = \mathrm{Spec}(A)$, then $\widehat{X} = \mathrm{Spa}(\widehat{A}, \widehat{A})$ where $\widehat{A}$ is the π-adic completion of A whereas X^{an} is in general not affinoid. The following lemma is immediate from the construction and Definition 2.10:

Lemma 2.11. *The diamond functor $(-)^{\diamondsuit}$ on pre-adic spaces induces isomorphisms of functors $\widehat{X}^{\diamondsuit} = X^{\diamond}$ and $(X^{\mathrm{an}})^{\diamondsuit} = X^{\diamondsuit}$ from the category of schemes X locally of finite type over O towards the category of v-sheaves over* $\mathrm{Spd}\, O$.

For any O-scheme X there is an evident natural transformation $X^{\diamond} \to X^{\diamondsuit}$. If X is separated over O, this map is a monomorphism of small v-sheaves, and an open immersion if X is also locally of finite type over O. If X is proper over O, then the open immersion $X^{\diamond} \to X^{\diamondsuit}$ is an isomorphism because it is surjective on points by the valuative criterion for properness. In the following, let $X^{\diamondsuit}$ denote the common value $X^{\diamond} = X^{\diamondsuit}$ whenever X is proper over O.

The two diamond functors play different roles in the following sections: the "analytic" functor (-)$^{\diamondsuit}$ is the most natural to study $\mathbb{G}_m$-actions and nearby cycles while the "formal" functor (-)$^{\diamond}$ carries a specialization map which we will use to determine specialization triples.

A natural question is to what extent the associated v-sheaves $X^{\diamond}$ and $X^{\diamondsuit}$ reflect the geometry of X. In general, neither of the functors is full or faithful. For example, if A is any O-algebra and $\hat{A}$ its π-adic completion, then the natural morphism

$$\mathrm{Spec}(\hat{A})^{\diamond} \to \mathrm{Spec}(A)^{\diamond} \tag{2.10}$$

is an isomorphism (because $R^{\sharp,+}$ is π-complete by uniformity of affinoid perfectoid spaces). In particular, if F is the fraction field of O, and A an F-algebra, then $\mathrm{Spec}(A)^{\diamond} = \emptyset$. If $A = F[t]$, then

$$\mathrm{Spec}(F[t])^{\diamondsuit} = (\mathbb{A}^{1,\mathrm{ad}}_F)^{\diamondsuit} \tag{2.11}$$

is the rigid-analytic affine line over F, which has many non-algebraic automorphisms.

When we restrict to schemes over O for which $\pi = 0$, the situation is more clear. Both $\diamond$ and $\diamondsuit$ are fully faithful on perfect schemes and if we let Y denote the perfection of X, then $X^{\diamond} = Y^{\diamond}$ and $X^{\diamondsuit} = Y^{\diamondsuit}$ [SW20, Proposition 18.3.1], [Gle24, Theorem 2.32]. That is, up to a fully faithful embedding, both functors are the perfection functor. Nevertheless, we stress again that the essential images of the functors (-)$^{\diamond}$, (-)$^{\diamondsuit}$ on (perfect) schemes over k are different.

To prove the Scholze-Weinstein conjecture we need to work with schemes that are proper and flat over O, and their associated small v-sheaves. Therefore, we have to relate these two notions.

The functor $X \mapsto X^{\diamond} (= X^{\diamondsuit}$ if X is proper) from the category schemes over O to small v-sheaves over $\mathrm{Spd}\, O$ factors as the composition of the functor

$$\widehat{(\text{-})}_\pi \colon \mathrm{Sch}_O \to \mathrm{fSch}_O,\ Y \mapsto \widehat{Y}_\pi \tag{2.12}$$

of π-adic completion, the functor sending (locally) a formal scheme $\mathrm{Spf}(A)$ over $\mathrm{Spf}(O)$ (with locally finitely generated ideal of definition) to the (pre-)adic space $\mathrm{Spa}(A)$, and then the functor (-)$^{\diamondsuit}$ on pre-adic spaces over $\mathrm{Spa}(O)$. We also denote for a formal scheme Y (admitting locally a finite ideal of definition) by $Y^{\diamondsuit}$ the v-sheaf for the pre-adic space associated with Y.

On the category of schemes which are proper over O, the functor $\widehat{(\text{-})}_\pi$ of π-adic completion is fully faithful by Grothendieck's existence theorem [Sta23, Tag 08BF] or [Gro61, Théorème 5.4.1]. Let us note that π-adic completion maps schemes, which are flat over O, to formal schemes, which are flat over O.

Let us recall the following notions.

Definition 2.12 ([Sta23, Tag 0EUL])**.** A ring A is called semi-normal if for all $a, b \in A$ with $a^3 = b^2$ there exists a unique $c \in A$ with $a = c^2$ and $b = c^3$. Similarly, A is called absolutely weakly normal if it is semi-normal and if, for any prime ℓ and elements $a, b \in A$ with $\ell^\ell a = b^\ell$, there exists a unique $c \in A$ with $a = c^\ell$ and $b = \ell c$. Further, a ring A whose spectrum has finitely many irreducible components is called weakly normal if it is semi-normal and if, for any prime ℓ and elements $a, b \in A$ such that a is a nonzerodivisor, $\ell a \mid \ell b$ and $a \mid \ell b$, one has $a \mid b$.

Note that the last two properties are automatic for any prime ℓ, which is invertible in A, and that each semi-normal ring is reduced.

Since ring localizations preserve semi-normality, absolute weak normality or weak normality, they can be generalized to schemes (having locally finitely many irreducible components for the last property), see [Sta23, Tags 0EUN, 0H3I]. Moreover, given any scheme X, there exists a morphism $X^{\mathrm{sn}} \to X$ (respectively, $X^{\mathrm{awn}} \to X$) from a semi-normal (respectively, absolutely weakly normal) scheme, which is called the semi-normalization (respectively, absolutely weak normalization) of X, and which is also the initial morphism $Y \to X$, which is a universal homeomorphism inducing isomorphisms on each residue field (respectively, universal homeomorphism), see [Sta23, Tags 0EUS]. In comparison, when X has locally finitely many irreducible components, then the normalization $X^{\mathrm{n}} \to X$ factors uniquely through the weak normalization $X^{\mathrm{wn}} \to X$, which is also the initial homeomorphism $Y \to X$ admitting a factorization $X^{\mathrm{n}} \to Y \to X$, see [Sta23, Tags 0H3N].

If A is an $\mathbb{F}_p$-algebra, then A is absolutely weakly normal if and only if A is perfect [Sta23, 0EVV], and thus the absolute weak normalization agrees with the perfection of schemes over $\mathbb{F}_p$. From the universal property of X^{awn} and the fact that universal homeomorphisms are integral, radicial and surjective [Sta23, Tag 04DF], it is clear that normal, integral schemes X with perfect function field are absolutely weakly normal. In comparison, note that any normal scheme is weakly normal, without any condition on its function fields at generic points.

Lemma 2.13. *Let $Y = \mathrm{Spf}(A)$ be an affine formal scheme over O, and let $I \subset A$ be a finitely generated ideal of definition. Let $B := \widehat{A^{\mathrm{awn}}}_I$ be the I-adic completion of the absolute weak normalization of A. Then the natural map*

$$\mathrm{Spf}(B)^{\diamondsuit} \to \mathrm{Spf}(A)^{\diamondsuit} \tag{2.13}$$

is an isomorphism.

Proof. Let $\mathrm{Spa}(R, R^+) \to \mathrm{Spa}(A)$ be a morphism from some affinoid perfectoid space over O. Then R^+ is automatically I-adically complete. By the universal property of the absolute weak normalization and the fact that $\mathrm{Spf}(B)^{\diamondsuit}, \mathrm{Spf}(A)^{\diamondsuit}$ are v-sheaves it suffices to see that every affinoid perfectoid space admits a v-cover by one of the form $\mathrm{Spa}(R, R^+)$ with R^+ absolutely weakly normal. We can always choose (R, R^+) so that R^+ equals a product of perfectoid valuation rings with algebraically closed fraction fields. In this case, R^+ is absolutely weakly normal because the conditions in Definition 2.12 can be checked in each factor. Indeed, it is clear that any such factor is a normal, integral domain with algebraically closed, in particular perfect, fraction field. □

Definition 2.14. A formal scheme X topologically of finite type and flat over O is called weakly normal if it is locally of the form $\mathrm{Spf}(A)$ where A is a weakly normal, flat and π-adically complete topological algebra of the form $O\langle T_1, \ldots, T_n\rangle/I$ for some ideal $I \subset O\langle T_1, \ldots, T_n\rangle$.

To justify Definition 2.14, we need to prove that "weak normality" glues and localizes for the formal schemes that we work with. This is the content of the next statement.

Proposition 2.15. *Assume that A is flat and topologically of finite type over O. Let $\emptyset \neq U_{f_i} \subset \mathrm{Spf}(A)$ with $i \in \{1 \ldots, n\}$ be an open cover by distinguished open subsets with $U_i = \mathrm{Spf}(B_i)$. Then A is weakly normal if and only if all of the B_i are weakly normal.*

Proof. Since weak normality is compatible with localization A is weakly normal if and only if all of the $A[f_i^{-1}]$ are weakly normal. Now, B_i is the π-adic completion of $A[f_i^{-1}]$, in particular flat over it. We claim that $A[f_i^{-1}] \to B_i$ is a regular map [Sta23, Tag 07BZ] and that $A \to \prod_i B_i$ is regular and faithfully flat. Given these, the statement follows directly from [Man80, Proposition

III.3] since a regular map is a reduced and normal map. Let us prove the claim. Observe that all of the rings are Noetherian and excellent because they are obtained from O by adding variables, taking quotient by ideals, completing or localizing. By [Sta23, Tag 07C0], we may check regularity after localizing at a maximal ideal $\mathfrak{m} \subset B_i$. Consider the following maps of rings:

$$A_{(\mathfrak{m})} \to (B_i)_{(\mathfrak{m})} \to (\widehat{A_{(\mathfrak{m})}})_\pi \to \widehat{(A)_\mathfrak{m}} \tag{2.14}$$

They are all faithfully flat. Since A is excellent, the map $A_{(\mathfrak{m})} \to \widehat{(A)_\mathfrak{m}}$ is regular. By [Sta23, Tag 07QI], we conclude $A_{(\mathfrak{m})} \to (B_i)_{(\mathfrak{m})}$ is so as well. □

Variants of the following result appear in [SW20, Proposition 18.4.1] and [Lou20, IV, Theorem 4.6]:

Theorem 2.16. *The functor $X \mapsto X^\diamondsuit$ from the category of absolute weakly normal formal schemes flat, separated and topologically of finite type over O, to v-sheaves over* $\operatorname{Spd} O$ *is fully faithful.*

Proof. We begin by proving the case in which X and Y are affine formal schemes. Confusing a formal scheme with its associated adic space we may assume that $X = \operatorname{Spa}(A), Y = \operatorname{Spa}(B)$ with A, B absolutely weakly normal flat and topologically of finite type over O. Faithfulness follows from the fact that B admits an injection (as it is reduced) into a product of perfectoid valuation rings. For fullness, let $f\colon X^\diamondsuit \to Y^\diamondsuit$ be a morphism of small v-sheaves. We are seeking a morphism $\psi\colon X \to Y$ such that $\psi^\diamondsuit = f$. Let $K = O[\pi^{-1}]$ be the fraction field of O. As A, B are π-adic the generic fibers X_η, Y_η are given by $\operatorname{Spa}(A[\pi^{-1}], A')$, $\operatorname{Spa}(B[\pi^{-1}], B')$ with A', B' the integral closure of A, B in $A[\pi^{-1}]$, $B[\pi^{-1}]$. The localizations $A[\pi^{-1}]$ and $B[\pi^{-1}]$ are absolutely weakly normal by [Sta23, Tag 0EUM], and thus semi-normal. By [SW20, Proposition 10.2.3], we get a morphism $\psi_\eta\colon Y_\eta \to X_\eta$, so that $\psi_\eta^\diamondsuit = f_\eta$. Because A, B are topologically of finite type over O and reduced, the rings A', B' are finite over A, B, and thus in particular the subspace topology coming from $A[\pi^{-1}]$, $B[\pi^{-1}]$ is π-adic on A', B'. In particular, A', B' are Huber. By definition the map $\psi_\eta\colon Y_\eta \to X_\eta$ induces a morphism $\psi'\colon Y' := \operatorname{Spa}(B') \to X$ over O, so that $\psi'_\eta = \psi_\eta$. Denoting by $B'' \subset B'$ the (automatically closed as B is noetherian and B' finite over B) image of $A \widehat{\otimes}_O B$ in $B[\pi^{-1}]$, we even get $\psi''\colon Y'' := \operatorname{Spa}(B'') \to X$ such that the morphism $Y'' \to X \times_O Y$ is a closed embedding of formal schemes. It is easy to see that $(Y'')^\diamondsuit \to (X \times_{\operatorname{Spa}(O)} Y)^\diamondsuit \cong X^\diamondsuit \times_{\operatorname{Spd} O} Y^\diamondsuit$ is a closed immersion of v-sheaves. Inside $X^\diamondsuit \times_{\operatorname{Spd} O} Y^\diamondsuit$, we then have two closed sub-v-sheaves, namely $Y''^\diamondsuit$ induced by $\psi''^\diamondsuit$ and $Y^\diamondsuit \simeq \Gamma_f$ induced by the graph of f. In both of these closed sub-v-sheaves, the generic fiber is dense by Lemma 2.17 below (applied to B and B''), and they carry the same generic fiber. Therefore, the finite birational morphism $Y'' \to Y$ induced by the inclusion $B \subset B''$ becomes an isomorphism in the category of v-sheaves. Passing to special fibers, this implies that $\operatorname{Spec}(B''/\pi)^{\mathrm{perf}} \to \operatorname{Spec}(B/\pi)^{\mathrm{perf}}$ is an isomorphism [SW20, Proposition 18.3.1]. As $B''[\pi^{-1}] \cong B[\pi^{-1}]$ we can conclude that $\operatorname{Spec}(B'') \to \operatorname{Spec}(B)$ is a universal homeomorphism. Indeed, $B \to B''$ is integral, radicial (as can be checked on each fiber over $\operatorname{Spec}(O)$) and surjective. Since B is absolutely weakly normal, we get $B'' = B$ and thus $(\psi'')^\diamondsuit = f$.

We now extend the argument to the general case. To verify faithfulness one can easily argue locally on X and Y because if $X = \cup_{i\in I} X_i$ is an open cover by formal schemes, then $\cup_{i\in I} X_i^\diamondsuit$ is an open cover of $X^\diamondsuit$. Proving fullness is more subtle since one has to justify that for a map $f\colon X^\diamondsuit \to Y^\diamondsuit$ and an open subset $U \subset Y$ with $U = \operatorname{Spf}(A)$, the pullback $f^{-1}(U^\diamondsuit) \subset X^\diamondsuit$ is "classical". In other words,

$$f^{-1}(U^\diamondsuit) = V^\diamondsuit \tag{2.15}$$

for some open immersion of formal schemes $V \subset X$. Now, by [SW20, Proposition 18.3.1] the special fiber map $f \times_{\operatorname{Spd} O} \operatorname{Spd} k$ is induced by a map of perfect schemes $f_{\mathrm{red}}\colon X_{\mathrm{red}}^{\mathrm{perf}} \to Y_{\mathrm{red}}^{\mathrm{perf}}$.

Identifying $|X|$, $|Y|$ with $|X^{\mathrm{perf}}_{\mathrm{red}}|$ and $|Y^{\mathrm{perf}}_{\mathrm{red}}|$, we can construct V as $f^{-1}_{\mathrm{red}}(U_{\mathrm{red}})$. That the identity in Equation (2.15) holds will follow from functoriality of the specialization map considered in [Gle24]. Indeed, $U^{\diamondsuit} = \mathrm{sp}^{-1}_{Y^{\diamondsuit}}(U_{\mathrm{red}})$. $\square$

We used the following lemma. Here, for a Huber pair (A, A^+) over O the notation $\mathrm{Spd}(A, A^+)$ is a shorthand for $\mathrm{Spa}(A, A^+)^{\diamondsuit}$.

Lemma 2.17. *Suppose that B is a π-adically complete flat and topologically of finite type O-algebra, let B' denote the integral closure of B in $B[\pi^{-1}]$. Then the generic fiber $\mathrm{Spd}(B[\pi^{-1}], B')$ is a dense open subset of $\mathrm{Spd}(B)$.*

Proof. Let $X = \mathrm{Spa}(B)$ with B given the π-adic topology. Let Y be the punctured open unit ball over X. That is, $Y = \{y \in \mathrm{Spa}(B[\![t]\!]) \mid |t|_y \neq 0\}$, where $B[\![t]\!]$ is endowed with the (π, t)-adic topology. The map $Y^{\diamondsuit} \to X^{\diamondsuit}$ is a v-cover so it is enough to prove $|Y^{\diamondsuit}_\eta|$ is dense in $|Y^{\diamondsuit}|$. Now, Y is the diamond associated to an analytic adic space so $|Y| = |Y^{\diamondsuit}|$ by [SW20, Proposition 10.3.7]. Let $\mathrm{Spa}(R, R^+) \subset Y$ be a non-empty affinoid rational subset (with (R, R^+) a complete Huber pair). Since $B[\![t]\!]$ is noetherian, flat over B, and rational localizations are flat for Huber pairs admitting a noetherian ring of definition, we can conclude that R is flat over O. Now $\mathrm{Spa}(R, R^+)$ is a pseudorigid space over $\mathrm{Spa}(O)$ in the sense of [Lou17], and thus in particular R is a Jacobson ring [Lou17, Proposition 3.3.(3), 4.6]. By flatness of R over O we get that π is not nilpotent in R. There is a maximal ideal $\mathfrak{m} \subset R$ with $\pi \notin \mathfrak{m}$ as R is a Jacobson ring. By [Hub94, Lemma 1.4] there is an element $x \in \mathrm{Spa}(R, R^+)$ whose support ideal is $\mathfrak{m}$. In particular, this point lies in $\mathrm{Spa}(R, R^+) \cap Y_\eta \neq \emptyset$, which finishes the proof. $\square$

The following consequence is the main statement we need from this chapter.

Proposition 2.18.
(1) *Let X be a proper, flat scheme over O. Then the absolute weak normalization $X^{\mathrm{awn}} \to \mathrm{Spec}(O)$ is proper and flat, and the canonical morphism*
$$(X^{\mathrm{awn}})^{\diamondsuit} \to X^{\diamondsuit} \tag{2.16}$$
is an isomorphism
(2) *The functor $X \mapsto X^{\diamondsuit}$ is fully faithul when restricted to proper, flat and absolutely weakly normal schemes over O.*

Proof. Using Theorem 2.16 and Grothendieck's existence theorem as explained before there remain two statements to check: firstly that $X^{\mathrm{awn}} \to \mathrm{Spec}(O)$ is locally of finite type, and secondly that π-adic completion preserves absolute weak normality of O-algebras of finite type. The first follows from the fact that X is excellent (implying finiteness of the normalization of the reduction of X), and that the absolute weak normalization of an integral domain with field of fraction of characteristic 0 embeds into its normalization. The second follows by stability of absolute weak normality under regular ring homomorphisms, see [GT80, Proposition 5.1] and [Man80, Proposition III.3]. $\square$

2.3. **π-adic kimberlites.** As in Section 2.2, we let O be a complete discrete valuation ring, which is flat over $\mathbb{Z}_p$, with perfect residue field k (of characteristic p) and uniformizer $\pi \in O$. Let F denote its fraction field and C a completed algebraic closure of F. We denote by $\breve{F} \subset C$ the maximal unramified complete subextension with ring of integers $\breve{O}$ and algebraically closed residue field $\bar{k}/k$.

In [Gle24], the second named author introduced a set of axioms for a v-sheaf to have a well behaved specialization map to its reduced locus. The v-sheaves satisfying these axioms are called kimberlites [Gle24, Definition 4.35] and they mimic the behavior of formal schemes. Actually (under the very mild conditions of being separated and locally admitting a finitely generated ideal of definition), the v-sheaves associated to a formal scheme are always kimberlites [Gle24, Proposition 4.31][1] and the specialization map of the kimberlite attached to the formal scheme agrees with the traditional one.

On the other hand, in [Lou17] the third named author considers the functor from the category of formal schemes X over O to the category $\mathcal{C}$ of specialization triples $(X_\eta, X_s, \mathrm{sp}_{\breve{X}})$ where X_η is a rigid analytic space over F, X_s is a scheme over k and $\mathrm{sp}\colon |X_{\breve{\eta}}| \to |X_{\bar{s}}|$ is a continuous map on the underlying topological spaces. Here, $X_{\breve{\eta}}$ and $X_{\bar{s}}$ denotes the base change to $\breve{F}$ and $\bar{k}$ respectively. This functor turns out to be fully faithful when one restricts to X locally formally of finite type (that is, locally of the form $O[\![T_1, \dots, T_n]\!]\langle X_1, \dots, X_m\rangle/I$ for some ideal I), normal and flat over O, see [SW20, 18.4.2].

In this section we take this approach to study π-adic kimberlites. That is, to a π-adic kimberlite X over $\mathrm{Spd}\, O$ we attach a specialization triple $(X_\eta, X^{\mathrm{red}}, \mathrm{sp}_{\breve{X}})$ where now X_η a diamond over $\mathrm{Spd}(F)$, X^{red} a perfect scheme over $\mathrm{Spec}(k)$ and $\mathrm{sp}_{\breve{X}}\colon |X_{\breve{\eta}}| \to |X^{\mathrm{red}}_{\bar{k}}|$ a continuous map. More importantly, we discuss some conditions on X that make this functor fully faithful.

We start by giving a small review of the theory of specialization for kimberlites. Set $\mathrm{SchPerf}_k$ as the v-site of perfect schemes over k (subject to the usual set-theoretic constraints of fixing some cut-off cardinal), and $\widetilde{\mathrm{SchPerf}}_k$ the associated topos.

Definition 2.19 ([Gle24, Definition 3.12])**.** Given a v-sheaf X on $\mathrm{Perf}_{\mathbb{F}_p}$ over $\mathrm{Spd}\, O$, one defines X^{red} as the functor on $\mathrm{SchPerf}_k$ given by $Y \mapsto \mathrm{Hom}(Y^\diamond, X)$.

Thus, if $Y = \mathrm{Spec}(A)$ is an affine perfect scheme, then $X^{\mathrm{red}}(\mathrm{Spec}(A)) = X(\mathrm{Spd}(A))$. By [Gle24, Proposition 3.7], X^{red} is in fact a small v-sheaf on $\mathrm{SchPerf}_k$. The functor $(\text{-})^\diamond\colon \mathrm{Spec}(A) \mapsto \mathrm{Spd}(A)$ extends to small scheme-theoretic v-sheaves and the pair $(\diamond, (\text{-})^{\mathrm{red}})$ forms an adjunction, see [Gle24, Definition 3.12].

For formal schemes over O, the reduction functor is simply the functor that assigns the perfection of the reduced locus [Gle24, Proposition 3.18]. More precisely, if (B, B) is a formal Huber pair over O, that is B is a complete I-adic O-algebra (with I finitely generated), then $\mathrm{Spd}(B)^{\mathrm{red}} = \mathrm{Spec}(B/I)^{\mathrm{perf}}$.

Definition 2.20. (1) A map of v-sheaves $X \to Y$ is said to be formally adic if the following diagram is Cartesian:

$$\begin{tikzcd} (X^{\mathrm{red}})^\diamond \arrow[r] \arrow[d] & X \arrow[d] \\ (Y^{\mathrm{red}})^\diamond \arrow[r] & Y \end{tikzcd}$$

(2) A v-sheaf X over $\mathrm{Spd}\, O$ is π-adic if the structure morphism $X \to \mathrm{Spd}\, O$ is formally adic.

If $\mathrm{Spa}(A, A^+)$ is an affinoid adic space, we let $\mathrm{Spd}(A, A^+)$ denote the associated v-sheaf given by homomorphisms to untilts, see [SW20, Subsection 18.1]. If $A = A^+$, we abbreviate this by $\mathrm{Spd}(A)$.

Definition 2.21. Given a v-sheaf X, a map $f\colon \mathrm{Spa}(R, R^+) \to X$ from an affinoid perfectoid space formalizes if it factors through a map $g\colon \mathrm{Spd}(R^+) \to X$. Any such g is called a formalization of f. The map f v-formalizes if there is a v-cover $h\colon \mathrm{Spa}(S, S^+) \to \mathrm{Spa}(R, R^+)$ such that $f \circ h$ formalizes.

Proposition 2.22. *For a small v-sheaf X, the following are equivalent:*

[1] The reference provided here only shows that the v-sheaves associated to formal schemes are valuative prekimberlites, but the additional axiom that the analytic locus is a spatial diamond is easily verified [Gle24, Remark 4.37].

(1) *There is a set I, a family of formal Huber pairs $(B_i, B_i)_{\{i\in I\}}$ and a v-cover*
$$\coprod_{i\in I} \mathrm{Spd}(B_i) \to X. \tag{2.17}$$
(2) *There is a set J, a family of perfectoid Huber pairs $(R_j, R_j^+)_{\{j\in J\}}$ and a v-cover*
$$\coprod_{j\in J} \mathrm{Spd}(R_j^+) \to X. \tag{2.18}$$
(3) *For any perfectoid Huber pair (R, R^+) all the maps $f\colon \mathrm{Spa}(R, R^+) \to X$ v-formalize.*

Proof. This is [Gle24, Lemma 4.7]. □

Any v-sheaf satisfying the conditions in Proposition 2.22 is said to be *v-locally formal* or alternatively *v-formalizing*.

Definition 2.23. A small v-sheaf X over $\mathrm{Spd}\, O$ is a π-*adic prekimberlite* if it is v-locally formal, the structure map $X \to \mathrm{Spd}\, O$ is separated and formally adic, and if X^{red} is represented on $\mathrm{SchPerf}_k$ by a perfect scheme.

The more general definition of a prekimberlite is given in [Gle24, Definition 4.15], and we justify below why π-adic prekimberlites are a special type of prekimberlite. For this reason, in our context, we can take Definition 2.25 as our definition.

Proposition 2.24. *A small v-sheaf X equipped with a separated morphism $X \to \mathrm{Spd}\, O$ is a π-adic prekimberlite if and only if X is a prekimberlite and the map $X \to \mathrm{Spd}\, O$ is formally adic.*

Proof. Formal adicness implies that $X^{\mathrm{an}} = X_\eta$ and $(X^{\mathrm{red}})^\diamond = X \times_{\mathrm{Spd}\, O} \mathrm{Spd}\, k$. From this it is clear how one definition translates to the other except that to prove X is a prekimberlite we need to justify why it is formally separated. Now, the argument given in [Gle24, Proposition 3.29] applies with the role of $\mathbb{Z}_p$ exchanged for O. □

To any prekimberlite X, in particular to any π-adic prekimberlite, one attaches a topological specialization map $\mathrm{sp}_X : |X| \to |X^{\mathrm{red}}|$ [Gle24, Definition 4.12], and a v-sheaf theoretic specialization map $\mathrm{SP}_X : X \to (X^{\mathrm{red}})^{\diamond/\circ}$ due to Heuer, see [Gle24, Section 4.4]. Here $(X^{\mathrm{red}})^{\diamond/\circ}$ is as in [Gle24, Definition 4.23].

Definition 2.25. A π-adic prekimberlite is a π-adic kimberlite if X_η is a locally spatial diamond, the restriction of sp_X to $|X_\eta| \subseteq |X|$ is a quasi-compact map and SP_X is partially proper.

Remark 2.26. The more general definition of a kimberlite is given in [Gle24, Definition 4.35]. Just as in Proposition 2.24 and with the same argument one can see that a small v-sheaf X equipped with a separated morphism $X \to \mathrm{Spd}\, O$ is a π-adic kimberlite if and only if X is a kimberlite and the map $X \to \mathrm{Spd}\, O$ is formally adic.

If $f\colon S \to T$ is a map of locally spectral spaces, then we call f spectral if for any quasi-compact open $U \subset S, V \subset T$ with $f(U) \subset V$ the induced map $f\colon U \to V$ of spectral spaces is spectral, that is, quasi-compact.

In what follows we consider the restriction of the topological specialization map $|X| \to |X^{\mathrm{red}}|$ to $|X_\eta| \subseteq |X|$. By abuse of notation, we still use sp_X to denote the map $\mathrm{sp}_X : |X_\eta| \to |X^{\mathrm{red}}|$.

Proposition 2.27. *The following statements hold:*

(1) *The rule $X \mapsto (X_\eta, X^{\mathrm{red}}, \mathrm{sp}_X)$ is functorial when X varies along π-adic prekimberlites.*
(2) *If X is a π-adic prekimberlite, and X_η is a locally spatial diamond then the specialization map $\mathrm{sp}_X\colon |X_\eta| \to |X^{\mathrm{red}}|$ is spectral.*
(3) *If X is a π-adic kimberlite the map sp_X is a closed map.*

Proof. Functoriality is [Gle24, Proposition 4.14] specialized to the π-adic case considered here. The same argument as in [Gle24, Theorem 4.40] shows that the map is spectral. The last statement is [Gle24, Theorem 4.40.(2)]. $\square$

One of the main features of kimberlites is that, as with formal schemes, they come with a notion of tubular neighborhoods (or completion at a point).

Definition 2.28 ([Gle24, 4.18])**.** Given a π-adic prekimberlite X and a locally closed subset $S \subset |X^{\mathrm{red}}|$, one defines $\widehat{X}_{/S}$ as the v-sheaf making the following diagram Cartesian:

$$\begin{tikzcd} \widehat{X}_{/S} \arrow[r] \arrow[d] & X \arrow[d] \\ \underline{|S|} \arrow[r] & \underline{|X^{\mathrm{red}}|} \end{tikzcd}$$

Here $\widehat{X}_{/S}$ is called the formal neighborhood of X around S, and $(\widehat{X}_{/S})_\eta$ the tubular neighborhood of X around S.

Here, the right vertical arrow is the composition of the natural map $X \to \underline{|X|}$ and the map $\underline{|X|} \to \underline{|X^{\mathrm{red}}|}$ mentioned in Proposition 2.27. We will mostly use tubular neighborhoods when $S = \{x\}$ is a closed (and constructible) point in X^{red}.

Example 2.29. For any π-adic prekimberlite X and any locally closed subset $S \subset |X^{\mathrm{red}}|$, one has inclusions

$$|S^\diamond| \subset |\widehat{X}_{/S}| \subset \mathrm{sp}_X^{-1}(S), \tag{2.19}$$

which are strict in general. For example, let $X = \mathrm{Spd}(A)$ with A a perfect k-algebra and let $S \subset \mathrm{Spec}(A) = X^{\mathrm{red}}$ the Zariski closed subset defined by a finitely generated ideal $I \subset A$ with generators $a_1, \ldots, a_n$. Then, $S^\diamond$ is the locus in $\mathrm{Spd}(A)$ where $a_1 = \cdots = a_n = 0$, $\widehat{X}_{/S}$ is the (open) locus in $\mathrm{Spd}(A, A)$ where $a_1, \ldots, a_n$ are all topologically nilpotent and $\mathrm{sp}_X^{-1}(S)$ is the closed subset of points for which $|a_i| < 1$. With this description it is immediate to verify the containment of (2.19).

Now, the complement $\mathrm{sp}_X^{-1}(S) \setminus |\widehat{X}_{/S}|$ consists of those higher rank points $(A, A) \to (C, C^+)$, for which at least one of $a_i^{-1} \in C^\circ \setminus C^+$. Note the associated point $(A, A) \to (C, C^\circ)$ is not in $\mathrm{sp}_X^{-1}(S)$. In particular, $\mathrm{sp}_X^{-1}(S)$ is usually not weakly generalizing and does not define a closed subsheaf.

Proposition 2.30. *If $S \subset |X^{\mathrm{red}}|$ is locally closed and constructible, then $\widehat{X}_{/S} \to X$ is an open immersion.*

Proof. This is proved in [Gle24, Proposition 4.22]. $\square$

We now introduce a weak form of flatness over O for π-adic kimberlites.

Definition 2.31. A π-adic kimberlite X over $\mathrm{Spd}\, O$ is said to be *flat* if there is a set I, a family of F-perfectoid Huber pairs $\{(R_i^\sharp, R_i^{\sharp+})\}_{i\in I}$ and a v-cover over $\mathrm{Spd}\, O$

$$\coprod_{i \in I} \mathrm{Spd}(R_i^{\sharp+}) \to X. \tag{2.20}$$

We now construct our first examples of flat π-adic kimberlites.

Proposition 2.32. *Let $f\colon A \to B$ be a map of complete π-adic algebras that are flat over O. Suppose that A is integrally closed in $A[\pi^{-1}]$ and that $\mathrm{Spd}(B[\pi^{-1}], B) \to \mathrm{Spd}(A[\pi^{-1}], A)$ is a v-cover. Then $\mathrm{Spd}(B) \to \mathrm{Spd}(A)$ is also a v-cover. In particular, for any such A the v-sheaf $\mathrm{Spd}(A)$ is a flat π-adic kimberlite.*

Proof. By [Gle24, Lemma 2.26], the map $\mathrm{Spd}(B) \to \mathrm{Spd}(A)$ is quasi-compact, so it is enough to prove $|\mathrm{Spd}(B)| \to |\mathrm{Spd}(A)|$ is surjective by [Sch17, Lemma 12.11]. Surjectivity on the generic fiber follows from the hypothesis. On the special fiber, we use [Gle24, Lemma 3.5, Proposition 3.7] to prove instead that the map $\mathrm{Spa}(B/\pi) \to \mathrm{Spa}(A/\pi)$ is surjective.

Let $x \in \mathrm{Spa}(A/\pi)$ and let $\mathrm{Spa}(k(x), k(x)^+) \to \mathrm{Spa}(A/\pi)$ the affinoid residue field map. Let $\mathfrak{p}_x \in \mathrm{Spec}(A/\pi)$ denote the support ideal of x. Since A is integrally closed in $A[\pi^{-1}]$, the pair $(A[\pi^{-1}], A)$ is a complete Tate Huber pair and we have a surjective specialization map $\mathrm{sp}_A\colon \mathrm{Spa}(A[\pi^{-1}], A) \to \mathrm{Spec}(A/\pi)$ by [Gle24, Proposition 4.2], [Bha17, Remark 7.4.12]. Let $y \in \mathrm{Spa}(A[\pi^{-1}], A)$ with $\mathrm{sp}_A(y) = \mathfrak{p}_x$. We obtain a map $\mathrm{Spa}(R[p^{-1}], R) \to \mathrm{Spa}(A[\pi^{-1}], A)$ with $R := k(y)^+$. The residue field of R is $k(x)$ and we can consider $R^+ \subset R$ defined as $R \times_{k(x)} k(x)^+$. This promotes to a map $\mathrm{Spa}(R^+) \to \mathrm{Spa}(A)$. As $\mathrm{Spd}(B[\pi^{-1}], B) \to \mathrm{Spd}(A[\pi^{-1}], A)$ is a v-cover we can find a v-cover of $\mathrm{Spa}(C, C^+) \to \mathrm{Spa}(R[\pi^{-1}], R^+)$ with (C, C^+) a perfectoid field and a commutative diagram

$$\begin{tikzcd} \mathrm{Spa}(C^+) \arrow[r] \arrow[d] & \mathrm{Spa}(R^+) \arrow[d] \\ \mathrm{Spa}(B) \arrow[r] & \mathrm{Spa}(A). \end{tikzcd}$$

The map $\mathrm{Spa}(C^+) \to \mathrm{Spa}(R^+)$ is easily seen to be surjective since it is an extension of valuation rings. So x lies in the image of $\mathrm{Spa}(B/\pi)$ as we needed to show.

That $\mathrm{Spd}(A)$ is a valuative prekimberlite for A as above follows from [Gle24, Proposition 4.31]. To show it is a kimberlite it suffices to know that $\mathrm{Spd}(A)_\eta$ is a spatial diamond, which follows from [Sch17, Lemma 15.6]. Indeed, in this case the specialization map is automatically quasi-compact [Gle24, Remark 4.37]. Now, we may always find a cover by an affinoid perfectoid space $\mathrm{Spd}(P, P^+) \to \mathrm{Spd}(A[p^{-1}], A)$ by [Sch17, Lemma 15.3]. What we have shown so far implies that $\mathrm{Spd}(P^+) \to \mathrm{Spd}(A)$ is also a v-cover. This finishes the proof. □

Proposition 2.33. *If A is the π-adic completion of a flat and finite type algebra over O, then* $\mathrm{Spd}(A)$ *is a flat π-adic kimberlite.*

Proof. We may assume that A is reduced as passing to the absolute weak normalization does not change $\mathrm{Spd}(A)$ by Theorem 2.16 and Proposition 2.18. As A is noetherian and quasi-excellent, the integral closure of A in its total ring of fractions is therefore a finite A-module. In particular, the integral closure A' of A in $A[p^{-1}]$ is finite over A. Thus, we can conclude that $\mathrm{Spd}(A')$ (with A' given the π-adic topology) is flat by Proposition 2.32 and the map $\mathrm{Spd}(A') \to \mathrm{Spd}(A)$ is a v-cover since it is isomorphism over $\mathrm{Spd}(F)$ (this uses that the π-adic topology on A' agrees with the subspace topology on $A[\pi^{-1}]$) and the map $\mathrm{Spec}(A'/\pi) \to \mathrm{Spec}(A/\pi)$ is proper and surjective (here we use again [Gle24, Lemma 3.5, Proposition 3.7] as in Proposition 2.32). □

Remark 2.34. A careful inspection of the proof of Proposition 2.32 above allows us to conclude that a π-adic formal Huber pair (A, A) will give rise to a flat π-adic kimberlite $\mathrm{Spd}(A)$ if and only if the specialization map

$$\mathrm{sp}_A\colon \{x \in \mathrm{Spa}(A) \mid |\pi|_x \neq 0\} \subset \mathrm{Spd}(A) \to \mathrm{Spec}(A/\pi) \tag{2.21}$$

is surjective. The hypothesis taken in Proposition 2.32 are easy to verify assumptions that ensure this happens. Without assuming flatness of A, this might not hold since for a discrete and perfect O-algebra A in characteristic p, the v-sheaf $\mathrm{Spd}(A)$ is a π-adic kimberlite that is not flat.

We can relate flatness for π-adic kimberlites to surjectivity of the specialization map.

Proposition 2.35. *Let X be a π-adic kimberlite over* $\mathrm{Spd}\, O$.

(1) *If X is flat, then the specialization map* $\mathrm{sp}\colon |X_\eta| \to |X^{\mathrm{red}}|$ *is surjective.*
(2) *Conversely, if $X \to \mathrm{Spd}\, O$ is proper and* $\mathrm{sp}\colon |X_\eta| \to |X^{\mathrm{red}}|$ *surjective, then X is flat over* $\mathrm{Spd}\, O$.

Proof. The first statement reduces to the case $X = \mathrm{Spd}(R_i^{\sharp+})$ for $(R_i^\sharp, R_i^{\sharp+})$ a perfectoid Huber pair over F, where it follows from [Gle24, Proposition 4.2]. Let us prove the second. It follows from the hypothesis that X_η is quasi-compact over $\mathrm{Spd}\, F$, and thus we may find a v-cover $\mathrm{Spa}(R, R^+) \to X_\eta$ by affinoid perfectoid. Refining the cover if necessary we may assume this map factors through a map $\mathrm{Spd}(R^+) \to X$ because X is v-formalizing. Since X is quasi-separated over $\mathrm{Spd}\, O$ and $\mathrm{Spd}(R^+)$ is quasi-compact over $\mathrm{Spd}\, O$ (see [Gle24, Lemma 2.26]), we may conclude that $\mathrm{Spd}(R^+)$ is quasi-compact over X. To prove it is a v-cover, it is therefore enough to prove that the map of topological spaces is surjective. On the generic fiber this is clear. Using [Gle24, Lemma 3.5, Proposition 3.7], we need to show $\mathrm{Spec}((R^+/\pi)^{\mathrm{perf}}) \to X^{\mathrm{red}}$ is a scheme-theoretic v-cover, or equivalently that the map of the associated adic spectra induced by the morphism of schemes is surjective.

The proof now follows a similar argument to the one given in Proposition 2.32. Given a point $x \in |(X^{\mathrm{red}})^{\mathrm{ad}}|$ in the adic spectrum of X with affinoid residue field $\mathrm{Spa}(k(x), k(x)^+)$ we consider the point in $\mathfrak{p}_x \in |X^{\mathrm{red}}|$ corresponding to the support of x. By surjectivity of the specialization map there is a point $y \in |X_\eta|$ with $\mathrm{sp}_X(y) = \mathfrak{p}_x$. Represent y by a map $\mathrm{Spa}(C, C^+) \to X_\eta$ with (C, C^+) a perfectoid affinoid field over F. Replacing $\mathrm{Spd}(C, C^+)$ by a v-cover we may assume this map factors over a map $\mathrm{Spd}(C^+, C^+) \to X$. In particular, it promotes to a map $\mathrm{Spd}(C^+) \to X$. The closed point of $\mathrm{Spd}(C, C^+)$ specializes to a point with the same support as x. Let $\kappa(y)$ be the residue field of C^+. Then $\kappa(y)$ is a field extension of $k(x)$, and we can find a valuation ring $\kappa(y)^+ \subset \kappa(y)$ making $\kappa(y)^+/k(x)^+$ an extension of valuation rings. By pullback along the surjection $C^+ \twoheadrightarrow \kappa(y)$ we may construct from $\kappa(y)^+$ an open and bounded valuation $C_1^+ \subset C^+$. Since X_η is partially proper we may extend $\mathrm{Spd}(C, C^+)$ to a map $\mathrm{Spd}(C^+, C_1^+) \to X_\eta$. After possibly replacing $\mathrm{Spa}(C, C_1^+)$ by a v-cover, we may assume it factors through $\mathrm{Spa}(R, R^+)$. Then the map extends to $\mathrm{Spd}(C_1^+) \to \mathrm{Spd}(R^+) \to X$. The map of adic spectra $\mathrm{Spec}((C_1^+/\pi)^{\mathrm{perf}})^{\mathrm{ad}} = \mathrm{Spa}((C_1^+/\pi)^{\mathrm{perf}}, (C_1^+/\pi)^{\mathrm{perf}}) \to (X^{\mathrm{red}})^{\mathrm{ad}}$ has x in its image as we wanted to show. □

We now discuss some ad hoc hypothesis on π-adic kimberlites that allow us to recover them from their specialization triple.

Definition 2.36. Let $\mathcal{K}$ be the full subcategory of v-sheaves over $\mathrm{Spd}\, O$ consisting of flat π-adic kimberlites X that are quasi-compact and separated over $\mathrm{Spd}\, O$ and satisfy the following properties:
(1) The $\mathrm{Spd}\, C$-valued points of X define a dense subset of $|X_C|$.
(2) The reduction X^{red} is a perfect k-scheme perfectly of finite type.
(3) Every section $\mathrm{Spd}\, C \to X_C$ formalizes to a map $\mathrm{Spd}\, O_C \to X_{O_C}$.

Our main theorem about the category $\mathcal{K}$ is the following.

Theorem 2.37. *When restricted to the category $\mathcal{K}$ of Definition 2.36, the functor sending a π-adic kimberlite to its generic fiber is faithful and the functor that sends it to its specialization triple*
$$X \mapsto (X_\eta, X^{\mathrm{red}}, \mathrm{sp}_{\breve{X}}) \tag{2.22}$$
is fully faithful. Here, $\mathrm{sp}_{\breve{X}}$ *denotes the specialization map associated with the base change* $\breve{X} := X \times_{\mathrm{Spd}(O)} \mathrm{Spd}(\breve{O})$.

Proof. Let us prove faithfulness. Let $f, g\colon X \to Y$ be two maps such that $f_\eta = g_\eta$. Since X is flat and quasi-compact we may replace it by a cover of the form $\mathrm{Spd}(R^+)$. Since Y is separated and π-adic the map $\Delta\colon Y \to Y \times_{\mathrm{Spd}\, O} Y$ is formally adic and a closed immersion. The pullback of Δ by (f, g) is closed and formally adic subsheaf of $\mathrm{Spd}(R^+)$ with the same generic fiber. We may finish by arguing as in the proof of [Gle24, Proposition 4.9].

Let us prove the map is full. Fix a map $f := (f_\eta, f^{\mathrm{red}})$ of triples

$$f\colon (X_\eta, X^{\mathrm{red}}, \mathrm{sp}_{\check{X}}) \to (Y_\eta, Y^{\mathrm{red}}, \mathrm{sp}_{\check{Y}}) \tag{2.23}$$

and let $W = X \times_{\mathrm{Spd}\, O} Y$. Let $g\colon \mathrm{Spa}(R, R^+) \to X_\eta$ be a formalizable v-cover which extends to a surjection $\mathrm{Spd}(R^+) \to X$ and for which $f \circ g$ is also formalizable (this is possible using Proposition 2.32). Let $(g, f \circ g)\colon \mathrm{Spd}(R^+) \to W$ be the induced map and define Z as the sheaf-theoretic image of $(g, f \circ g)$ in W. We have a projection map $Z \to X$ and we wish to prove that it is an isomorphism. Observe that the graph morphism $(\mathrm{id}, f_\eta)\colon X_\eta \to W_\eta$ already identifies X_η with Z_η. In particular, $Z(C)$ is dense inside $|Z_C|$ by our assumption on X.

By construction Z is v-locally formal since $\mathrm{Spd}(R^+, R^+)$ surjects onto it. Moreover, since $Z \subset W$ and W is separated over O, we see that Z is also separated over O. Let us prove that Z if formally π-adic and that Z^{red} is isomorphic to X^{red}.

We claim that $Z_s \subset W_s = (W^{\mathrm{red}})^\diamond$ factors through the graph of $(f^{\mathrm{red}})^\diamond$. Indeed, since X_{O_C} and Y_{O_C} formalize C-sections, for any map $q\colon \mathrm{Spd}\, C \to Z_\eta$ we obtain maps $q_x^{\mathrm{red}}\colon \mathrm{Spec}(\bar{k}) \to X^{\mathrm{red}}$ and $q_y^{\mathrm{red}}\colon \mathrm{Spec}(\bar{k}) \to Y^{\mathrm{red}}$ intertwinned under f^{red}. This holds because $|f_{\bar{k}}^{\mathrm{red}}| \circ \mathrm{sp}_{\check{X}} = \mathrm{sp}_{\check{Y}} \circ |f_{\check{\eta}}|$ by assumption and $\bar{k}$-sections of $Y_{\bar{k}}^{\mathrm{red}}$ are determined by the closed point in $|Y_{\bar{k}}^{\mathrm{red}}|$ that they induce. This shows that $\mathrm{sp}_W(q) \in \Gamma(f^{\mathrm{red}})$. In particular, $\mathrm{sp}_W(|Z_\eta|) \subset \overline{\mathrm{sp}_W(Z(C))} \subset |W^{\mathrm{red}}|$ is contained in $\Gamma(f^{\mathrm{red}})$. By [Gle24, Proposition 4.2], we know that the specialization map $\mathrm{Spd}(R, R^+) \to \mathrm{Spec}((R^+/\pi)^{\mathrm{perf}})$ is surjective. Because $(g, f_\eta \circ g)\colon \mathrm{Spd}(R, R^+) \to W$ has image $|Z_\eta|$ (on topological spaces), this implies by naturality of the specialization map that the morphism $g^{\mathrm{red}}\colon \mathrm{Spec}(R^+/\pi)^{\mathrm{perf}} \to W^{\mathrm{red}}$ factors through $\Gamma(f^{\mathrm{red}})$ as well. Consequently, $Z_s \to W_s$ factors through $\Gamma(f^{\mathrm{red}})^\diamond$. On the other hand, since $\mathrm{Spd}(R^+) \to X$ is surjective the projection map

$$(\mathrm{Spec}(R^+/p)^{\mathrm{perf}})^\diamond \to (X^{\mathrm{red}})^\diamond \tag{2.24}$$

is a surjection. This implies that the morphism $\mathrm{Spec}((R^+/\pi)^{\mathrm{perf}}) \to W_{\mathrm{red}}$ surjects onto $\Gamma(f^{\mathrm{red}})$, and this in turn implies that $Z_s \to \Gamma(f^{\mathrm{red}})^\diamond$ is an isomorphism, as it is a monomorphism and surjective. In particular, we get that $Z_s \cong (Z^{\mathrm{red}})^\diamond$, that is, Z is formally π-adic.

As we have seen the map $Z \to X$ is an isomorphism on the generic fiber and on the special fiber. Since $\mathrm{Spd}(R^+) \to Z$ is surjective Z is quasi-compact over $\mathrm{Spd}\, O$, which is enough to conclude $Z \to X$ is an isomorphism (by [Sch17, Lemma 12.5], note that $Z \to X$ is quasi-compact, as X is qcqs over $\mathrm{Spd}\, O$). □

It is also relevant to relate this to a notion of topological flatness that appears in [PR24].

Lemma 2.38. *Let X be a proper π-adic prekimberlite over $\mathrm{Spd}\, O$ satisfying conditions (1)-(3) of Definition 2.36 and such that X_η is a spatial diamond. Then the following hold.*

(1) *X is a π-adic kimberlite.*
(2) *If $|X_\eta|$ is a dense open[2] subset of $|X|$, then X is flat, thus lies in $\mathcal{K}$.*

Proof. By [Gle24, Proposition 4.32] the map $\mathrm{SP}_X : X \to (X^{\mathrm{red}})^{\diamond/\circ}$ is partially proper. By Proposition 2.27.(2) the specialization map $\mathrm{sp} : |X| \to |X^{\mathrm{red}}|$ is a spectral map, but since both topological spaces are qcqs it is a quasi-compact map. This shows that X is a π-adic kimberlite.

[2]The converse, however, fails. Indeed, let $O\langle t\rangle \subset V$ be a higher rank valuation ring endowed with its π-adic topology. Then $\mathrm{Spd}(V)$ is a flat π-adic kimberlite. As in [Gle24, Definition 2.1, Proposition 2.19], one can prove that the locus $N_{t \ll 1}$ where t is topologically nilpotent is an open subset of $|\mathrm{Spd}(V)|$ that does not meet the generic fiber.

Now, by Proposition 2.27.(3) the specialization map sends closed subsets to closed subsets. Since X^{red} is perfectly of finite type, it suffices to prove surjectivity of the map $X(\mathrm{Spd}\, C) \to X^{\mathrm{red}}(\bar{k})$ induced by sp.

For this, take the associated formal neighborhood $\widehat{X}_{/x}$ over a closed point x of the reduction, which can be represented by $\mathrm{Spec}\bar{k} \to X^{\mathrm{red}}$ uniquely up to Galois automorphisms. It is a non-empty open by [Gle24, Proposition 4.22]. Hence, it must have topologically dense generic fiber, which is in particular non-empty. By hypothesis, we can find a C-valued point mapping to x. □

The following statement gives a v-sheaf theoretic criterion to determine when a weakly normal scheme is already normal.

Proposition 2.39. *Let A be a flat, weakly normal and topologically of finite type π-adically complete domain over O. Suppose that $A[\pi^{-1}]$ is normal and that, for every closed point $x \in \mathrm{Spec}(A/\pi)$, the diamond $(\widehat{\mathrm{Spd}(A)}_{/x})_\eta$ (equivalently, the rigid analytic fiber of the formal affine scheme $\widehat{\mathrm{Spec}(A)}_{/x}$) is connected. Then A is normal.*

Proof. First off, one has $\widehat{\mathrm{Spd}(A)}_{/x} = (\widehat{\mathrm{Spec}(A)}_{/x})^\diamond$ by [Gle24, Proposition 4.19]. Since taking generic fibers commutes with $\diamond$, we see that the connectedness of $(\widehat{\mathrm{Spd}(A)}_{/x})_\eta$ is equivalently to the connectedness of the rigid analytic fiber of $\widehat{\mathrm{Spec}(A)}_{/x}$.

Now, let B denote the integral closure of A in $A[\pi^{-1}]$. Since $A[\pi^{-1}]$ is normal, B is also normal and B is a finite A-algebra. We claim that $f\colon \mathrm{Spd}(B) \to \mathrm{Spd}(A)$ is an isomorphism, so that $A = B$ by Theorem 2.16. By quasi-compactness, it is enough to check this on the generic and special fibers. The generic case follows from the definition of B. We need to prove $\mathrm{Spec}(B/\pi)^{\mathrm{perf}} \cong \mathrm{Spec}(A/\pi)^{\mathrm{perf}}$ which amounts to proving that the fibers at closed points consists of singletons. Let $x \in \mathrm{Spec}(A/\pi)$ denote a closed point. By [Gle24, Proposition 4.20], we have an identification

$$(\widehat{\mathrm{Spd}(B)}_{/f^{-1}(x)})_\eta \cong (\widehat{\mathrm{Spd}(A)}_{/x})_\eta \tag{2.25}$$

In turn we also have $\coprod_{y\in f^{-1}(x)}(\widehat{\mathrm{Spd}(B)}_{/y})_\eta \cong (\widehat{\mathrm{Spd}(B)}_{/f^{-1}(x)})_\eta$. By Proposition 2.33 and Proposition 2.35 for all $y \in f^{-1}(x)$ the tubular neighborhood $(\widehat{\mathrm{Spd}(B)}_{/y})_\eta$ is a non-empty open subset of $(\widehat{\mathrm{Spd}(A)}_{/x})_\eta$. Since we assumed this to be connected we can conclude $f^{-1}(x)$ contains a unique element. □

3. The affine flag variety

In this section, we discuss some relevant material on perfect schemes and Witt vector affine flag varieties. Namely, we review the calculation of the Picard group by He–Zhou [HZ20], the definition of canonical finite type deperfections of Schubert perfect schemes and apply a Stein factorization argument to construct a comparison isomorphism between the p-adic canonical deperfections of depth 0 Schubert perfect schemes with the corresponding weakly normal Schubert schemes in the equicharacteristic situation. In particular, we prove [Zhu17b, Conjecture III] on their singularities in this case.

3.1. **Perfect schemes.** Here, we present some facts on perfect schemes that we will need later. Let p be a prime number. All our schemes in this subsection will be assumed to lie over $\mathbb{F}_p$.

The basic theory of perfect schemes is discussed in [Zhu17a, A.] and [BS17, Section 3]. In particular, we will use the notions of a perfectly finitely presented map between qcqs perfect schemes [BS17, Proposition 3.11] of a perfectly proper morphism [BS17, Defintion 3.14], [Zhu17a,

Appendix A.18]. If k is a perfect field, we occasionally call a separated, perfectly finitely presented scheme X over k a perfect k-variety [Zhu17a, Remark A.14].

A morphism $Y \to X$ of perfect schemes is called perfectly smooth if, étale locally on Y, there exists étale morphisms to the perfection of some relative affine space over X, see [Zhu17a, Definition A.25].

Given any normal finite type k-scheme Y, its perfection Y_{perf} is normal as it is a filtered colimit of normal schemes along affine transition maps. Conversely, if X is a qcqs normal perfect scheme perfectly of finite type, then using [Sta23, Tag 01ZA] (and finiteness for integral closures of schemes of finite type over a field), then we can write X as the filtered colimit of perfections of normal schemes $Y_i, i \in I$, which are of finite type over k.

The following result gives a topological criterion for normality of perfect schemes. We stress that perfectness is crucial as one sees, for example, by looking at the normalization morphisms of the cuspidal curve.

Lemma 3.1. *Let $f \colon Y \to X$ be a surjective, perfectly proper morphism between qcqs integral perfect schemes. Assume that Y is normal and f birational. Then X is normal if and only if the geometric fibers of f are connected.*

Proof. If all geometric fibers of f are connected, then the natural map $\mathcal{O}_X \to f_*\mathcal{O}_Y$ is an isomorphism, see [BS17, Proposition 6.1], [Zhu17a, Lemma A.21]. Thus,

$$\mathcal{O}_X(U) \cong \mathcal{O}_Y(f^{-1}(U)),$$

for any open affine $U \subset X$. As $\mathcal{O}_Y(V)$ is a normal ring for any open subset $V \subset Y$, the claim follows (here we use that Y is integral [Sta23, Tag 0358]).

Conversely, we can write f as the perfection of a proper, finitely presented morphism $f_0 \colon Y_0 \to X$ by [BS17, Proposition 3.13, Corollary 3.15]. Let $g_0 \colon Y_0 \to Z_0 = \underline{\mathrm{Spec}}((f_0)_*(\mathcal{O}_{Y_0}))$ be the Stein factorization of f_0, see [Sta23, Tag 03H2]. Perfecting again, we get a factorization $f = h \circ g$ with $g \colon Y \to Z := (Z_0)_{\mathrm{perf}}$ having connected geometric fibers, and $h \colon Z \to X$ an integral, dominant morphism of integral schemes inducing an isomorphism at generic points (because f is birational). As X is normal we obtain that $X \cong Z$, which implies the claim. □

We now turn to Picard groups of perfect schemes. Given any qcqs perfect k-scheme X, we have $\mathrm{Pic}(X) \cong \mathrm{Pic}(X_0)[1/p]$ for any preferred choice of finite type deperfection X_0, cf. [BS17, Lemma 3.5]. In particular, the Picard groups of perfect schemes are always uniquely p-divisible.

If X is perfectly finitely presented over some perfect field k and $X_0 \to \mathrm{Spec}(k)$ a finitely presented model for X, then the localized Weil divisor class group $\mathrm{Cl}(X_0)[1/p]$ only depends on X and not on X_0, and we set

$$\mathrm{Cl}(X) := \mathrm{Cl}(X_0)[1/p]. \tag{3.1}$$

If X is normal, then by [Sta23, 0BE8] (and passage to the limit over Frobenius for some normal model) there exists a natural, injective map

$$\mathrm{Pic}(X) \hookrightarrow \mathrm{Cl}(X). \tag{3.2}$$

Let us recall that a line bundle on a (qcqs) scheme is semi-ample if some positive power of it is globally generated.

Proposition 3.2. *Let X be a perfectly proper perfect k-scheme and $\mathcal{L}$ be a semi-ample line bundle on X. There is a unique perfectly proper surjection $X \to Y$ of perfect k-schemes with connected geometric fibers such that all sufficiently divisible powers of $\mathcal{L}$ descend uniquely to ample line bundles on Y.*

Proof. By semi-ampleness of $\mathcal{L}$, we can take X_0 to be a finite type deperfection of X over k, and let $\mathcal{L}_0$ be a base point free line bundle on X_0 whose pullback to X is a power of $\mathcal{L}$. Let Y_0 be the Stein factorization of the canonical morphism

$$X_0 \to Z_0 \subset \mathbb{P}(\Gamma(X_0, \mathcal{L}_0)), \tag{3.3}$$

where Z_0 is the (scheme-theoretic) image of X_0. Clearly, $\mathcal{L}_0$ descends by construction to an ample line bundle on Y_0, pulling back $\mathcal{O}(1)$ on the right side of (3.3). After taking perfections, we get $X \to Y$ with the desired properties (see [BS17, Proposition 6.1] for unique descent of line bundles).

In order to prove uniqueness of Y, we proceed as in [BS17, Proof of Theorem 8.3]. The morphism $X \to Y$ is a v-cover (by properness), hence Y is determined by the closed subscheme $X \times_Y X \subset X \times_{\mathrm{Spec}(k)} X$. To identify this closed (and necessarily reduced) subscheme it suffices to identify the geometric fibers of the map $X \to Y$ in terms of $\mathcal{L}$, and we only have to argue on k-valued points as these are dense inside $X \times_Y X$. We claim that two k-rational points of X lie in the same fiber over Y if and only if both points can be linked by a chain of closed integral perfect k-curves C, such that the restriction $\mathcal{L}|_C$ is torsion in $\mathrm{Pic}(C)$.

By connectedness of the fibers and the definition of $X \to Y$, every two points in it can be linked by such a chain of integral perfect k-curves C on which $\mathcal{L}$ is torsion. Conversely, given an integral perfect k-curve $C \subset X$ whose image in Y is not a point, all sufficiently large powers of $\mathcal{L}$ restrict to an ample line bundle on C. Indeed, after passing to a finitely presented deperfection of C over k the morphism $C_0 \to Y$ is finite and pullback of ample line bundles along affine morphisms are ample. □

Next, we discuss finite type deperfections. Let k be a perfect field and let X be a qcqs perfect k-scheme of perfectly finite presentation. For each of the finitely many generic points $\eta \in X$, fix a subfield $k(\eta_0) \subset k(\eta)$, which is finitely generated over k and has perfection $k(\eta)$. Then there exists a unique (up to unique isomorphism) weakly normal finite type k-scheme X_0 such that $(X_0)_{\mathrm{perf}} \cong X$ and for each generic point $\eta_0 \in |X_0| \cong |X|$ the function field of X_0 at η_0 identifies with $k(\eta_0)$, see [Zhu17a, Proposition A.15]. Note that X_0 also reflects normality of X, that is, X normal implies X_0 normal.

For group actions we can draw the following consequence.

Proposition 3.3. *Let G be an affine perfect k-group of perfectly finite presentation and X a qcqs perfect k-scheme of perfectly finite presentation equipped with a G-action with finitely many orbits.*

1. *Any reduced deperfection G_0 of G is a smooth affine k-group.*
2. *For such G_0, there are unique weakly normal deperfections X_0 with G_0-action, whose fixers on the open orbits are also smooth.*

Proof. The first item is [Zhu17a, Lemma A.26]. For the second item, we notice that X has a dense open subset U consisting of the disjoint union of its maximal orbits, cf. [Zhu17a, Proposition A.32]. Having constructed the unique deperfection U_0 with the desired properties, it has a unique extension to a deperfection X_0 of X by [Zhu17a, Lemma A.15]. Furthermore, the action map $G \times X \to X$ also deperfects, because it does so over a dense open (and X_0 is weakly normal).

Therefore, we may and do assume that $X = G/H$ is a single orbit around a certain k-valued point x. But then taking $H_0 \subset G_0$ to be the unique reduced closed subscheme whose perfection recovers $H \subset G$, we get a G_0-orbit $X_0 = G_0/H_0$ deperfecting X with smooth fixers. Uniqueness is clear. □

Proposition 3.3 will be useful for constructing finite type deperfections for Schubert varieties in Witt vector affine Grassmannians, see Section 3.3.

3.2. **Affine flag varieties.** We now study the geometry of Witt vector affine flag varieties. Assume that k is a perfect field of characteristic $p > 0$ and that F is a complete discretely valued field with residue field k and ring of integers O. Exceptionally, we allow $F \cong k(\!(\pi)\!)$ to be a Laurent series field, since it is needed in Section 3.3.

We denote by $\mathrm{Alg}_k^{\mathrm{perf}}$ the category of perfect k-algebras. For $R \in \mathrm{Alg}_k^{\mathrm{perf}}$, we denote by $W_O(R)$ the associated ring of O-Witt vectors, see [FF, Section 1.2.1]: if O is p-adic, then $W_O(R) = W(R) \otimes_{W(k)} O$; if $O \cong k[\![\pi]\!]$, then $W_O(R) = R\widehat{\otimes}_k O \cong R[\![\pi]\!]$.

Moreover, we fix a (connected) reductive F-group G and a parahoric O-model $\mathcal{G}$ in the sense of Bruhat–Tits. We note that, over the completion $\breve{F}$ of the maximal unramified extension of F, the group $G_{\breve{F}}$ is automatically quasi-split by Steinberg's theorem, see [Ser94, Chapitre III.2.3]. We let $\mathcal{G}_k = \mathcal{G} \otimes_O k$ be the special fiber of $\mathcal{G}$.

Recall the definition of the Witt vector affine flag variety associated to $\mathcal{G}$.

Definition 3.4. (1) The loop group of G is the functor

$$L_k G\colon \mathrm{Alg}_k^{\mathrm{perf}} \to (\mathrm{Sets}),\ R \mapsto G(W_O(R) \otimes_O F). \tag{3.4}$$

(2) The positive loop group of $\mathcal{G}$ is the functor

$$L_k^+ \mathcal{G}\colon \mathrm{Alg}_k^{\mathrm{perf}} \to (\mathrm{Sets}),\ R \mapsto \mathcal{G}(W_O(R)) \tag{3.5}$$

(3) The affine flag variety for $\mathcal{G}$ is the quotient (for the étale topology)

$$\mathcal{F}\ell_{\mathcal{G}} := L_k G / L_k^+ \mathcal{G}. \tag{3.6}$$

Because any $\mathcal{G}$-torsor on $W_O(R)$ can be trivialized over $W_O(R')$ for some with $R \to R'$ étale, the affine flag variety $\mathcal{F}\ell_{\mathcal{G}}$ is equivalently the functor on perfect k-algebras R that classifies $\mathcal{G}$-torsors $\mathcal{P}$ on $\mathrm{Spec}(W_O(R))$ together with a trivialization over $\mathrm{Spec}(W_O(R) \otimes_O F)$.

We have the following crucial representability result, see [BS17, Corollary 9.6].

Theorem 3.5 (Bhatt-Scholze)**.** *The functor $\mathcal{F}\ell_{\mathcal{G}}$ is representable by an ind-(perfectly projective) ind-(perfect k-scheme).*

Representability as an ind-(perfect algebraic space) was previously proved by Zhu, [Zhu17a], but is not sufficient for our purpose.

Fix an auxiliary maximal split F-torus A, a maximal $\breve{F}$-split F-torus $A \subset S \subset G$ whose connected Néron O-model $\mathcal{S}$ is contained in $\mathcal{G}$, see [BT84, Proposition 5.1.10]. Let $T \subset G$ be the centralizer of S, and let $\mathcal{T}$ be the connected Néron O-model of T. This yields the Iwahori–Weyl group

$$\tilde{W} := N_G(T)(\breve{F})/\mathcal{T}(\breve{O}) \tag{3.7}$$

associated with S, see [HR08, Definition 7]. By [HR08, Lemma 14], there exists a short exact sequence

$$1 \to W_{\mathrm{af}} \to \tilde{W} \to \pi_1(G)_I \to 1 \tag{3.8}$$

with $W_{\mathrm{af}} \subset \tilde{W}$ the affine Weyl group, I the absolute Galois group of $\breve{F}$, and $\pi_1(G)$ Borovoi's algebraic fundamental group of G. The choice of an alcove in the apartment for S yields a splitting $W_{\mathrm{af}} \rtimes \pi_1(G)_I$ of the sequence. By declaring the elements of $\pi_1(G)_I$ to have length 0 and to be pairwise incomparable, we can further extend the length function and the Bruhat partial order on the Coxeter group W_{af} to $\tilde{W}$.

By the Cartan decomposition, we may identify the double coclasses

$$\mathcal{G}(\breve{O})\backslash G(\breve{F})/\mathcal{G}(\breve{O}) \cong W_{\mathcal{G}}\backslash \tilde{W}/W_{\mathcal{G}} \tag{3.9}$$

where $W_{\mathcal{G}} := (N_G(T)(\breve{F}) \cap \mathcal{G}(\breve{O}))/\mathcal{T}(\breve{O})$ is the Weyl O-group of $\mathcal{G}$ relative to its maximal O-torus $\mathcal{S}$, see also [HR08, Proposition 8]. This double coset carries a natural action of the Galois group $\mathrm{Gal}(\breve{F}/F)$.

Definition 3.6. Given a finite subset $W \subset W_{\mathcal{G}}\backslash\tilde{W}/W_{\mathcal{G}}$ with reflex field[3] k_W, one defines the associated Schubert perfect k_W-scheme $\mathcal{F}\ell_{\mathcal{G},W} \subset \mathcal{F}\ell_{\mathcal{G}}$ as the closure of the Schubert perfect orbit $\mathcal{F}\ell^{\circ}_{\mathcal{G},W}$, the étale descent to k_W of the union of the $L^+_{\bar{k}}\mathcal{G}$-orbits of the maximal elements $w \in W$.

If $W = \{w\}$, then these are perfect k_w-varieties denoted by $\mathcal{F}\ell_{\mathcal{G},w}$, respectively $\mathcal{F}\ell^{\circ}_{\mathcal{G},w}$, which are usually called the Schubert perfect variety, respectively Schubert perfect orbit associated with w. More generally, if we fix an Iwahori $\mathcal{I}$ dilating $\mathcal{G}$ and containing $\mathcal{S}$, then its $L^+_{\bar{k}}\mathcal{I}$-orbits are enumerated by $\tilde{W}/W_{\mathcal{G}}$. Given some finite subset $W \subset \tilde{W}/W_{\mathcal{G}}$, we can define in the same manner

$$\mathcal{F}\ell^{\circ}_{(\mathcal{I},\mathcal{G}),W} \subset \mathcal{F}\ell_{(\mathcal{I},\mathcal{G}),W} \tag{3.10}$$

the finite disjoint union of the $L^+_{\bar{k}}\mathcal{I}$-orbits corresponding to W and their closure inside $\mathcal{F}\ell_{\mathcal{G}}$. The latter is called an Iwahori–Schubert perfect scheme. We observe that Schubert perfect schemes are always Iwahori–Schubert (but the converse is false). Indeed, given $w \in W_{\mathcal{G}}\backslash\tilde{W}/W_{\mathcal{G}}$ with lift $\dot{w} \in \tilde{W}/W_{\mathcal{G}}$ of maximal length, we have

$$\mathcal{F}\ell_{(\mathcal{I},\mathcal{G}),\dot{w}} = \mathcal{F}\ell_{\mathcal{G},w}. \tag{3.11}$$

Here we recall that the length function and Bruhat partial order on $\tilde{W}$ induces one on the cosets $W_{\mathcal{G}}\backslash\tilde{W}/W_{\mathcal{G}}$, respectively $\tilde{W}/W_{\mathcal{G}}$ compatibly with the dimensions and closure relations of Schubert varieties, respectively Iwahori–Schubert varieties, see [Ric13, Section 1, Proposition 2.8] for details and proofs in the equicharacteristic situation (the arguments translate literally).

Proposition 3.7. *For each $w \in W_{\mathcal{G}}\backslash\tilde{W}/W_{\mathcal{G}}$, the Schubert perfect variety $\mathcal{F}\ell_{\mathcal{G},w}$ is normal and $\mathcal{F}\ell^{\circ}_{\mathcal{G},w}$ is a perfectly smooth dense open with connected fixers.*

Proof. Let $\mathscr{B}(G,F)$ be the Bruhat-Tits building of G, and let $\mathbf{f} \subset \mathscr{B}(G,F)$ be the facet associated to $\mathcal{G}$, see [BT84]. Given $w \in \tilde{W}/W_{\mathcal{G}}$, the stabilizer of $wL^+_{\bar{k}}\mathcal{G} \in \mathcal{F}\ell_{\mathcal{G}}$ is $L^+_{\bar{k}}\mathcal{G} \cap wL^+_{\bar{k}}\mathcal{G}w^{-1}$, which is the positive loop group associated to the parahoric group scheme, which is the connected fixer of $\mathbf{f} \cup w(\mathbf{f})$. In particular, this stabilizer is pro-(perfectly smooth and connected). We deduce that $\mathcal{F}\ell^{\circ}_{\mathcal{G},w}$ is perfectly smooth.

Fix an auxiliary Iwahori $\mathcal{I}$ dilating $\mathcal{G}$ and containing $\mathcal{S}$. This yields the subgroup functor $L^+_{\bar{k}}\mathcal{I} \subset L^+_{\bar{k}}\mathcal{G}$ and, as explained before, we know that $\mathcal{F}\ell_{\mathcal{G},w} = \mathcal{F}\ell_{(\mathcal{I},\mathcal{G}),w^{\mathbf{f}}}$ where $w^{\mathbf{f}}$ is the maximal lift of w to $\tilde{W}/W_{\mathcal{G}}$. Let ${}_{\mathbf{f}}w^{\mathbf{f}}$ be the minimal lift to $\tilde{W}$, write it as $w_{\mathrm{af}}\tau$ with $w_{\mathrm{af}} \in W_{\mathrm{af}}, \tau \in \pi_1(G)_I$, and fix some reduced word $\dot{w}$ in simple reflections (along the alcove defined by $\mathcal{I}$) for w_{af}.

Now consider the Demazure variety

$$\mathcal{D}_{\mathcal{I},\bar{k},\dot{w}} := L^+_{\bar{k}}\mathcal{P}_{i_1} \times^{L^+_{\bar{k}}\mathcal{I}} \cdots \times^{L^+_{\bar{k}}\mathcal{I}} L^+_{\bar{k}}\mathcal{P}_{i_n}/L^+_{\bar{k}}\mathcal{I}, \tag{3.12}$$

where $L^+_{\bar{k}}\mathcal{I} \subset L^+_{\bar{k}}\mathcal{P}_{i_j}$ are the minimal parahoric overgroups attached to the simple reflections. It follows by induction that the geometric fibers of the birational resolution (induced by multiplication)

$$\pi_{\dot{w}} \colon \mathcal{D}_{\mathcal{I},\bar{k},\dot{w}} \to \mathcal{F}\ell_{\mathcal{G},w} \tag{3.13}$$

are connected. As $\mathcal{D}_{\mathcal{I},\bar{k},\dot{w}}$ is perfectly smooth over $\bar{k}$, normality becomes a consequence of Lemma 3.1. $\square$

The Picard group of Schubert perfect schemes over $\bar{k}$ can be explicitly determined, see [HZ20, Theorem 3.1] for the case when $\mathcal{G} = \mathcal{I}$ is Iwahori and $W = \{w\}$.

---

[3] Concretely, the residue field defined by the $\mathrm{Gal}(\breve{F}/F)$-stabilizer of W.

Theorem 3.8 (He–Zhou)**.** *The homomorphism*

$$\mathrm{Pic}(\mathcal{F}\ell_{(\mathcal{I},\mathcal{G}),\bar{k},W}) \to \mathrm{Pic}(\mathcal{F}\ell_{(\mathcal{I},\mathcal{G}),\bar{k},\mathbb{S}_W}) \cong \mathbb{Z}[p^{-1}]^{|\mathbb{S}_W|} \tag{3.14}$$

is a bijection where $\mathbb{S}_W$ *is the set of all length* 1 *elements in* $W \subset \tilde{W}/W_{\mathcal{G}}$*. (Note that* $\mathcal{F}\ell_{(\mathcal{I},\mathcal{G}),\bar{k}\mathbb{S}_W} \cong \mathbb{P}^{1,\mathrm{perf}}_{\bar{k}}$ *if* $\mathbb{S}_W$ *is a singleton.)*

Proof. To reduce the question to Iwahori–Schubert perfect varieties, we contemplate the Mayer–Vietoris sequence

$$1 \to \mathcal{O}^\times_{\mathcal{F}\ell_{(\mathcal{I},\mathcal{G}),\bar{k},W_0}} \to \mathcal{O}^\times_{\mathcal{F}\ell_{(\mathcal{I},\mathcal{G}),\bar{k},W_1}} \oplus \mathcal{O}^\times_{\mathcal{F}\ell_{(\mathcal{I},\mathcal{G}),\bar{k},W_2}} \to \mathcal{O}^\times_{\mathcal{F}\ell_{(\mathcal{I},\mathcal{G}),\bar{k},W_3}} \to 1 \tag{3.15}$$

where the subsets W_i are closed for the Bruhat order $W_0 = W_1 \cup W_2$ and $W_3 = W_1 \cap W_2$. Since we may and do assume all these Schubert perfect schemes to be contained in a single connected component of $\mathcal{F}\ell_{\mathcal{G},\bar{k}}$ (which implies $H^0(\mathcal{O}^\times) \cong \bar{k}^\times$ by perfectly properness), we get a natural isomorphism

$$\mathrm{Pic}(\mathcal{F}\ell_{(\mathcal{I},\mathcal{G}),\bar{k},W_0}) \cong \mathrm{Pic}(\mathcal{F}\ell_{(\mathcal{I},\mathcal{G}),\bar{k},W_1}) \times_{\mathrm{Pic}(\mathcal{F}\ell_{(\mathcal{I},\mathcal{G}),\bar{k},W_3})} \mathrm{Pic}(\mathcal{F}\ell_{(\mathcal{I},\mathcal{G}),\bar{k},W_2}). \tag{3.16}$$

By definition $S_0 = S_1 \cup_{S_3} S_2$, which implies that it suffices to show the claim for $X = \mathcal{F}\ell_{(\mathcal{I},\mathcal{G}),\bar{k},w}$ an Iwahori–Schubert perfect variety.

Injectivity can be reduced to Demazure varieties, see [BS17, Theorem 6.1]. The Demazure varieties $\mathcal{D}_{\mathcal{I},\bar{k},\dot{w}}$ are $\mathbb{P}^{1,\mathrm{perf}}_{\bar{k}}$-fibrations and can be handled directly, see [HZ20, Proposition 3.4]. To treat surjectivity, it suffices to descend certain line bundles on $\mathcal{D}_{\mathcal{I},\bar{k},\dot{w}}$ back to the Iwahori–Schubert varieties. By [BS17, Theorem 6.13], it remains to check that restriction of $\mathcal{L}$ to geometric fibers is trivial. For this, see [HZ20, Proposition 3.9]. □

The choice of a $\mathbb{Z}[p^{-1}]$-basis in $\mathrm{Pic}(\mathcal{F}\ell_{\mathcal{G},\bar{k},W})$ seems arbitrary, due to p-divisibility. However, using the deperfection $\mathcal{I} \otimes_O \bar{k}$ for the quotient $R \in \mathrm{Alg}^{\mathrm{perf}}_{\bar{k}} \mapsto \mathcal{I}(R)$ of $L^+_{\bar{k}}\mathcal{I}$, the perfect curve $\mathcal{F}\ell_{(\mathcal{I},\mathcal{G}),\bar{k},\mathbb{S}_W}$ has a canonical equivariant deperfection, see Proposition 3.3, yielding a natural $\mathbb{Z}$-lattice.

Remark 3.9. During the proof, we have also determined the Picard group of the Demazure varieties $\mathcal{D}_{\mathcal{I},\bar{k},\dot{w}}$, or more generally those of the convolutions

$$\mathcal{F}\ell_{\mathcal{I},\bar{k},W_1} \tilde{\times} \dots \tilde{\times} \mathcal{F}\ell_{\mathcal{I},\bar{k},W_{n-1}} \tilde{\times} \mathcal{F}\ell_{(\mathcal{I},\mathcal{G}),\bar{k},W_n} \tag{3.17}$$

of Iwahori–Schubert perfect schemes, where at most the last one is not at full level.

Together with Proposition 3.2, this tells us how to recover, for instance, the perfect Schubert variety $\mathcal{F}\ell_{\mathcal{G},\bar{k},w}$ just from its Demazure resolution and the sub-$\mathbb{Z}[p^{-1}]$-module $\mathrm{Pic}(\mathcal{F}\ell_{\mathcal{G},\bar{k},w}) \subset \mathrm{Pic}(\mathcal{D}_{\mathcal{I},\bar{k},\dot{w}})$: take any $\mathcal{L}$ on $\mathcal{D}_{\mathcal{I},\bar{k},\dot{w}}$ which is the pullback of a line bundle on $\mathcal{F}\ell_{\mathcal{G},\bar{k},w}$ whose restriction to $\mathcal{F}\ell_{\mathcal{G},\bar{k},s}$ has positive degree for each $s \in \mathbb{S}_W$.

We now turn to equivariant automorphisms of (connected) Schubert schemes.

Proposition 3.10. *The group of* $L^+_{\bar{k}}\mathcal{G}$*-equivariant automorphisms of a connected Schubert perfect scheme* $\mathcal{F}\ell_{\mathcal{G},\bar{k},W}$ *is trivial. In particular, the stabilizers of* $\bar{k}$*-valued points in* $\mathcal{F}\ell_{\mathcal{G},\bar{k},W}$ *for the* $L^+_{\bar{k}}\mathcal{G}$*-action are self-normalizing subgroups of* $L^+_{\bar{k}}\mathcal{G}$*, that is, they agree with their normalizers.*

Proof. We prove the more general statement for Iwahori–Schubert perfect schemes. Consider the disjoint irreducible components in the dense open $\mathcal{F}\ell^\circ_{(\mathcal{I},\mathcal{G}),\bar{k},W} \subset \mathcal{F}\ell_{(\mathcal{I},\mathcal{G}),\bar{k},W}$. These will be permuted under any equivariant automorphism σ. Moreover, σ preserves the $\bar{k}$-valued points of $\mathcal{F}\ell_{(\mathcal{I},\mathcal{G}),\bar{k},W}$ fixed under $\mathcal{S}(\breve{O})$. For the entire flag variety, we claim that the $\mathcal{S}(\breve{O})$-fixed points in $G(\breve{F})/\mathcal{G}(\breve{O})$ lie in the image of $N(\breve{F})$. Indeed, let $[g] \in G(\breve{F})/\mathcal{G}(\breve{O})$ be a fixed point. Then $g\mathbf{f}$ is a $\mathcal{S}(\breve{O})$-stable facet (with $\mathbf{f}$ the facet determined by $\mathcal{G}$), hence contained in $\mathscr{A}(G, S, \breve{F})$ by [BT84, Proposition 5.1.37]. Multiplying on the left by a suitable element of $N(\breve{F})$, we can trivialize $[g]$, that is, $[g] \in N(\breve{F})\mathcal{G}(\breve{O})/\mathcal{G}(\breve{O})$.

Now, observe that the $L^+_{\bar{k}}\mathcal{I}$-fixer of some $w \in \tilde{W}/W_{\mathcal{G}}$ equals $L^+_{\bar{k}}\mathcal{I} \cap wL^+_{\bar{k}}\mathcal{G}w^{-1}$, and it suffices to recover w from this subgroup alone. Indeed, then σ must preserve w, and then $\mathcal{F}\ell_{(\mathcal{I},\mathcal{G}),\bar{k},W}$ pointwise by $L^+_{\bar{k}}\mathcal{I}$-equivariance as $w \in \tilde{W}/W_{\mathcal{G}} \cap \mathcal{F}\ell_{(\mathcal{I},\mathcal{G}),\bar{k},W}$ was arbitrary. If $\mathbf{a}$ is the alcove fixed by $\mathcal{I}(\breve{O})$ and $\tilde{w} \in \tilde{W}$ the minimal lift of w, then birationality of the Demazure resolution $\pi_{\dot{w}}$ implies

$$L^+_{\bar{k}}\mathcal{I} \cap wL^+_{\bar{k}}\mathcal{G}w^{-1} = L^+_{\bar{k}}\mathcal{I} \cap \tilde{w}L^+_{\bar{k}}\mathcal{I}\tilde{w}^{-1}. \tag{3.18}$$

Note that the right side is the Bruhat–Tits group attached to $\mathbf{a} \cup \tilde{w}(\mathbf{a})$. We need to recover $\tilde{w}$. Moreover, because $\mathcal{F}\ell_{(\mathcal{I},\mathcal{G}),\bar{k},W}$ was assumed to be connected, all $\tilde{w}$ considered here project to the same constant $\tau \in \pi_1(G)$, so it is enough to get $w_{\mathrm{af}} \in W_{\mathrm{af}}$ if $\tilde{w} = w_{\mathrm{af}}\tau$.

By [BT84, Corollaire 5.1.39], the fixed point set of $L^+_{\bar{k}}\mathcal{I} \cap \tilde{w}L^+_{\bar{k}}\mathcal{I}\tilde{w}^{-1}$ inside $\mathscr{B}(G, F)$ equals the closed convex hull of $\mathbf{a} \cup \tilde{w}(\mathbf{a})$. In turn, every alcove inside this closed convex hull lies in some minimal gallery connecting $\mathbf{a}$ to $\tilde{w}(\mathbf{a})$ by [BT72, Lemme 2.4.4]. Since a minimal gallery describes a unique word of simple reflections necessary to move from one alcove to another, this gives back the affine transformation w_{af}. □

Among Schubert schemes, we are especially interested in the μ-admissible locus. Recall that C is a completed algebraic closure of F and that I denotes the inertia group of F. Moreover, let $B \subset G_{\breve{F}}$ be a Borel containing $T_{\breve{F}}$. Recall that the inverse of the Kottwitz morphism [Kot97, Equation (7.2.1)] induces an isomorphism of the coinvariants

$$X_*(T)_I \cong T(\breve{F})/\mathcal{T}(\breve{O}),\ \nu_I \mapsto \nu_I(\pi), \tag{3.19}$$

not depending on the choice of uniformizer $\pi \in O$, under which we may regard the former as the subgroup of $\tilde{W}$ acting by translation on the standard apartment, see also [HR08, Proposition 13].

Definition 3.11. Let μ be a geometric conjugacy class of cocharacters with reflex field E. The μ-admissible locus is the Schubert perfect k_E-scheme

$$\mathcal{A}_{\mathcal{G},\mu} = \mathcal{F}\ell_{\mathcal{G},\{\lambda_I(\pi)\}}, \tag{3.20}$$

where $\lambda \in X_*(T)$ runs over all representatives of μ and $\lambda_I \in X_*(T)_I$ denotes the associated coinvariant under I.

Note that $\mathcal{A}_{\mathcal{G},\mu}$ is geometrically connected because the finite Weyl group acts trivially on $\pi_1(G)_I$. It does not depend on the choice of T. By a result of Haines [Hai18, Theorem 4.2], we know that $\mathcal{A}^\circ_{\mathcal{G},\mu} := \mathcal{F}\ell^\circ_{\mathcal{G},\{\lambda_I(\pi)\}}$, where λ now runs over the rational conjugates of μ, that is, all those which are contained in a closed Weyl chamber attached to $w_0Bw_0^{-1}$ for some $w_0 \in W_0$, the finite Weyl group of $G_{\breve{F}}$ with respect to $S_{\breve{F}}$.

It will turn out that $\mathcal{A}_{\mathcal{G},\bar{k},\mu}$ is functorial in $(\mathcal{G}, \mu)$, as soon as we develop a theory of local models $\mathcal{M}_{\mathcal{G},\mu}$, see Definition 4.11, and calculate their special fibers, confer Theorem 6.16. The admissible locus also admits the following representation-theoretic interpretation in terms of representations of the Langlands dual group $\widehat{G}$ (here, taken over any algebraically closed field) with dual torus $\widehat{T}$:

Lemma 3.12. *Let $\widehat{\lambda}^I \in X^*(\widehat{T}^I) \cong X_*(T)_I$ be running through the set of restrictions of all weights $\widehat{\lambda}$ for $\widehat{T}$ occurring in a finite dimensional algebraic representation of $\widehat{G}$ with fixed highest weight $\widehat{\mu} = \mu$. Then $\mathcal{A}_{\mathcal{G},\mu} = \mathcal{F}\ell_{\mathcal{G},\{\widehat{\lambda}^I(\pi)\}}$.*

Proof. Being a $\widehat{G}$-representation, V contains all the weights $\widehat{\lambda}$ conjugate to $\widehat{\mu}$ under the absolute Weyl group with the same non-zero multiplicity. Under $X^*(\widehat{T}) \cong X_*(T)$, these correspond to the conjugates of μ compatibly with the projection to $X^*(\widehat{T}^I) \cong X_*(T)_I$. Hence, the lemma follows from the definition of the admissible locus. □

Example 3.13. The basic example of the admissible locus occurs for $G = \mathrm{GL}_2$, $\mu = (1, 0)$ and $\mathcal{G} = \mathcal{I}$ an Iwahori. In this case, $\mathcal{A}_{\mathcal{G},\mu}$ is the union of two copies of $\mathbb{P}^{1,\mathrm{perf}}_k$ intersecting transversally at a point. More generally, one can enumerate the Iwahori–Schubert orbits of the translated to the neutral component admissible locus $\mathcal{A}_{\mathcal{I},\mu}$ in terms of alcoves in the standard apartment $\mathscr{A}(G, S, F)$. For pictures in the case of unitary groups of split rank 2, the reader is referred to the introduction of [PR09]. For further examples, see the survey [PRS13].

3.3. **Canonical deperfections.** Now, we wish to introduce equivariant deperfections of the Schubert perfect schemes $\mathcal{F}\ell_{\mathcal{G},W}$ following Proposition 3.3 and discuss their geometry, at least for certain W. We are especially interested in admissible loci $\mathcal{A}_{\mathcal{G},\mu}$ for μ minuscule.

First, recall that the congruence quotient $L^{\leq n}_{\bar{k}}\mathcal{G}$ of $L^+_{\bar{k}}\mathcal{G}$ has a deperfection $\mathrm{Gr}_n\mathcal{G}$, given by $(n+1)$-truncated Witt vectors and which is called the Greenberg realization. We denote by $L^{>n}_{\bar{k}}\mathcal{G}$ the kernel of $L^+_{\bar{k}}\mathcal{G} \to L^{\leq n}_{\bar{k}}\mathcal{G}$.

Definition 3.14. Let n be the smallest nonnegative integer such that $L^{>n}_{\bar{k}}\mathcal{G}$ acts trivially on $\mathcal{F}\ell_{\mathcal{G},\bar{k},W}$ and call it the associated depth. The canonical deperfection[4] $\mathcal{F}\ell^{\mathrm{can}}_{\mathcal{G},W}$ of the perfect Schubert scheme $\mathcal{F}\ell_{\mathcal{G},W}$ is the finite type k_W-scheme with $\mathrm{Gr}_n\mathcal{G}$-action determined by Proposition 3.3.

Assume the $L^+_{\bar{k}}\mathcal{G}$-action on $\mathcal{F}\ell_{\mathcal{G},\bar{k},W}$ factors through $L^0_{\bar{k}}\mathcal{G} = \mathcal{G}^{\mathrm{perf}}_{\bar{k}}$. For $V \leq W$, we get a deperfection

$$\mathcal{F}\ell^{\mathrm{can}}_{\mathcal{G},\bar{k},V} \to \mathcal{F}\ell^{\mathrm{can}}_{\mathcal{G},\bar{k},W} \tag{3.21}$$

of the closed immersion of perfect Schubert schemes, because the image is a finite type deperfection with smaller function fields, as it carries a $\mathcal{G}_{\bar{k}}$-action.

However, it is not clear that the finite type morphism is a closed immersion. To know more about the geometry of $\mathcal{F}\ell^{\mathrm{can}}_{\mathcal{G},W}$, we exploit the picture in the equicharacteristic situation.

Assume G is adjoint, and also Assumption 1.9 for G over $\breve{F}$, that is, if $p = 2$, then G has no odd unitary factors over $\breve{F}$. Then, for every parahoric $\breve{O}$-group $\mathcal{G}$ attached to a facet in $\mathscr{A}(G, S, \breve{F})$, we find smooth, affine, fiberwise connected $\breve{O}[\![t]\!]$-lifts $\underline{\mathcal{G}}$ in the sense of [FHLR22, Proposition 2.8]. Note that the $\bar{k}[\![t]\!]$-reductions $\mathcal{G}'$ are parahoric models of some adjoint connected reductive $\bar{k}(\!(t)\!)$-group G' attached to a facet in some apartment $\mathscr{A}(G', S', \bar{k}(\!(t)\!)) \cong \mathscr{A}(G, S, F)$, see [FHLR22, Lemma 2.7].

In particular, these come with isomorphisms

$$\mathcal{G} \otimes_{\breve{O}} \bar{k} \cong \mathcal{G}' \otimes_{\bar{k}[\![t]\!]} \bar{k}, \tag{3.22}$$

that are functorial as we vary $\mathcal{G}$ among parahoric models attached to a facet in $\mathscr{A}(G, S, \breve{F})$, and which we now exploit to compare their Schubert schemes. Let us note that the loop groups $L^+_{\bar{k}}\mathcal{G}$ and its analogue $L^+_{\bar{k}}(\mathcal{G}')$ in the equicharacteristic setting admit natural surjections on $\mathcal{G}^{\mathrm{perf}}_{\bar{k}} \cong \mathcal{G}'^{,\mathrm{perf}}_{\bar{k}}$. Below, we use subscripts (-)$'$ to denote the perfections of the equicharacteristic loop groups and Schubert varieties for $\mathcal{G}'$.

[4]In the equicharacteristic situation, one recovers the weak normalization of classical Schubert schemes, which turn out to be the classical ones under Assumption 1.9 and also $p \nmid |\pi_1(G_{\mathrm{der}})|$, see [FHLR22, Section 4.1] and [HLR24].

Lemma 3.15. *Under the above constraints, there are unique equivariant isomorphisms*

$$\mathcal{F}\ell^{\mathrm{can}}_{\mathcal{G},\bar{k},W} \cong \mathcal{F}\ell^{\mathrm{can}}_{\mathcal{G}',\bar{k},W'} \tag{3.23}$$

for all connected $\mathcal{F}\ell_{\mathcal{G},\bar{k},W}$ of depth 0, that is, whose $L^+_{\bar{k}}\mathcal{G}$-action factors through $\mathcal{G}^{\mathrm{perf}}_{\bar{k}}$.

Proof. As $\mathcal{G}_{\bar{k}} \cong \mathcal{G}'_{\bar{k}}$, it suffices by Proposition 3.3 to produce equivariant isomorphism $\mathcal{F}\ell_{\mathcal{G},\bar{k},W} \cong \mathcal{F}\ell_{\mathcal{G}',\bar{k},W'}$ of the perfect Schubert schemes. During the proof, we fix an auxiliary Iwahori $\mathcal{I}$ dilated from $\mathcal{G}$ and consider the corresponding Iwahori–Schubert perfect scheme $\mathcal{F}\ell_{(\mathcal{I},\mathcal{G}),\bar{k},W}$.

First assume that $W = \{w\}$. The perfect variety $\mathcal{F}\ell_{(\mathcal{I},\mathcal{G}),\bar{k},w}$ can be resolved via a Demazure variety $\mathcal{D}_{\mathcal{I},\bar{k},\dot{w}}$. If s is the first letter of the word $\dot{w}$ and $\dot{v}$ is the word obtained from deleting the first letter, we get

$$\mathcal{D}_{\mathcal{I},\bar{k},\dot{w}} = \mathcal{F}\ell_{\mathcal{I},\bar{k},s} \tilde{\times} \mathcal{D}_{\mathcal{I},\bar{k},\dot{v}} \tag{3.24}$$

where $L^+_{\bar{k}}\mathcal{I} \subset L^+_{\bar{k}}\mathcal{P}$ is the minimal parahoric corresponding to s. We claim that the action of $L^+_{\bar{k}}\mathcal{I}$ on $\mathcal{D}_{\mathcal{I},\bar{k},\dot{v}}$ is trivial when restricted to the normal subgroup $L^{\geq 1}_{\bar{k}}\mathcal{P}$. Otherwise, let α be the negative simple affine root corresponding to s and observe that $L^+_{\bar{k}}\mathcal{U}_{\alpha+1}$ acts non-trivially on $\mathcal{D}_{\mathcal{I},\bar{k},\dot{v}}$. But conjugating by s yields that $L^+_{\bar{k}}\mathcal{U}_{-\alpha+1} \subset L^{\geq 1}_{\bar{k}}\mathcal{I}$ does not act trivially on $\mathcal{D}_{\mathcal{I},\bar{k},\dot{w}}$.

Arguing inductively on $\dot{w}$, and exploting the above claim, we reach at an $\mathcal{I}^{\mathrm{perf}}_{\bar{k}}$-equivariant identification of the Demazure perfect varieties

$$\mathcal{D}_{\mathcal{I},\bar{k},\dot{w}} \cong \mathcal{D}_{\mathcal{I}',\bar{k},\dot{w}'} \tag{3.25}$$

bounded by $\dot{w}$ resp. $\dot{w}'$ and attached to $\mathcal{I}$, respectively the $\bar{k}(\!(t)\!)$-reduction $\mathcal{I}'$ of the Iwahori $\breve{O}[\![t]\!]$-lift. In the case $l(\dot{w}) = 1$, then we get the unique equivariant identification of one-dimensional Iwahori–Schubert perfect varieties, which are perfected projective lines.

As the Picard group of the Demazure varieties have already been determined, see Theorem 3.8, respectively [HZ20, Section 3.2] for the equicharacteristic case, the previous isomorphism descends uniquely by Proposition 3.2 to an $\mathcal{I}^{\mathrm{perf}}_{\bar{k}}$-equivariant identification

$$\mathcal{F}\ell_{(\mathcal{I},\mathcal{G}),\bar{k},w} \cong \mathcal{F}\ell_{(\mathcal{I}',\mathcal{G}'),\bar{k},w'} \tag{3.26}$$

of the perfect Schubert varieties, see Remark 3.9. If the left side is stable under $\mathcal{G}_{\bar{k}}$, then we need to show the map is not only $\mathcal{I}^{\mathrm{perf}}_{\bar{k}}$-equivariant, but furthermore $\mathcal{G}^{\mathrm{perf}}_{\bar{k}}$-equivariant.

Let $\bar{Q} \subset \mathcal{G}_{\bar{k}}$ denote the image of $\mathcal{I}_{\bar{k}}$. By assumption, the $\mathcal{I}^{\mathrm{perf}}_{\bar{k}}$-action on both sides factors through the perfection of $\bar{Q}$. Using the convolution product

$$\mathcal{G}^{\mathrm{perf}}_{V} \times^{\bar{Q}^{\mathrm{perf}}} \mathcal{F}\ell_{\mathcal{G},\bar{k},w}, \tag{3.27}$$

we get a perfect $\bar{k}$-variety mapping $\mathcal{I}^{\mathrm{perf}}_{\bar{k}}$-equivariantly to $\mathcal{F}\ell_{\mathcal{G},\bar{k},w}$ and that can be identified with its equicharacteristic analogue in a $\mathcal{G}^{\mathrm{perf}}_{\bar{k}}$-equivariant fashion. Since $\mathcal{G}^{\mathrm{perf}}_{\bar{k}}/\bar{Q}^{\mathrm{perf}} \subset \mathcal{F}\ell_{\mathcal{I},\bar{k}}$ is a Schubert perfect variety at Iwahori level, we know its Picard group by Theorem 3.8 and Remark 3.9. Applying again Proposition 3.2, we not only recover the original isomorphism, by Proposition 3.10, but also conclude it is $\mathcal{G}^{\mathrm{perf}}_{\bar{k}}$-equivariant and the unique such map.

For a general W as in the statement, we now can glue the above isomorphism to non-irreducible Schubert perfect schemes

$$\mathcal{F}\ell_{\mathcal{G},\bar{k},W} \cong \mathcal{F}\ell_{\mathcal{G}',\bar{k},W'} \tag{3.28}$$

appealing again to Proposition 3.10. □

From now on G is no longer assumed to be adjoint. We approach the canonical admissible locus $\mathcal{A}^{\mathrm{can}}_{\mathcal{G},\mu}$ for minuscule μ, that is, the canonical perfection of the admissible locus, with our comparison result, describing its singularities (thereby confirming [Zhu17a, Conjecture III] for Schubert varieties in the admissible locus) and computing its coherent cohomology.

Theorem 3.16. *Let μ be minuscule and assume Assumption 1.9. Then, $\mathcal{A}^{\mathrm{can}}_{\mathcal{G},\mu}$ is Cohen–Macaulay and Frobenius split compatibly with its $\mathcal{G}_{\bar{k}}$-stable reduced $\bar{k}$-subschemes.*

Moreover, for every ample line bundle $\mathcal{L}$ on $\mathcal{F}\ell_{\mathcal{G},\bar{k}}$ that descends to $\mathcal{A}^{\mathrm{can}}_{\mathcal{G},\bar{k},\mu}$, there is an equality

$$\dim_{\bar{k}} \mathrm{H}^0(\mathcal{A}^{\mathrm{can}}_{\mathcal{G},\bar{k},\mu}, \mathcal{L}) = \dim_C \mathrm{H}^0(\mathcal{F}_{G,C,\mu}, \mathcal{O}(c_{\mathcal{L}})). \tag{3.29}$$

Here, $\mathcal{F}_{G,\mu} = G_E/P_\mu^-$ is the classical flag variety attached to μ, the central charge $c_{\mathcal{L}}$ is given by Kac–Moody coefficients, see [PR08, Section 10] and [BS17, Section 10], and the line bundle $\mathcal{O}(c_{\mathcal{L}})$ is the corresponding power of the ample generator of $\mathrm{Pic}(\mathcal{F}_{G,C,\mu})$.

Proof. We want to apply Lemma 3.15, in order to reduce the statements to the equicharacteristic situation, where we refer to [FHLR22, Theorem 4.1, Theorem 4.25].

In order to do this, we first notice that there is an equivariant isomorphism $\mathcal{A}^{\mathrm{can}}_{\mathcal{G},\bar{k},\mu} \cong \mathcal{A}^{\mathrm{can}}_{\mathcal{G}_{\mathrm{ad}},\bar{k},\mu_{\mathrm{ad}}}$ via the natural map. Here, μ_{ad} denotes the composition of μ with $G_C \to G_{\mathrm{ad}C}$. Indeed, this can be checked on perfections and then at the level of geometric points, where it follows from the assertion that $\mathcal{F}\ell_{\mathcal{G},\bar{k}} \to \mathcal{F}\ell_{\mathcal{G}_{\mathrm{ad}},\bar{k}}$ is an open and closed immersion.

We still have to show that $\mathcal{A}^{\mathrm{can}}_{\mathcal{G},\bar{k},\mu}$ has minimal depth, that is, $L^+_{\bar{k}}\mathcal{G}$ acts via $\mathcal{G}_{\bar{k}}$. Since $L^{\geq 1}_{\bar{k}}\mathcal{G}$ is a normal subgroup, it suffices to check that it fixes each of the sections λ defining the admissible locus. By the combinatorial dictionary, see our proof of Proposition 3.10, it suffices to show that $|a(\lambda_I)| \leq 1$, that is, the translation λ_I moves every affine root to a parallel one at distance at most one. By definition, one has

$$a(\lambda_I) = [K : \breve{F}]^{-1} \sum_{\sigma \in \mathrm{Gal}(K/\breve{F})} \sigma\tilde{a}(\lambda), \tag{3.30}$$

where K is a finite Galois extension of $\breve{F}$ splitting $G_{\breve{F}}$, and $\tilde{a}$ is an absolute root restricting to a, so its absolute value is at most 1, since λ is minuscule. $\square$

Remark 3.17. After the first version of this paper was written, Assumption 1.9 was removed from Theorem 3.16 in the equicharacteristic setting in [Lou23b]. These results will be extended to the mixed characteristic setting in work in preparation of Cass and the third author [CL25]. We remark that Cohen–Macaulayness would follow from a positive solution of [FHLR22, Conjecture 3.6].

4. Affine Grassmannians and local models

In this section, we start by gathering several basic facts on the B^+_{dR}-affine Grassmannian over $\mathrm{Spd}\, C$. Most of them were established in [SW20, Lecture XIX] and [FS21, Chapters VI.2, VI.5], but our approach is sufficiently different and relevant to later sections that it merits some elaboration.

Then, we introduce the main objects of study of this article, to wit the local models

$$\mathcal{M}_{\mathcal{G},\mu} \subset \mathrm{Gr}_{\mathcal{G},O_E} \tag{4.1}$$

defined for every $\mu \in X_*(T)$ via v-closures of Schubert diamonds in a Beilinson–Drinfeld Grassmannian. We dedicate the rest of the section to showing that $\mathcal{M}_{\mathcal{G},\mu}$ is an $L^+_{O_E}\mathcal{G}$-stable flat, proper π-adic kimberlite with good finiteness properties. In particular, its special fiber will be shown to be representable by some connected Schubert perfect scheme $\mathcal{F}\ell_{\mathcal{G},W}$.

4.1. **The B^+_{dR}-affine Grassmannian.** In this section, we fix a complete discretely valued field $F/\mathbb{Q}_p$ with perfect residue field k, ring of integers O and uniformizer π, a complete algebraic closure C/F with ring of integers O_C and residue field $\bar{k} = k_C$. We denote by $\breve{F} \subset C$ the maximal unramified complete subextension with ring of integers $\breve{O}$ and the same residue field $\bar{k} = k_{\breve{F}}$. Further, we fix a (connected) reductive F-group G and a maximal $\breve{F}$-split F-torus $S \subset G$ containing a maximal F-split torus, see [BT84, Proposition 5.1.10]. As G is quasi-split over $\breve{F}$ by Steinberg's theorem, the centralizer T of S is a maximal torus. Also, we fix a Borel subgroup $B \subset G_{\breve{F}}$ containing $T_{\breve{F}}$.

For any affinoid perfectoid space $\mathrm{Spa}(R, R^+)$ in characteristic p equipped with a map to $\mathrm{Spd}\,\mathbb{Z}_p$, let $B^+_{\mathrm{dR}}(R^\sharp)$, respectively $B_{\mathrm{dR}}(R^\sharp)$, be the rings of de Rham periods formed using O-Witt vectors. For convenience, we set $B^+_{\mathrm{dR}} := B^+_{\mathrm{dR}}(C)$ and $B_{\mathrm{dR}} := B_{\mathrm{dR}}(C)$. The B^+_{dR}-loop group of G is the group functor over $\mathrm{Spd}\,F$ given by

$$LG\colon (R, R^+) \mapsto G(B_{\mathrm{dR}}(R^\sharp)), \tag{4.2}$$

and the positive loop group is the subgroup functor

$$L^+G\colon (R, R^+) \mapsto G(B^+_{\mathrm{dR}}(R^\sharp)). \tag{4.3}$$

Their v-sheaf quotient

$$\mathrm{Gr}_G := LG/L^+G \tag{4.4}$$

is called the B^+_{dR}-affine Grassmannian. Similarly to Section 3.2, $\mathrm{Gr}_G(R, R^+)$ parametrizes G-torsors on the spectrum $\mathrm{Spec}(B^+_{\mathrm{dR}}(R^\sharp))$ with a trivialization over $\mathrm{Spec}(B_{\mathrm{dR}}(R^\sharp))$. Here, we are primarily interested in the geometry and work therefore over $\mathrm{Spd}\,C$. The base changes are denoted by L_CG, L^+_CG and $\mathrm{Gr}_{G,C}$, for convenience.

As an auxiliary first step, we study the affine flag variety and then translate the results to the affine Grassmannian. For this, the Iwahori group B^+_{dR}-model $\mathcal{I}$ is given as the dilatation of $G \otimes_F B^+_{\mathrm{dR}}$ along the subscheme $B_C \subset G_C$ of its special fiber. Define

$$L^+_C\mathcal{I}\colon (R, R^+) \mapsto \mathcal{I}(B^+_{\mathrm{dR}}(R^\sharp)) \tag{4.5}$$

which is a subgroup v-sheaf of L^+G. It gives rise to the B^+_{dR}-affine flag variety

$$\mathcal{F}\ell_{\mathcal{I},C} := L_CG/L^+_C\mathcal{I}, \tag{4.6}$$

viewed as a v-sheaf over $\mathrm{Spd}\,C$.

We recall that $\mathrm{Gr}_{G,C} \to \mathrm{Spd}\,C$ is an increasing union of proper, spatial diamonds by [SW20, Lecture XIX]. The same holds for $\mathcal{F}\ell_{\mathcal{I},C} \to \mathrm{Spd}\,C$, as the projection

$$\mathcal{F}\ell_{\mathcal{I},C} \to \mathrm{Gr}_{G,C} \tag{4.7}$$

is a proper, cohomologically smooth $(G_C/B_C)^\diamondsuit$-fibration. The following discussion is parallel to parts of Section 3.2 but simplified by the fact that we consider $G \otimes_F B^+_{\mathrm{dR}}$ which is a (split) reductive group over B^+_{dR} (and not some parahoric group scheme). The geometry of the affine flag variety $\mathcal{F}\ell_{\mathcal{I},C}$ or, better, the v-stack quotient

$$\mathrm{Hk}_{\mathcal{I},C} := L^+_C\mathcal{I}\backslash\mathcal{F}\ell_{\mathcal{I},C} = L^+_C\mathcal{I}\backslash L_CG/L^+_C\mathcal{I} \tag{4.8}$$

is reflected in the Iwahori-Weyl group of $G(B_{\mathrm{dR}})$,

$$\tilde{W}_{\mathrm{dR}} := N_G(T)(B_{\mathrm{dR}})/T(B^+_{\mathrm{dR}}), \tag{4.9}$$

where $N_G(T)$ denotes the normalizer of T in G. There is a canonical map $\underline{\tilde{W}_{\mathrm{dR}}} \to \mathcal{F}\ell_{\mathcal{I},C}$ because $T(B^+_{\mathrm{dR}}) \subset \mathcal{I}(B^+_{\mathrm{dR}})$.

Lemma 4.1. *The map $\underline{\tilde{W}_{\mathrm{dR}}} \to \mathcal{F}\ell_{\mathcal{I},C}$ induces a bijection*

$$\tilde{W}_{\mathrm{dR}} \cong |\mathrm{Hk}_{\mathcal{I},C}|. \tag{4.10}$$

Proof. Every point of $|\mathrm{Hk}_{\mathcal{I},C}|$ is represented by a map $\mathrm{Spa}(K,K^+)\to L_CG$ with K algebraically closed perfectoid. Two K-valued points have the same underlying element in $|\mathrm{Hk}_{\mathcal{I},C}|$ if, v-locally, they lie in the same double coset

$$\mathcal{I}(B_{\mathrm{dR}}^+(K^\sharp))\backslash G(B_{\mathrm{dR}}(K^\sharp))/\mathcal{I}(B_{\mathrm{dR}}^+(K^\sharp)). \tag{4.11}$$

The identification now follows from the Bruhat decomposition which is independent of $K^\sharp$. □

Let $\mathbf{a}\subset\mathscr{A}(G,T,B_{\mathrm{dR}})$ be the alcove defined by $\mathcal{I}$ in the apartment for T of the Bruhat–Tits building of $G(B_{\mathrm{dR}})$. Let $\mathbb{S}\subset\tilde{W}_{\mathrm{dR}}$ be the set of simple reflections along the walls bounding $\mathbf{a}$. The affine Weyl group $W_{\mathrm{dR,af}}\subset\tilde{W}_{\mathrm{dR}}$ is the subgroup generated by the elements in $\mathbb{S}$. Then, W_{af} is a Coxeter group which only depends on the Bruhat–Tits building of $G(B_{\mathrm{dR}})$. As in (3.8) there is a canonical short exact sequence

$$1\to W_{\mathrm{dR,af}}\to\tilde{W}_{\mathrm{dR}}\to\pi_1(G)\to 1, \tag{4.12}$$

which is naturally split by taking the stabilizer $\Omega_{\mathbf{a}}\subset\tilde{W}_{\mathrm{dR}}$ of the alcove $\mathbf{a}$. Thus, we can write each $w\in\tilde{W}_{\mathrm{dR}}$ uniquely as $w=w_{\mathrm{af}}\tau$ with $\tau\in\Omega_{\mathbf{a}}$ and $w_{\mathrm{af}}\in W_{\mathrm{dR,af}}$.

Lemma 4.2. *Equip $\pi_1(G)$ with the discrete topology. The morphism*

$$|\mathrm{Hk}_{\mathcal{I},C}|\to\pi_1(G) \tag{4.13}$$

is locally constant, thus underlies a morphism $\mathrm{Hk}_{\mathcal{I},C}\to\underline{\pi_1(G)}$ of small v-stacks.

Proof. Here, we follow the argument behind the proof of [PR08, Theorem 5.1]. If $G=G_{\mathrm{sc}}$ is simply connected, then we see that, for every algebraically closed perfectoid field K/C, the group $L_CG(K)$ is generated by its affine root subgroups $L_CU_a(K)$.

But, since L_CU_a is connected (choose a pinning), we conclude that L_CG, hence also $\mathcal{F}\ell_{\mathcal{I},C}$ and $\mathrm{Hk}_{\mathcal{I},C}$ are connected. If $G=T$ is a torus, then we see easily that $\mathrm{Gr}_{T,C}$ equals the v-sheaf $\underline{X_*(T)}$ compatibly with the map above.

Now, suppose that $G_{\mathrm{der}}=G_{\mathrm{sc}}$. Then, $\pi_1(G)$ identifies with the fundamental group of the abelian quotient G/G_{der}, so the claim is clear. Finally, for a general group G, consider the z-extension

$$1\to T_{\mathrm{sc}}\to\tilde{G}\to G\to 1, \tag{4.14}$$

where $\tilde{G}$ is given by the pushout of $(G_{\mathrm{sc}}\rtimes T)$ along the morphism $\ker(T_{\mathrm{sc}}\to T)\to T_{\mathrm{sc}}$, where T_{sc} is the preimage of the maximal torus $T\subset G$ under the map $G_{\mathrm{sc}}\to G$. Using the fact that T_{sc} is an induced torus, we see that the Hecke stacks and the π_1's lie in a similar exact sequence, which yields the claim. □

For $\tau\in\pi_1(G)$, we denote by $\mathcal{F}\ell^{\tau}_{\mathcal{I},C}$ the fiber over τ of the morphism $\mathcal{F}\ell_{\mathcal{I},C}\to\underline{\pi_1(G)}$. We note that right translation by a representative of τ in L_CG induces an isomorphism

$$\mathcal{F}\ell^{1}_{\mathcal{I},C}\xrightarrow{\cong}\mathcal{F}\ell^{\tau}_{\mathcal{I},C}. \tag{4.15}$$

Moreover, $\mathcal{F}\ell^{1}_{\mathcal{I},C}$ is canonically isomorphic to the affine flag variety $\mathcal{F}\ell_{\mathcal{I}_{\mathrm{sc}},C}$ of the simply connected cover G_{sc}. Namely, the transition morphism $\mathcal{F}\ell_{\mathcal{I}_{\mathrm{sc}},C}\to\mathcal{F}\ell^{1}_{\mathcal{I},C}$ is bijective by checking on geometric points ([Sch17, Lemma 12.5]) and using the Bruhat decomposition, hence must be an isomorphism as both $\mathcal{F}\ell_{\mathcal{I}_{\mathrm{sc}},C},\mathcal{F}\ell^{1}_{\mathcal{I},C}$ are ind-proper over $\mathrm{Spd}\,C$.

Definition 4.3. Let $w\in\tilde{W}_{\mathrm{dR}}$. The Schubert cell $\mathcal{F}\ell^{\circ}_{\mathcal{I},C,w}\subset\mathcal{F}\ell_{\mathcal{I},C}$ is the v-sheaf-theoretic image of the orbit map

$$L^+_C\mathcal{I}\to\mathcal{F}\ell_{\mathcal{I},C},\ i\mapsto iw. \tag{4.16}$$

The Schubert variety is the v-closure $\mathcal{F}\ell_{\mathcal{I},C,w}:=\mathcal{F}\ell^{\circ,\mathrm{cl}}_{\mathcal{I},C,w}$ in the sense of Section 2.1.

By Proposition 2.8, we know that the underlying topological space of $\mathcal{F}\ell_{I,w}$ is the weakly generalizing closure of $|\mathcal{F}\ell^\circ_{\mathcal{I},C,w}|$ inside $|\mathcal{F}\ell_{\mathcal{I},C}|$. But, $\mathcal{F}\ell^\circ_{\mathcal{I},C,w}$ is possibly ill-behaved because $L^+_C\mathcal{I}$ is not quasicompact. As we show in Proposition 4.5 and Corollary 4.6 for the affine Grassmannian, our definition is equivalent to the pointwise definition in [FS21, Definition VI.2.2]. We start with the case of simple reflections:

Lemma 4.4. *Let $s \in \mathbb{S}$ be a simple reflection. Then there is an isomorphism $\mathcal{F}\ell_{\mathcal{I},C,s} \simeq (\mathbb{P}^1_C)^\diamondsuit$ that restricts to $\mathcal{F}\ell^\circ_{\mathcal{I},C,s} \simeq (\mathbb{A}^1_C)^\diamondsuit$. In particular, $\mathcal{F}\ell^\circ_{\mathcal{I},C,s}$ is a topologically dense open subset of $\mathcal{F}\ell_{\mathcal{I},C,s}$.*

Proof. Let $\mathcal{P}_s$ be the parahoric group scheme over B^+_{dR} associated to the wall of $\mathbf{a}$ defining s. The maximal reductive quotient H of its special fiber over C has semisimple rank 1. Using [BT84, Théorème 4.6.33], we see that $L^+_C\mathcal{I}$ is the preimage of $Q^\diamondsuit$ under $L^+\mathcal{P}_s \to H^\diamondsuit$ for some Borel subgroup $Q \subset H$. Thus, there are isomorphisms $L^+_C\mathcal{P}_s/L^+_C\mathcal{I} \simeq (H/Q)^\diamondsuit \simeq (\mathbb{P}^1_C)^\diamondsuit$ which can be made explicit via the choice of a pinning. This implies that the monomorphism

$$L^+_C\mathcal{P}_s/L^+_C\mathcal{I} \subset \mathcal{F}\ell_{\mathcal{I},C} \tag{4.17}$$

is a closed embedding, as $\mathcal{F}\ell_{\mathcal{I},C}$ is separated and $(\mathbb{P}^1_C)^\diamondsuit$ is proper over $\operatorname{Spd} C$. The isomorphism $\mathcal{F}\ell^\circ_{\mathcal{I},C,s} \simeq (\mathbb{A}^1_C)^\diamondsuit$ is now clear, since this is the only non-trivial $Q^\diamondsuit$-orbit in $(\mathbb{P}^1_C)^\diamondsuit$. ☐

In order to treat more general $w = w_{\mathrm{af}}\tau \in \tilde{W}_{\mathrm{dR}} \cong W_{\mathrm{dR,af}} \rtimes \Omega_{\mathfrak{a}}$, we invoke Demazure resolutions as follows. Let $\dot{w} = s_1 \dots s_n$ be a reduced word for $w_{\mathrm{af}} = w\tau^{-1}$ with $s_i \in \mathbb{S}$ and consider the Demazure variety

$$\mathcal{D}_{C,\dot{w}} := L^+_C\mathcal{P}_1 \times^{L^+_C\mathcal{I}} \dots \times^{L^+_C\mathcal{I}} L^+_C\mathcal{P}_n/L^+_C\mathcal{I} \tag{4.18}$$

which will also be denoted by $\mathcal{F}\ell_{\mathcal{I},C,s_1}\tilde{\times}\dots\tilde{\times}\mathcal{F}\ell_{\mathcal{I},C,s_n}$. It is connected and cohomologically smooth over $\operatorname{Spd} C$ (being an iterated $\mathbb{P}^1_C$-fibration), and the twisted product

$$\mathcal{D}^\circ_{C,\dot{w}} = \mathcal{F}\ell^\circ_{\mathcal{I},C,s_1}\tilde{\times}\dots\tilde{\times}\mathcal{F}\ell^\circ_{\mathcal{I},C,s_n} \tag{4.19}$$

of the open cells is topologically dense by induction on n, starting with Lemma 4.4 and using that $L^+_C\mathcal{I}$ is pro-(cohomologically smooth) over $\operatorname{Spd} C$. It carries, moreover, a natural morphism (induced by multiplication)

$$\pi_{\dot{w}} \colon \mathcal{D}_{C,\dot{w}} \to \mathcal{F}\ell_{\mathcal{I},C} \tag{4.20}$$

which necessarily maps onto $\mathcal{F}\ell_{\mathcal{I},C,w_{\mathrm{af}}}$, by properness, $L^+_C\mathcal{I}$-equivariance and the fact that $\dot{w}$ maps to w. After translation by τ, we may regard this as a resolution of $\mathcal{F}\ell_{\mathcal{I},C,w}$, which is thus in particular connected.

For the next result, we note that $\tilde{W}_{\mathrm{dR}}$, in analogy to the discussion following (3.8), is equipped with a length function and Bruhat partial order induced from the quasi-Coxeter structure on $\tilde{W}_{\mathrm{dR}} \cong W_{\mathrm{dR,af}} \rtimes \Omega_{\mathfrak{a}}$.

Proposition 4.5. *Let $w \in \tilde{W}_{\mathrm{dR}}$. Then $\mathcal{F}\ell^\circ_{\mathcal{I},C,w}$, respectively $\mathcal{F}\ell_{\mathcal{I},C,w}$, agrees with the subfunctor of all maps $S \to \mathcal{F}\ell_{\mathcal{I},C}$ such that for all geometric points $S' = \operatorname{Spa}(K, K^+) \to S$, the induced point $S' \to \mathcal{F}\ell_{\mathcal{I},C} \to \mathrm{Hk}_{\mathcal{I},C}$ is given by w, respectively by v for some $v \leq w$. In particular, $\mathcal{F}\ell^\circ_{\mathcal{I},C,w} \subset \mathcal{F}\ell_{\mathcal{I},C,w}$ is a topologically dense open.*

Proof. Observe that the first assertion cannot be verified at geometric points because $L^+_C\mathcal{I}$ is not quasicompact. However, we see from Lemma 4.4 that the result holds for simple reflections. Indeed, we even have by Bruhat–Tits combinatorics

$$L^+_C\mathcal{U}_{\alpha_s} \cdot s = \mathcal{F}\ell^\circ_{\mathcal{I},C,s}, \tag{4.21}$$

where α_s denotes the positive simple affine root associated to the simple reflection s, and $\mathcal{U}_{\alpha_s}$ is the corresponding B^+_{dR}-model of the affine root group. Pulling across the reflections s_i appearing in the convolution product of $\mathcal{D}_{C,\dot{w}}$, we see that $\mathcal{F}\ell^\circ_{\mathcal{I},C,w}$ surjects to the v-sheaf image of $\mathcal{D}^\circ_{C,\dot{w}}$ along $\pi_{\dot{w}}$. This v-sheaf image identifies with the pointwise description of $\mathcal{F}\ell^\circ_{\mathcal{I},C,w}$ by quasicompactness of $\pi_{\dot{w}}$ and bijectivity at geometric points.

Similarly, the v-sheaf image of $\mathcal{D}_{C,\dot{w}}$ along $\pi_{\dot{w}}$ is a proper closed sub-v-sheaf of $\mathcal{F}\ell_{\mathcal{I},C}$. By generalities of Tits system, see [BT72, 1.2.6], this v-sheaf image coincides with the desired pointwise description of $\mathcal{F}\ell_{\mathcal{I},C,w}$. Pulling back again via the quotient map $\pi_{\dot{w}}$, we see that $\mathcal{F}\ell^\circ_{\mathcal{I},C,w} \subset \mathcal{F}\ell_{\mathcal{I},C,w}$ is a topologically dense open of the closed v-sheaf image of $\pi_{\dot{w}}$. □

As a corollary, we get that the bijection $|\mathrm{Hk}_{\mathcal{I},C}| \cong \tilde{W}_{\mathrm{dR}}$ from Lemma 4.1 is a homeomorphism where $\tilde{W}_{\mathrm{dR}}$ is endowed with order topology via its Bruhat order, and also that $\pi_0(\mathcal{F}\ell_{\mathcal{I},C}) = \pi_0(\mathrm{Gr}_{G,C}) = \pi_1(G)$ via Lemma 4.2.

Now, we apply our results to the affine Grassmannian Gr_G. Note that there is the group isomorphism

$$X_*(T) \cong T(B_{\mathrm{dR}})/T(B^+_{\mathrm{dR}}),\ \chi \mapsto \chi(\xi) \tag{4.22}$$

which is independent of the choice of uniformizer $\xi \in B^+_{\mathrm{dR}}$. Then the Cartan decomposition induces a bijection

$$|\mathrm{Hk}_{G,C}| \simeq X_*(T)_+, \tag{4.23}$$

where $\mathrm{Hk}_{G,C} = L^+_C G\backslash L_C G/L^+_C G$ denotes the Hecke stack. Therefore, we get a Schubert cell $\mathrm{Gr}^\circ_{G,C,\mu} \subset \mathrm{Gr}_{G,C}$ defined as the v-sheaf-theoretic image of the orbit map and the Schubert cell $\mathrm{Gr}_{G,C,\mu}$ defined as its closure, for each $\mu \in X_*(T)_+$, compare with Definition 4.3.

Corollary 4.6. *Let $\mu \in X_*(T)_+$. Then $\mathrm{Gr}^\circ_{G,C,\mu}$, respectively $\mathrm{Gr}_{G,C,\mu}$ agrees with the subfunctor of all maps $S \to \mathrm{Gr}_{G,C,\mu}$ such that for all geometric points $S' = \mathrm{Spa}(K, K^+) \to S$, the induced point $S' \to \mathrm{Gr}_{G,C} \to \mathrm{Hk}_{G,C}$ is given by μ, respectively by some $\lambda \le \mu$ in the dominance order on $X_*(T)_+$. In particular, $\mathrm{Gr}^\circ_{G,C,\mu} \subset \mathrm{Gr}_{G,C,\mu}$ is a topologically dense open.*

Proof. This formally follows from Proposition 4.5 by using the projection $\mathcal{F}\ell_{\mathcal{I},C} \to \mathrm{Gr}_{G,C}$ from (4.7) and noting that the dominance order on $X_*(T)_+$ is induced by the Bruhat order, see [Ric13, Corollaries 1.8, 2.10] for similar arguments. We leave the details to the reader. □

We also have the following fact which says that the $\mathrm{Spd}\, C$-valued points are dense in $\mathrm{Gr}_{G,C,\mu}$ even for the constructible topology.

Corollary 4.7. *Let $\mu \in X_*(T)_+$. The spatial diamond $\mathrm{Gr}_{G,C,\mu}$ has enough C-facets in the sense of* [Gle24, Definition 4.50].

Proof. Taking the preimage under the projection $\mathcal{F}\ell_{\mathcal{I},C} \to \mathrm{Gr}_{G,C}$ from (4.7), this reduces to the analogous assertion for $\mathcal{F}\ell_{\mathcal{I},C,w}$ for some $w \in \tilde{W}_{\mathrm{dR}}$. Since the Demazure resolution is a v-cover, it is enough to prove that $\mathcal{D}_{C,\dot{w}}$ has enough C-facets. This in turn can be proved inductively on the length of $\dot{w}$. If $\dot{w} = s \cdot \dot{v}$ then $\mathcal{D}_{C,\dot{w}}$ is a pro-étale $(\mathbb{P}^1_C)^\diamondsuit$-bundle over $\mathcal{D}_{C,\dot{v}}$. We may find a pro-étale cover

$$X \to \mathcal{D}_{C,\dot{v}} \tag{4.24}$$

with $X \times_{\mathcal{D}_{C,\dot{v}}} \mathcal{D}_{C,\dot{w}} = X \times_{\mathrm{Spd}\, C} (\mathbb{P}^1_C)^\diamondsuit$. Following the arguments given in [Gle24, Lemma 5.16, Proposition 5.21], we may even assume that X has enough facets over $\mathrm{Spd}\, C$. By [Gle24, Proposition 4.51.(2)], $\mathcal{D}_{C,\dot{w}}$ also has enough facets. □

We conclude with some motivation for our later discussion of representability.

Proposition 4.8. *Let $\mu \in X_*(T)_+$. The v-sheaf $\mathrm{Gr}_{G,C,\mu}$ is representable by a projective C-scheme $\mathcal{F}_{G,C,\mu}$ if and only if μ is minuscule.*

Proof. If μ is minuscule, then the L_C^+G-action factors through $G_C^\diamondsuit$ and the Białynicki-Birula map gives an isomorphism $\mathrm{Gr}_{G,C,\mu} \simeq (G_C/P_\mu)^\diamondsuit$, see [SW20, Proposition 19.4.2].

Now suppose that μ is not minuscule, so that $\langle \mu, \theta \rangle \geq 2$ for the highest root θ of G_C. Then, we are going to show that the $L_C^+\mathcal{I}$-orbit of the point μ is not representable by a rigid space. First, we notice that this orbit is isomorphic to $L_C^+\mathcal{I}/H_\mu$ for the subgroup v-sheaf $H_\mu := L_C^+\mathcal{I} \cap \xi^\mu L_C^+\mathcal{I}\xi^{-\mu}$. Note that the positive loop group $L_C^+B^-$ of the negative Borel $B^- \subset G_{\breve{F}}$ is contained in the stabilizer H_μ, so we deduce that the $L_C^+\mathcal{I}$-orbit is even a $L_C^+\mathcal{U}$-orbit, where $\mathcal{U} \subset \mathcal{I}$ denotes the flat closure over B_{dR}^+ of the unipotent radical $U \subset G_{\breve{F}}$ of the fixed positive Borel $B \subset G_{\breve{F}}$. We are going to filter this space further by using the structure of root groups. Fix an "ordre grignotant" on the set Φ^+ of positive roots in G_C in the sense of [BT84, Definition 3.1.2], that is, a descending sequence $\Psi_i \subset \Phi^+$ for $0 \leq i < m = \dim(U)$, of subsets closed under summable roots, such that $\Psi_0 = \Phi^+$, $\Psi_{m-1} = \{\theta\}$ and Ψ_i is obtained from Ψ_{i-1} by deleting one of the smallest roots α_i. Then, [BT84, Proposition 3.3.2] yields associated root group models $\mathcal{U}_i := \mathcal{U}_{\Psi_i}$ such that $\mathcal{U}_i \subset \mathcal{U}_{i-1}$ is a normal subgroup with quotient isomorphic to $\mathcal{U}_{\alpha_i}$. We now set $X_i := L_C^+\mathcal{U}_i/H_\mu \cap L_C^+\mathcal{U}_i$ for the corresponding orbits and we can realize X_{i-1} as a pro-étale fibration over the Banach–Colmez space $\mathcal{BC}(B_{\mathrm{dR}}^+/\xi^{\langle \mu, \alpha_i \rangle})$ with fiber given by X_i. Considering the filtration by powers of ξ on the $L_C^+\mathcal{U}_{\alpha_i}$, we can even refine this further to a filtration by orbits Y_{ij} for $0 \leq j < \langle \mu, \alpha_i \rangle$ such that Y_{ij} is the fiber of a pro-étale fibration $Y_{i,j-1} \to \mathbb{G}_a^\diamondsuit$. If $\mathcal{F}_{G,C,\mu}$ were representable by a rigid space, it would follow by descending induction that all the Y_{ij} are representable by rigid spaces. In particular, representability of $\mathcal{F}_{G,C,\mu}$ implies representability of the Banach-Colmez space $\mathcal{BC}(B_{\mathrm{dR}}^+/\xi^2) \cong Y_{m,\langle \mu, \theta \rangle - 2}$ where we invoke the inequality $\langle \mu, \theta \rangle \geq 2$. However, this is a non-split self-extension of $\mathbb{G}_a^\diamondsuit$ which does not even split étale locally, so cannot be representable ([SW20, Example 15.2.9.5.]). Indeed, if the extension were split étale locally, it would actually split on the nose, as $H^1(\mathbb{A}_C^1, \mathcal{O})$ is trivial. However, if

$$X := \mathrm{Spa}(C\langle T^{\pm 1}\rangle, O_C\langle T^{\pm 1}\rangle) \subset \mathbb{A}_C^1 \tag{4.25}$$

is the affinoid torus, the element $T \in \mathbb{A}_C^1(X)$ does not admit a lift to $\mathcal{BC}(B_{\mathrm{dR}}^+/\xi^2)(X)$ as we now show. Let $X' = \mathrm{Spa}(C\langle T^{\pm 1/p^\infty}\rangle, O_C\langle T^{\pm 1/p^\infty}\rangle)$ be the usual perfectoid $\mathbb{Z}_p(1)$-cover of X. Elements in $\mathcal{BC}(B_{\mathrm{dR}}^+/\xi^2)(X')$ can be represented by $[a] + [b]\xi$ with $a, b \in (C\langle T^{\pm 1/p^\infty}\rangle)^\flat$. Assume now that

$$x := [a] + [b]\xi$$

maps to T, that means, $a^\sharp = T$ in $\mathbb{A}_C^1(X')$ (cf. [SW20, Section 6.2] for the map $(-)^\sharp$), and assume that x is invariant under $\mathbb{Z}_p(1)$. Let $g \in \mathbb{Z}_p(1)$. Then g acts on $[a]$ via multiplication with $[g]$ if we identify $\mathbb{Z}_p(1) \subset C^\flat$. Using the (-)$^\sharp$-map, we get

$$a^\sharp \left(\frac{[g]-1}{\xi} \right)^\sharp = b^\sharp - g(b^\sharp). \tag{4.26}$$

But, if $g \in \mathbb{Z}_p(1)$ is a generator, then $a^\sharp(\frac{[g]-1}{\xi})^\sharp = cT$ with $c \in C$ non-zero. Writing $b^\sharp$ as a power series in the g-eigenvectors T^n with $n \in \mathbb{Z}[1/p]$, then shows that (4.26) can not hold because $g \in \mathbb{Z}_p(1)$ fixes T. This finishes the argument. $\square$

4.2. **Local models.** We continue with the notation of Section 4.1, and additionally let $\mathcal{G}$ be a parahoric O-model of G.

We work with the moduli space $\mathrm{Gr}_{\mathcal{G}}$ of $\mathcal{G}$-torsors over $\mathrm{Spec}(B_{\mathrm{dR}}^+)$ trivialized over $\mathrm{Spec}(B_{\mathrm{dR}})$, see [SW20, Definition 20.3.1], which is the Beilinson–Drinfeld Grassmannian over $\mathrm{Spd}\, O$. A crucial result of Scholze–Weinstein concerns its ind-properness, see [SW20, Theorems 19.3.4, 20.3.6, 21.2.1].

Theorem 4.9 (Scholze–Weinstein)**.** *The structure morphism of the Beilinson–Drinfeld Grassmannian*
$$\mathrm{Gr}_{\mathcal{G}} \to \mathrm{Spd}\, O \tag{4.27}$$
is ind-proper and ind-representable in spatial diamonds.

In [Ans22, Section 12] and then later in [Gle24, Section 5], the first named and second named author respectively constructed and studied the specialization map for $\mathrm{Gr}_{\mathcal{G}}$, see also [Gle24, Theorem 5.1].

Again, we have natural loop groups at hand, namely
$$L_O\mathcal{G}\colon (R, R^+) \mapsto \mathcal{G}(B_{\mathrm{dR}}(R^\sharp)) \tag{4.28}$$
and
$$L_O^+\mathcal{G}\colon (R, R^+) \mapsto \mathcal{G}(B_{\mathrm{dR}}^+(R^\sharp)), \tag{4.29}$$
where $(R^\sharp, R^{\sharp+})$ denotes an untilt of (R, R^+) over (O, O) and $B_{\mathrm{dR}}^{(+)}(R^\sharp)$ the ring of de Rham periods formed using O-Witt vectors. These define v-sheaves over $\mathrm{Spd}\, O$ and the base changes to $\mathrm{Spd}\, F$, respectively $\mathrm{Spd}\, k$ recover the loop groups $L^+G \subset LG$ from Section 4.1, respectively the v-sheaves $(L_k^+\mathcal{G})^\diamondsuit \subset (L_k G)^\diamondsuit$ associated with the loop groups from Section 3.2. Their base changes to O_C are denoted $L_{O_C}\mathcal{G}$, $L_{O_C}^+\mathcal{G}$ and $\mathrm{Gr}_{\mathcal{G},O_C}$.

Lemma 4.10. *There is a natural isomorphism*
$$L_O\mathcal{G}/L_O^+\mathcal{G} \cong \mathrm{Gr}_{\mathcal{G}}, \tag{4.30}$$
where the left side is a quotient for the étale topology. In particular, on geometric fibers
$$\mathrm{Gr}_G \cong \mathrm{Gr}_{\mathcal{G}} \times_{\mathrm{Spd}\, O} \mathrm{Spd}\, F, \qquad \mathcal{F}\ell_{\mathcal{G}}^\diamondsuit \cong \mathrm{Gr}_{\mathcal{G}} \times_{\mathrm{Spd}\, O} \mathrm{Spd}\, k, \tag{4.31}$$
where Gr_G is the affine Grassmannian from Section 4.1 and $\mathcal{F}\ell_{\mathcal{G}}^\diamondsuit$ the v-sheaf attached to the Witt vector partial affine flag variety from Section 3.2.

Proof. For the uniformization (4.30), see [SW20, Proposition 20.3.2]. The isomorphisms (4.31) are given by base change from (4.30) by unwinding the definitions. $\square$

Let μ be a conjugacy class of cocharacters of G_C, with field of definition $E \subset C$. We denote by O_E its ring of integers with residue field k_E. We wish to construct a closed sub-v-sheaf
$$\mathcal{M}_{\mathcal{G},\mu} \subset \mathrm{Gr}_{\mathcal{G}}|_{\mathrm{Spd}\, O_E} \tag{4.32}$$
prolonging the Schubert diamonds $\mathrm{Gr}_{G,\mu}$ which are the descent to $\mathrm{Spd}\, E$ of the ones we studied in the previous subsection.

Definition 4.11. Let μ be a conjugacy class of cocharacters in G_C. The local model $\mathcal{M}_{\mathcal{G},\mu}$ is the v-closure of $\mathrm{Gr}_{G,\mu}$ inside $\mathrm{Gr}_{\mathcal{G}}|_{\mathrm{Spd}\, O_E}$.

A priori $\mathcal{M}_{\mathcal{G},\mu}$ does not admit a moduli problem description for general parahoric $\mathcal{G}$, so its structure could be harder to parse. Let us give some examples where the local model $\mathcal{M}_{\mathcal{G},\mu}$ is relatively well understood.

Example 4.12. If $\mathcal{G}$ is reductive, then $\mathcal{M}_{\mathcal{G},\mu}$ is the integral Schubert variety over $\mathrm{Spd}\, O_E$ and generalizes the objects introduced in Section 4.1, see [SW20, Proposition 20.3.6] and [FS21, VI.1]. If $G = T$ is a torus, then the explicit description of $\mathrm{Gr}_{\mathcal{T}}$ furnishes an identity $\mathcal{M}_{\mathcal{T},\mu} = \mathrm{Spd}\, O_E$, see [SW20, Proposition 21.3.1].

We need to show permanence of the local model under the $L_O^+\mathcal{G}$-action.

Proposition 4.13. *The natural action map $L_O^+\mathcal{G} \times \mathrm{Gr}_{\mathcal{G}} \to \mathrm{Gr}_{\mathcal{G}}$ restricts, after base change to $\mathrm{Spd}\, O_E$, to a group action on the closed sub-v-sheaf $\mathcal{M}_{\mathcal{G},\mu}$. Moreover, the generic fiber of $\mathcal{M}_{\mathcal{G},\mu}$ is topologically dense.*

Proof. By embedding $\mathcal{G}$ in GL_n, we may always find a quasi-compact closed subsheaf $X \subset \mathrm{Gr}_{\mathcal{G},O_E}$ with $\mathcal{M}_{\mathcal{G},\mu} \subset X$, stable under $L^+_{O_E}\mathcal{G}$ whose action factors through a congruence quotient $L^{\leq N}_{O_E}\mathcal{G}$, where N is a sufficiently large positive integer (necessarily at least $\langle 2\rho, \mu\rangle$ as one sees by restricting to $\mathrm{Gr}_{G,\mu}$).

The structure map $L^{\leq N}_{O_E}\mathcal{G} \to \operatorname{Spd} O_E$ is partially proper, cohomologically smooth and consequently universally open, see [Sch17, Proposition 23.11]. By Corollary 2.9,

$$L^{\leq N}_{O_E}\mathcal{G} \times_{\operatorname{Spd} O_E} \mathcal{M}_{\mathcal{G},\mu} = (L^{\leq N}_{E} G \times_{\operatorname{Spd} E} \mathrm{Gr}_{G,\mu})^{\mathrm{cl}} \tag{4.33}$$

as closed sub-v-sheaves of $L^{\leq N}_{O_E}\mathcal{G} \times_{\operatorname{Spd} O_E} X$. Now, we have seen that $L^+_E G$ respects $\mathrm{Gr}_{G,\mu}$, so the multiplication map $L^{\leq N}_E G \times_{\operatorname{Spd} E} \mathrm{Gr}_{G,\mu} \to X$ factors through $\mathrm{Gr}_{G,\mu}$. This also implies that the integral action map

$$L^{\leq N}_{O_E}\mathcal{G} \times_{\operatorname{Spd} O_E} \mathcal{M}_{\mathcal{G},\mu} \to X \tag{4.34}$$

factors through the closure $\mathcal{M}_{\mathcal{G},\mu}$, as we desired.

For the last claim, we consider the restricted variant of the Hecke v-stack

$$\mathrm{Hk}^{\leq N}_{\mathcal{G},\mu} := \big[L^{\leq N}_{O_E}\mathcal{G} \backslash \mathcal{M}_{\mathcal{G},\mu}\big]. \tag{4.35}$$

Its underlying topological space has the extra special property that every subset is weakly generalizing, since for every perfectoid affinoid field (K, K^+) the Bruhat decomposition over $B_{\mathrm{dR}}(K)$ is insensitive to variation of K^+. Now, the projection map

$$\mathrm{pr}\colon \mathcal{M}_{\mathcal{G},\mu} \to \mathrm{Hk}^{\leq N}_{\mathcal{G},\mu} \tag{4.36}$$

is cohomologically smooth and consequently open. Therefore, by the same argument of Corollary 2.9, we see that both the usual topological and the weakly generalizing closure commute with pullback along $|\mathrm{pr}|$. It results that $|\mathrm{Gr}_{G,\mu}|$ is a dense open of $|\mathcal{M}_{\mathcal{G},\mu}|$. ☐

We can now prove the following structural properties of $\mathcal{M}_{\mathcal{G},\mu}$.

Proposition 4.14. *With notation as in Definition 2.36, $\mathcal{M}_{\mathcal{G},\mu} \in \mathcal{K}$. More specifically, the local model $\mathcal{M}_{\mathcal{G},\mu}$ is a flat π-adic kimberlite over O_E with enough facets over C and O_C-formalizable C-sections. Moreover, the special fiber*

$$\mathcal{M}_{\mathcal{G},\mu,k_E} := \mathcal{M}_{\mathcal{G},\mu}|_{\operatorname{Spd}(k_E)} \tag{4.37}$$

is of the form $\mathcal{F}\ell^{\diamondsuit}_{\mathcal{G},W}$ for a connected perfect Schubert scheme $\mathcal{F}\ell_{\mathcal{G},W}$. In particular, $\mathcal{M}^{\mathrm{red}}_{\mathcal{G},\mu} = \mathcal{F}\ell_{\mathcal{G},W}$ is perfectly proper and perfectly finitely presented perfect k-scheme.

Proof. Choosing a closed embedding $\mathcal{G} \hookrightarrow \mathrm{GL}_n$, we may find a cocharacter ν of GL_n giving rise to a closed immersion

$$\mathcal{M}_{\mathcal{G},\mu} \hookrightarrow \mathcal{M}_{\mathrm{GL}_n,\nu}, \tag{4.38}$$

so that local models are proper v-sheaves and, in particular, quasi-compact. By [Gle24, Proposition 4.41.(3), Proposition 2.2.5] to prove $\mathcal{M}_{\mathcal{G},\mu}$ is a kimberlite, it suffices to prove it is π-adic. By [Gle24, Proposition 3.32], we may reduce to proving that the special fiber of $\mathcal{M}_{\mathcal{G},\mu}$ is represented by a v-sheaf of the form $X^{\diamond}$ for X a perfect scheme.

By Proposition 4.13, $\mathcal{M}_{\mathcal{G},\mu,\bar{k}}$, is of the form $\cup_{i\in I}\mathcal{F}\ell^{\diamond}_{\mathcal{G},\bar{k},W_i}$ with finite subsets $W_i \subset \tilde{W}$, where we have used the fact that $\mathcal{F}\ell_{\mathcal{G},W}$ is perfectly proper, in order to deduce $\mathcal{F}\ell^{\diamondsuit}_{\mathcal{G},W} = \mathcal{F}\ell^{\diamond}_{\mathcal{G},W}$. By quasi-compactness, we get $\mathcal{M}_{\mathcal{G},\mu,k_E} = \mathcal{F}\ell^{\diamond}_{\mathcal{G},W}$ for some finite subset $W \subset \tilde{W}$, which finishes the proof that $\mathcal{M}_{\mathcal{G},\mu}$ is a π-adic kimberlite.

That $\mathcal{M}_{\mathcal{G},\mu}$ has O_C-formalizable C-sections follows from the main theorem of [Ans22]. Indeed, as in [Ans22, Section 12] any C-point of $\mathrm{Gr}_{\mathcal{G}}$ produces canonically a $\mathcal{G}$-torsor $\mathcal{P}$ over $\operatorname{Spec}(A_{\mathrm{inf}})$ together with a trivialization $\alpha : \mathcal{P} \dashrightarrow \mathcal{G}$ over $\operatorname{Spec}(A_{\mathrm{inf}})\backslash V(\xi)$. Any map $\operatorname{Spa}(R, R^+) \to \operatorname{Spd} O_C$ induces a map $f : \operatorname{Spec}(B_{\mathrm{dR}}(R^\sharp)) \to \operatorname{Spec}(A_{\mathrm{inf}})$ and the pullback $f^*\alpha \in \mathrm{Gr}_{\mathcal{G}}(R, R^+)$ provides a map functors $\operatorname{Spd} O_C \to \mathrm{Gr}_{\mathcal{G}}$ extending the original C-point. Since $\operatorname{Spd} C \subseteq \operatorname{Spd} O_C$ is dense and $\mathcal{M}_{\mathcal{G},\mu} \subseteq \mathrm{Gr}_{\mathcal{G}}$ is closed, if the original C-point factors through $\mathcal{M}_{\mathcal{G},\mu}$ then the corresponding O_C-point also does.

We explained in Corollary 4.7 that $\mathcal{M}_{\mathcal{G},\mu}$ has enough C-facets. Together with Proposition 4.13 and Lemma 2.38, flatness follows. By Proposition 2.35, $|\mathcal{F}\ell_{\mathcal{G},W}| = \mathrm{sp}(|\mathrm{Gr}_{G,\mu}|)$ and since $\mathrm{Gr}_{G,\mu}$ is connected $\mathcal{F}\ell_{\mathcal{G},W}$ is also connected. □

Remark 4.15. It follows that the base change $\mathcal{M}_{\mathcal{G},\mu}|_{\operatorname{Spd} O_C}$ is still topologically flat, and hence agrees with the v-closure $\mathcal{M}_{\mathcal{G},O_C,\mu}$ of $\mathrm{Gr}_{G,C,\mu}$ inside $\mathrm{Gr}_{\mathcal{G},O_C}$. Indeed, repeating the argument of Proposition 4.13 over $\operatorname{Spd} O_C$, we see that the special fiber of $\mathcal{M}_{\mathcal{G},O_C,\mu}$ is represented by a Schubert perfect scheme. But a $\operatorname{Spd} k$-valued point of $\mathcal{M}_{\mathcal{G},\mu}$ is a specialization of some $\operatorname{Spd} C$-valued point by Proposition 4.14, hence equality of both closures is clear.

Next, we analyse some functoriality behavior of $\mathcal{M}_{\mathcal{G},\mu}$ in the pair $(\mathcal{G},\mu)$. Here, by definition, a map $(\mathcal{G}_1,\mu_1) \to (\mathcal{G}_2,\mu_2)$ is a morphism of O-group schemes $\mathcal{G}_1 \to \mathcal{G}_2$ such that the image of μ_1 in $G_{2,C}$ lies in the same conjugacy class as μ_2.

Proposition 4.16. *The association $(\mathcal{G},\mu) \mapsto \mathcal{M}_{\mathcal{G},O_C,\mu}$, see Remark 4.15, is functorial, preserves closed embeddings and direct products, and induces isomorphisms $\mathcal{M}_{\mathcal{G},O_C,\mu} \cong \mathcal{M}_{\mathcal{G}_{\mathrm{ad}},O_C,\mu_{\mathrm{ad}}}$, where μ_{ad} denotes the composite of μ with $G_C \to G_{\mathrm{ad},C}$.*

Proof. Functoriality follows from that of $\mathrm{Gr}_{G,C,\mu}$ and the definition using v-closures, see Remark 4.15. For the claim regarding closed embeddings and central extensions, we refer to [Lou20, IV, Proposition 4.16, Corollary 4.17]: one checks injectivity at geometric points, using Lemma 4.17. As for direct products, it suffices to check equality at the level of $\operatorname{Spd} k$-valued points by Proposition 4.14, and this is easy because the generic fiber was already a product. □

Lemma 4.17. *The specialization map induces a bijection*

$$\pi_0(\mathrm{Gr}_{\mathcal{G},\breve{O}}) \xrightarrow{\cong} \pi_0(\mathcal{F}\ell_{\mathcal{G},\bar{k}}) \cong \pi_1(G)_I, \tag{4.39}$$

where I is the absolute Galois group of $\breve{F}$.

Proof. For the final bijection, see [Zhu17a, Proposition 1.21]. The first is a consequence of proper base change [Sch17, Theorem 19.2, Remark 19.3] applied to $f\colon \mathrm{Gr}_{\mathcal{G},\breve{O}} \to \operatorname{Spd} \breve{O}$ and the base change $i\colon \operatorname{Spd} \bar{k} \to \operatorname{Spd} \breve{O}$, using that the 0-th cohomology group computes connected components: for some coefficient ring, say, $\Lambda = \mathbb{Z}/\ell$ with $\ell \neq p$, we apply the proper base change $i^* R^0 f_* \Lambda_X \cong R^0(f')_*(i')^*\Lambda_X$ where Λ_X is the constant sheaf supported on increasing closed $\breve{O}$-proper sub-v-sheaves $X \subset \mathrm{Gr}_{\mathcal{G},\breve{O}}$. Passing to global sections, the second computes $\Lambda^{\pi_0(X_{\bar{k}})}$ by definition whereas the first computes $\Lambda^{\pi_0(X)}$ by using the v-cover $\operatorname{Spd} O_C \to \operatorname{Spd} \breve{O}$. Finally, we use [Sch17, Section 27] to pass between $\mathcal{F}\ell^{\diamondsuit}_{\mathcal{G},\bar{k}}$ and $\mathcal{F}\ell_{\mathcal{G},\bar{k}}$. □

5. Geometry of multiplicative group actions

Our approach to the Scholze–Weinstein conjecture requires determining the special fiber of local models in terms of admissible loci. We follow the general strategy of Haines and the fourth named author [HR21] of calculating the support of nearby cycles using hyperbolic localization. This requires translating the results from [HR21, Section 5] to the v-sheaf Beilinson-Drinfeld Grassmannian. For basic facts pertaining to $\mathbb{G}_m^{\diamondsuit}$-actions on small v-stacks, the reader is referred to [FS21, Chapter IV.6].

5.1. **Over O.** As in Section 4.1, we continue to fix a complete discretely valued field $F/\mathbb{Q}_p$ with ring of integers O and perfect residue field k, a complete algebraic closure C/F and a connected reductive F-group G with parahoric model $\mathcal{G}$ over O containing the connected Néron model $\mathcal{S}$ of the maximally F-split maximal $\breve{F}$-split F-torus S.

Fix a cocharacter $\lambda\colon \mathbb{G}_m \to \mathcal{S} \subset \mathcal{G}$ defined over O. After base change to F, this induces a Levi $M = M_\lambda$ with Lie algebra $\operatorname{Lie} M = (\operatorname{Lie} G)_{\lambda=0}$, a parabolic subgroup $P = P_\lambda^+$ with Lie algebra $\operatorname{Lie} P = (\operatorname{Lie} G)_{\lambda\geq 0}$ and an unipotent subgroup $U = U_\lambda^+$ with Lie algebra $\operatorname{Lie} U = (\operatorname{Lie} G)_{\lambda>0}$ fitting in a semi-direct product decomposition $P = M \ltimes U$. Since λ is defined over O, the decomposition $P = M \ltimes U$ extends to O-models $\mathcal{P} = \mathcal{M} \ltimes \mathcal{U}$, admitting analogous descriptions for their Lie algebras and being equipped with homomorphisms

$$\mathcal{M} \longleftarrow \mathcal{P} \longrightarrow \mathcal{G}. \tag{5.1}$$

The O-group schemes $\mathcal{P}$, $\mathcal{M}$, $\mathcal{U}$ are smooth affine with connected fibers, and $\mathcal{M}$ is a parahoric O-model of the Levi subgroup M, see [HR21, Lemma 4.5] and also [CGP15, Section 2.1], [KP23, Section 6.2] for proofs of these claims.

By functoriality, (5.1) induces maps of ind-(spatial $\operatorname{Spd} O$-diamonds)

$$\mathrm{Gr}_{\mathcal{M}} \longleftarrow \mathrm{Gr}_{\mathcal{P}} \longrightarrow \mathrm{Gr}_{\mathcal{G}}, \tag{5.2}$$

where $\mathrm{Gr}_{\mathcal{M}} \to \operatorname{Spd} O$ and $\mathrm{Gr}_{\mathcal{G}} \to \operatorname{Spd} O$ are ind-proper by Theorem 4.9. On the other hand, the cocharacter λ induces a cocharacter

$$\mathbb{G}_m^\diamond \xrightarrow{[\text{-}]} L_O^+\mathbb{G}_m \xrightarrow{L_O^+\lambda} L_O^+\mathcal{G}, \tag{5.3}$$

where [-] denotes the Teichmüller lift. Thus, we obtain a left action of $\mathbb{G}_m^\diamond$ on $\mathrm{Gr}_{\mathcal{G}}$.

Lemma 5.1. *The $\mathbb{G}_m^\diamond$-action on $\mathrm{Gr}_{\mathcal{G}}$ satisfies* [FS21, Hypothesis IV.6.1].

Proof. Choosing a closed immersion $\mathcal{G} \hookrightarrow \mathrm{GL}_{n,O}$ of group schemes, we reduce to the case $\mathcal{G} = \mathrm{GL}_{n,O}$, using Theorem 4.9 to see that the induced map $\mathrm{Gr}_{\mathcal{G}} \hookrightarrow \mathrm{Gr}_{\mathrm{GL}_{n,O}}$ is a closed immersion. Then the lemma is a special case of [FS21, Proposition VI.3.1]. □

Consequently, we obtain a $\mathbb{G}_m^\diamond$-equivariant diagram

$$(\mathrm{Gr}_{\mathcal{G}})^0 \longleftarrow (\mathrm{Gr}_{\mathcal{G}})^+ \longrightarrow \mathrm{Gr}_{\mathcal{G}}, \tag{5.4}$$

where $(\mathrm{Gr}_{\mathcal{G}})^0 = (\mathrm{Gr}_{\mathcal{G}})^{\mathbb{G}_m^\diamond}$ denotes the fixed points and $(\mathrm{Gr}_{\mathcal{G}})^+$ the attractor classifying $\mathbb{G}_m^\diamond$-equivariant maps $(\mathbb{A}^1)^{\diamond,+} \to \mathrm{Gr}_{\mathcal{G}}$ over $\operatorname{Spd} O$, see also (6.3) below. The map $(\mathrm{Gr}_{\mathcal{G}})^+ \to (\mathrm{Gr}_{\mathcal{G}})^0$ is the Białynicki-Birula map given by evaluating at the zero section. Our aim is to understand the relation of (5.2) with (5.4). The following result is the analogue of [HR21, Theorem 5.6, Theorem 5.19] in the context of ind-schemes:

Theorem 5.2. *The maps* (5.2) *and* (5.4) *fit into a commutative diagram of ind-(spatial* $\operatorname{Spd} O$*-diamonds)*

$$\begin{tikzcd} \mathrm{Gr}_{\mathcal{M}} \arrow[d, "\iota^0"'] & \mathrm{Gr}_{\mathcal{P}} \arrow[l] \arrow[r] \arrow[d, "\iota^+"'] & \mathrm{Gr}_{\mathcal{G}} \arrow[d, "\mathrm{id}"'] \\ (\mathrm{Gr}_{\mathcal{G}})^0 & (\mathrm{Gr}_{\mathcal{G}})^+ \arrow[l] \arrow[r] & \mathrm{Gr}_{\mathcal{G}}, \end{tikzcd} \tag{5.5}$$

with the following properties.

1. *The maps ι^0, ι^+ are open and closed immersions.*
2. *Their base changes ι^0_F, ι^+_F are isomorphisms.*
3. *If $\mathcal{G}_{\breve{O}}$ is very special parahoric (for example, reductive), then ι^0, ι^+ are isomorphisms.*

In particular, the complements $(\mathrm{Gr}_{\mathcal{G}})^0 \setminus \iota^0(\mathrm{Gr}_{\mathcal{M}})$, $(\mathrm{Gr}_{\mathcal{G}})^+ \setminus \iota^+(\mathrm{Gr}_{\mathcal{P}})$ are concentrated over $\operatorname{Spd} k$*, that is, their base change to* $\operatorname{Spd} F$ *is empty.*

If $\mathcal{G}$ is reductive, then Theorem 5.2 is proved in [FS21, Proposition VI.3.1]. Indeed, in this case ι^0, ι^+ are isomorphisms. In general, we follow the strategy of [HR21]:

Proof of Theorem 5.2. First, we construct the maps ι^0, ι^+. The closed subgroup $\mathcal{M} \hookrightarrow \mathcal{G}$ induces a closed immersion $\mathrm{Gr}_{\mathcal{M}} \hookrightarrow \mathrm{Gr}_{\mathcal{G}}$ (using Theorem 4.9) that is $\mathbb{G}_m^\diamondsuit$-equivariant for the trivial action on the source. Thus, the map factors though the fixed points, defining the necessarily closed immersion $\iota^0 \colon \mathrm{Gr}_{\mathcal{M}} \hookrightarrow (\mathrm{Gr}_{\mathcal{G}})^0$. The construction of ι^+ is more delicate and proceeds as follows. Pick a closed immersion $\mathcal{G} \hookrightarrow \mathrm{GL}_{n,O}$ of O-group schemes such that the fppf quotient $\mathrm{GL}_{n,O}/\mathcal{G}$ is quasi-affine, see [PR08, Proposition 1.3]. The cocharacter $\lambda' \colon \mathbb{G}_m \xrightarrow{\lambda} \mathcal{G} \hookrightarrow \mathrm{GL}_{n,O}$ induces parabolic subgroup $\mathcal{P}'$ with Lie algebra $\mathrm{Lie}\,\mathcal{P}' = (\mathrm{Lie}\,\mathrm{GL}_{n,O})_{\lambda' \geq 0}$. The induced map on fppf quotients $\mathcal{P}'/\mathcal{P} \to \mathrm{GL}_{n,O}/\mathcal{G}$ is a monomorphism between finite type O-schemes, thus quasi-affine by Zariski's main theorem. By functoriality of the construction $\mathcal{G} \mapsto \mathrm{Gr}_{\mathcal{G}}$, we obtain a commutative diagram of ind-(spatial $\mathrm{Spd}\,O$-diamonds):

$$\begin{tikzcd}
\mathrm{Gr}_{\mathcal{P}} \arrow[d] \arrow[dr, dotted, "\iota^+"'] \arrow[rr, bend left] & & \\
& (\mathrm{Gr}_{\mathcal{G}})^+ \arrow[r] \arrow[d] & \mathrm{Gr}_{\mathcal{G}} \arrow[d] \\
\mathrm{Gr}_{\mathcal{P}'} \arrow[r, "\cong"] & (\mathrm{Gr}_{\mathrm{GL}_{n,O}})^+ \arrow[r] & \mathrm{Gr}_{\mathrm{GL}_{n,O}}
\end{tikzcd} \tag{5.6}$$

Using Theorem 4.9, the map $\mathrm{Gr}_{\mathcal{G}} \to \mathrm{Gr}_{\mathrm{GL}_{n,O}}$ is a closed immersion so that the square is Cartesian. This proves the existence of ι^+. Furthermore, the displayed map $\mathrm{Gr}_{\mathcal{P}'} \to (\mathrm{Gr}_{\mathrm{GL}_{n,O}})^+$ is an isomorphism by [FS21, Proposition VI.3.1]. As $\mathcal{P}'/\mathcal{P}$ is quasi-affine, so $\mathrm{Gr}_{\mathcal{P}} \to \mathrm{Gr}_{\mathcal{P}'}$ is a locally closed immersion (compare with the proof of [Zhu17b, Proposition 1.2.6] and [SW20, Lemma 19.1.5]), the map ι^+ is necessarily a locally closed immersion as well.

Now, part (2) is immediate from our construction and [FS21, Proposition VI.3.1] applied over $\mathrm{Spd}\,F$. For part (3), we observe that ι^0, ι^+ are bijective on geometric points if $\mathcal{G}$ is very special parahoric: by (2) for geometric points lying over $\mathrm{Spd}\,F$ and by the Iwasawa decomposition [KP23, Section 3.3] for geometric points lying over $\mathrm{Spd}\,k$. As ι^0, ι^+ are locally closed immersions, they must be isomorphisms, so (3) follows.

For (1), it remains to prove that ι^0, ι^+ are open immersions for general parahoric group schemes $\mathcal{G}$. For this, we may and do assume that k is algebraically closed. There are bijections of connected components

$$\pi_0((\mathrm{Gr}_{\mathcal{G}})^+) \xrightarrow{\cong} \pi_0((\mathrm{Gr}_{\mathcal{G}})^0) \xrightarrow{\cong} \pi_0((\mathcal{F}\ell_{\mathcal{G}})^0), \tag{5.7}$$

where the first holds by general properties of Białynicki-Birula maps (see the proof of [Ric19, Corollary 1.12]) and the second by proper base change as in the proof of Lemma 4.17. The fixed points $(\mathcal{F}\ell_{\mathcal{G}})^0$ in the Witt vector partial affine flag variety can be analyzed in analogy to [HR21, Section 4]: concretely, if $\mathcal{P}_{\mathrm{sc}} = \mathcal{M}_{\mathrm{sc}} \ltimes \mathcal{U}$ for $\mathcal{M}_{\mathrm{sc}}$ being the corresponding parahoric model of M_{sc}, then there is a disjoint union (on points) into connected locally closed sub-ind-schemes

$$\mathcal{F}\ell_{\mathcal{G}} = \bigcup_{[w]} \mathcal{S}_w, \ \mathcal{S}_w = L_k\mathcal{P}_{\mathrm{sc}} \cdot w \tag{5.8}$$

where $[w]$ runs through the double coset $W_{M,\mathrm{af}} \backslash \tilde{W}/W_{\mathcal{G}}$ and w denotes the image of a representative under the embedding $\tilde{W}/W_{\mathcal{G}} \hookrightarrow \mathcal{F}\ell_{\mathcal{G}}$. The image of $\mathcal{F}\ell_{\mathcal{P}} \hookrightarrow \mathcal{F}\ell_{\mathcal{G}}$ consists of those $\mathcal{S}_w$ for $[w]$ lying in $W_{M,\mathrm{af}} \backslash \tilde{W}_M/W_{\mathcal{M}}$. Passing to fixed points, the image of $\mathcal{F}\ell_{\mathcal{M}} \hookrightarrow (\mathcal{F}\ell_{\mathcal{G}})^0$ is the union of the $L_k\mathcal{M}_{\mathrm{sc}}$-orbits for these $[w]$. So the map $\pi_0(\mathcal{F}\ell_{\mathcal{M}}) \to \pi_0((\mathcal{F}\ell_{\mathcal{G}})^0)$ identifies with the injection

$$W_{M,\mathrm{af}} \backslash \tilde{W}_M/W_{\mathcal{M}} \hookrightarrow W_{M,\mathrm{af}} \backslash \tilde{W}/W_{\mathcal{G}}. \tag{5.9}$$

We let $\mathcal{C}_{\mathcal{G}}^0$, respectively $\mathcal{C}_{\mathcal{G}}^+$, be the open and closed sub-v-sheaf of $(\mathrm{Gr}_{\mathcal{G}})^0$, respectively of $(\mathrm{Gr}_{\mathcal{G}})^+$, consisting of those components belonging to $\mathrm{im}(\pi_0(\mathcal{F}\ell_{\mathcal{M}}) \hookrightarrow \pi_0(\mathcal{F}\ell_{\mathcal{G}}))$ under (5.7). Then the maps ι^0, ι^+ factor through $\mathcal{C}_{\mathcal{G}}^0$, respectively $\mathcal{C}_{\mathcal{G}}^+$ inducing locally closed immersions

$$\mathrm{Gr}_{\mathcal{M}} \hookrightarrow \mathcal{C}_{\mathcal{G}}^0,\ \mathrm{Gr}_{\mathcal{P}} \hookrightarrow \mathcal{C}_{\mathcal{G}}^+, \tag{5.10}$$

that are bijective on geometric points, hence isomorphisms. The theorem is thus proven. $\square$

5.2. **Semi-infinite orbits.** We end this section with a study of the stratification (5.8). Throughout, we assume that $k = \bar{k}$ is algebraically closed so that $F = \breve{F}$ and $O = \breve{O}$. Note that the torus S is then maximal F-split. The following lemma simplifies some arguments of [HR21, Theorem 6.12] and is used in the proof of Theorem 6.16 given in Section 6.5.

Lemma 5.3. *For every $w \in \tilde{W}/W_{\mathcal{G}}$, there is an O-cocharacter $\mathbb{G}_m \to \mathcal{S} \subset \mathcal{G}$ such that for the induced strata $\mathcal{S}_w \cap \mathcal{F}\ell_{\mathcal{G},w} = \{w\}$.*

Proof. Up to changing the Iwahori $L_k^+\mathcal{I} \subset L_k^+\mathcal{G}$, we may and do assume that the Iwahori–Schubert variety $\mathcal{F}\ell^{\circ}_{(\mathcal{I},\mathcal{G}),w}$ is a dense open of $\mathcal{F}\ell_{\mathcal{G},w}$. Notice that the closed complement of that dense open is stable under the $\mathbb{G}_m^{\mathrm{perf}}$-action, so it follows that the connected component of the fixed point w in the attractor $\mathcal{F}\ell_{\mathcal{G}}^+$ cuts $\mathcal{F}\ell_{\mathcal{G},w}$ inside $\mathcal{F}\ell^{\circ}_{(\mathcal{I},\mathcal{G}),w}$.

The reduced word $\dot{w} = s_1 \dots s_n$ determines a minimal gallery $\Gamma = (\mathbf{a}_0, \mathbf{a}_1, \dots, \mathbf{a}_n)$, where $\mathbf{a}_i = s_1 \dots s_i(\mathbf{a})$, going from the alcove $\mathbf{a}$ fixed by $\mathcal{I}(O)$ to its $\dot{w}$-conjugate. Let α_i be the unique positive affine root such that $\partial\alpha_i$ is the wall separating $\mathbf{a}_{i-1}$ and $\mathbf{a}_i$. We claim that

$$\mathcal{F}\ell^{\circ}_{(\mathcal{I},\mathcal{G}),w} = L_k^+\mathcal{U}_{\alpha_1} \cdot \dots \cdot L_k^+\mathcal{U}_{\alpha_n} w. \tag{5.11}$$

This follows by expanding the Demazure twisted product, pulling across the simple reflections to the right, compare with Proposition 4.5. Indeed, $s_{i-1} \dots s_1(\alpha_i)$ is by construction the positive simple affine root attached to s_i. We need to produce an O-cocharacter $\mathbb{G}_m \to \mathcal{S}$ whose induced $\mathbb{G}_m^{\mathrm{perf}}$-action repels every affine root group $L^+\mathcal{U}_{\alpha_i}$, because then it would also repel $\mathcal{F}\ell^{\circ}_{(\mathcal{I},\mathcal{G}),w}$ by (5.11). This follows now by [HN02, Corollary 5.6] but we give below a quick proof for the reader's convenience.

Consider the subset $\Phi_{G,w} \subset \Phi_G$ of all euclidean roots $a = \nabla\alpha$ which are gradients of the prescribed affine root α_i attached to Γ. If b denotes the barycenter of $\mathbf{a}$, then by definition $a(\dot{w}b - b) < 0$ is stricly negative for $a \in \Phi_{G,w}$. In particular, $\Phi_{G,w}$ consists entirely of negative B-roots where $S \subset B \subset G$ is a Borel subgroup whose closed Weyl chamber contains the vector $\dot{w}b - b$. So we may take any B-dominant regular coweight $\mathbb{G}_m \to S$ which uniquely extends to the desired $\mathbb{G}_m \to \mathcal{S}$ because S is F-split. $\square$

Now assume that λ is regular. Then $M_\lambda = T$ is a maximal torus, $P_\lambda = B$ a Borel subgroup with unipotent radical U defined over F. The stratification (5.8) becomes

$$\mathcal{F}\ell_{\mathcal{G}} = \bigcup_{w \in \tilde{W}/W_{\mathcal{G}}} \mathcal{S}_w,\ \mathcal{S}_w = L_k U \cdot w \tag{5.12}$$

and the strata $\mathcal{S}_w$ are called semi-infinite orbits, compare with [FS21, Proposition VI.3.1]. Recall that there is a semi-infinite Bruhat order $\leq^{\frac{\infty}{2}}$ on $\tilde{W}/W_{\mathcal{G}}$ defined by:

$$v \leq^{\frac{\infty}{2}} w \quad \Longleftrightarrow \quad \forall\, \nu_I \gg 0\colon\ \nu_I(\pi) \cdot v \leq \nu_I(\pi) \cdot w \tag{5.13}$$

where $X_*(T)_I \subset \tilde{W}$, $\nu_I \mapsto \nu_I(\pi)$ is viewed as a subgroup using the Kottwitz morphism, see (3.19), and where $\nu_I \gg 0$ means that ν_I is sufficiently B-dominant. This order was first introduced by Lusztig in [Lus80] and depends on the F-Borel subgroup B attracted by λ.

Proposition 5.4. *The ind-closure of $\mathcal{S}_w$ inside $\mathcal{F}\ell_{\mathcal{G}}$ is given by the perfect sub-ind-scheme whose geometric points factor through some $\mathcal{S}_v$ with $v \leq^{\frac{\infty}{2}} w$.*

Proof. Let $\mathcal{I} \to \mathcal{G}$ be an auxiliary Iwahori model, fixed for the remainder of the proof. Suppose there is a curve $\mathcal{C} \subset \mathcal{F}\ell_{\mathcal{G}}$ containing v and whose complement $\mathcal{C}^\circ := \mathcal{C} \setminus \{v\}$ is contained in $\mathcal{S}_w = L_k U \cdot w$. Now, notice that, for sufficiently dominant ν_I, one gets the inclusion

$$\nu_I(\pi) \cdot \mathcal{C}^\circ \subset L_k^+\mathcal{I} \cdot \nu_I(\pi) \cdot w, \tag{5.14}$$

because conjugation by $\nu_I(\pi)$ moves any given perfect k-subscheme of $L_k U$ inside the Iwahori loop group $L_k^+\mathcal{I} \subset L_k^+\mathcal{G}$. This implies the inequality $\nu_I(\pi)v \leq \nu_I(\pi)w$, and therefore $v \leq^{\frac{\infty}{2}} w$.

Conversely, assume that the inequality $v \leq^{\frac{\infty}{2}} w$ holds. By definition, for all sufficiently dominant translations $\nu_I \in X_*(T)_I$, the inequality $\nu_I(\pi) \cdot w \leq \nu_I(\pi) \cdot v$ holds in the Bruhat order. By enlarging ν_I if necessary, we may assume the $\mathcal{F}\ell^\circ_{(\mathcal{I},\mathcal{G}),\nu_I(\pi)\cdot w}$ is of the form $\prod_{\alpha\in\Gamma} L_k^+\mathcal{U}_\alpha \cdot \nu_I(\pi) \cdot w$ where all of the $\alpha \in \Gamma$ have positive gradient. There is a curve $\mathcal{C}$ in $\mathcal{F}\ell_{(\mathcal{I},\mathcal{G}),\nu_I(\pi)\cdot w}$ joining $\nu_I(\pi) \cdot w$ to $\nu_I(\pi) \cdot v$ since $\nu_I(\pi) \cdot v \leq \nu_I(\pi) \cdot w$. By our assumption on $\nu_I(\pi)$, we have $\mathcal{C}^\circ \subset \mathcal{S}_{\nu_I(\pi)\cdot w}$ for $\mathcal{C}^\circ := \mathcal{C} \cap \mathcal{F}\ell^\circ_{(\mathcal{I},\mathcal{G}),\nu_I(\pi)\cdot w}$. Now, the map $t_{\nu_I} \colon \mathcal{F}\ell_{(\mathcal{I},\mathcal{G}),w} \to \mathcal{F}\ell_{(\mathcal{I},\mathcal{G}),\nu_I(\pi)\cdot w}$ induced by left translation with $\nu_I(\pi)$ is an isomorphism and hence induces $\mathcal{S}_w \cong \mathcal{S}_{\nu_I(\pi)\cdot w}$. Then the curve $t_{\nu_I}^{-1}(\mathcal{C})$ joins w to v, and $t_{\nu_I}^{-1}(\mathcal{C}^\circ) \subset \mathcal{S}_w$. □

Next, we extend the equi-dimensionality of Mirković–Vilonen cycles [MV07, Theorem 3.2] (see also [Zhu17a, Corollary 2.8] and [FS21, Corollary VI.3.8]) from split groups to twisted groups as follows. We continue to assume that λ is regular and, additionally, that $\mathcal{G}$ is special parahoric. Then $X_*(T)_I = \tilde{W}/W_{\mathcal{G}}$ and (5.12) becomes

$$\mathcal{F}\ell_{\mathcal{G}} = \bigcup_{\nu_I \in X_*(T)_I} \mathcal{S}_{\nu_I},\ \mathcal{S}_{\nu_I} = L_k U \cdot \nu_I(\pi). \tag{5.15}$$

If $\mathcal{G}$ is reductive, then $\mathcal{F}\ell_{\mathcal{G}}$ is the Witt vector affine Grassmannian studied in [Zhu17a]. So, in general, $\mathcal{F}\ell_{\mathcal{G}}$ can be regarded as a twisted version when $\mathcal{G}$ is special parahoric. Indeed, $\mathcal{F}\ell_{\mathcal{G}} = \operatorname{colim} \mathcal{F}\ell_{\mathcal{G},\mu_I}$ where μ_I runs through $X_*(T)_{I,+}$, the image of the B-dominant cocharacters under the projection $X_*(T) \to X_*(T)_I$ equipped with the induced dominance order and $\mathcal{F}\ell_{\mathcal{G},\mu_I} := \mathcal{F}\ell_{\mathcal{G},\mu_I(\pi)}$, as is usual notation for (twisted) affine Grassmannians. Also, the semi-infinite Bruhat order on $X_*(T)_I$ specializes to the dominance relation, that is, $\nu_I' \leq^{\frac{\infty}{2}} \nu_I$ if and only if $\nu_I - \nu_I'$ is a sum of coinvariants of positive roots with non-negative coefficients.

Lemma 5.5. *For any $\nu_I \in X_*(T)_I$, the intersection $\mathcal{S}_{\nu_I} \cap \mathcal{F}\ell_{\mathcal{G},\mu_I}$ is non-empty if and only if ν_I lies in the $W_{\mathcal{G}}$-orbit of some $\mu_I' \in X_*(T)_{I,+}$ with $\mu_I' \leq \mu_I$. In this case, it is affine and equidimensional of dimension $\langle \rho_G, \nu + \mu \rangle$.*

Here $\rho_G \in X^*(T)$ denotes the half sum of the B-positive roots. We note that the pairing $\langle \rho_G, \nu + \mu \rangle$ is well-defined independently of the choice of lifts $\nu, \mu \in X_*(T)$ of ν_I, μ_I because ρ_G is I-invariant.

Proof of Lemma 5.5. The map $(\mathcal{F}\ell_{\mathcal{G},\mu_I})^+ \to \mathcal{F}\ell_{\mathcal{G},\mu_I}$ induces an isomorphism (see Theorem 5.2)

$$(\mathcal{F}\ell_{\mathcal{G},\mu_I})^+ \xrightarrow{\cong} \bigsqcup_{\nu_I \in X_*(T)_I} \mathcal{S}_{\nu_I} \cap \mathcal{F}\ell_{\mathcal{G},\mu_I}. \tag{5.16}$$

Under $(\mathcal{F}\ell_{\mathcal{G},\mu_I})^+ \to (\mathcal{F}\ell_{\mathcal{G},\mu_I})^0$, the component $\mathcal{S}_{\nu_I} \cap \mathcal{F}\ell_{\mathcal{G},\mu_I}$ contracts to $\underline{\{\nu_I\}} = \operatorname{Spec}(k)$. Thus, it is affine because Białynicki-Birula maps for schemes are affine by [Ric19, Corollary 1.12]. Also, $(\mathcal{F}\ell_{\mathcal{G},\mu_I})^0$ identifies with the constant scheme associated with the subset of $\nu_I \in X_*(T)_I$ lying in the $W_{\mathcal{G}}$-orbit of some $\mu_I' \in X_*(T)_{I,+}$ with $\mu_I' \leq \mu_I$. So only such ν_I contribute to (5.16), and as the Białynicki-Birula map has a section, the non-emptiness criterion for $\mathcal{S}_{\nu_I} \cap \mathcal{F}\ell_{\mathcal{G},\mu_I}$ holds true. Furthermore, the union over all $\mathcal{S}_{\nu_I'} \cap \mathcal{F}\ell_{\mathcal{G},\mu_I}$ with $\nu_I' \leq^{\frac{\infty}{2}} \nu_I$ is a closed perfect subscheme by Proposition 5.4.

As noted in [FS21, Corollary VI.3.8], the affineness implies the dimension formula once we show that $\mathcal{S}_{\mu_I^{\mathrm{anti}}} \cap \mathcal{F}\ell_{\mathcal{G},\mu_I}$ is a point where μ_I^{anti} is the antidominant element in the $W_{\mathcal{G}}$-orbit of μ_I. This follows from the proof of Lemma 5.3. □

6. Nearby cycles of étale sheaves

6.1. **Recollections.** In [Sch17], Scholze constructs a category of étale sheaves

$$\mathrm{D}(X,\Lambda) := \mathrm{D}_{\text{ét}}(X,\Lambda) \tag{6.1}$$

for all small v-stacks X. As coefficients Λ, we allow prime-to-p torsion rings or, by the adic formalism of [Sch17, Section 26], an ℓ-torsion free, complete ℓ-adic ring for $\ell \neq p$, or a ring of the form $\Lambda = \Lambda_0[\ell^{-1}]$ where Λ_0 is as in the previous case. In the final case, as this is not covered in [Sch17], we define the triangulated category

$$\mathrm{D}(X,\Lambda) := \mathrm{D}(X,\Lambda_0) \otimes_{\Lambda_0} \Lambda, \tag{6.2}$$

in analogy to the classical definition for schemes, for example, see [KW01, Appendix A]. The adic formalism of [Sch17, Section 26] carries over to the categories (6.2). Finally, we also allow Λ to be a filtered colimit of the aforementioned rings, with the obvious definition for the categories. This includes algebraic field extensions $L/\mathbb{Q}_\ell$ and their rings of integers O_L.

The categories of étale sheaves are equipped with the usual six functors formalism: the endofunctors $\otimes^{\mathbb{L}}$, $R\mathscr{H}om$ and functors Rf_*, f^* for a morphism $f\colon X \to Y$ of small v-stacks. If f is compactifiable and representable in locally spatial diamonds with $\dim.\mathrm{tr} f < \infty$, we have the functors $Rf_!$, $Rf^!$, completing the six functor formalism.

In general, the categories $\mathrm{D}(X,\Lambda)$ and the six functors are rather inexplicit, constructed through v-descent using Lurie's ∞-categorical machinery. Nevertheless, whenever $f\colon X \to Y$ is a morphism between locally spatial diamonds, then X and Y admit a well-defined étale site and Scholze's operations are very closely related to the operations that one can construct site-theoretically, see [Sch17, Proposition 14.15, Section 17].

When X and Y are locally spatial diamonds we say that an object $A \in \mathrm{D}(X,\Lambda)$ is ULA (=universally locally acyclic) with respect to f if, for all locally spatial diamonds $Y' \to Y$, the pullback $A' \in \mathrm{D}(X',\Lambda)$ is overconvergent along the fibers of $f' : X' = X \times_Y Y' \to Y'$ and $R(f' \circ j')_! j'^* A$ is perfect-constructible for all separated étale neighborhoods $j'\colon U' \to X'$ for which $f' \circ j'$ is quasi-compact, see [FS21, Definition IV.2.1]. If Λ is ℓ-adic as above, then a complex $A \in \mathrm{D}(X,\Lambda)$ is called perfect-constructible if $A \otimes^{\mathbb{L}}_\Lambda \Lambda/\ell$ is étale locally perfect-constant after passing to a constructible stratification, equivalently $A \otimes^{\mathbb{L}}_\Lambda \Lambda/\ell^n$ are so for all $n \geq 1$. Finally, if $\Lambda = \Lambda_0[\ell^{-1}]$ is as in (6.2), then an object in $\mathrm{D}(X,\Lambda)$ is called perfect-constructible if it admits a Λ_0-lattice which is so. For X and Y more general v-stacks (and $f\colon X \to Y$ representable in locally spatial diamonds), we call A ULA if it is ULA after any base change $S \to Y$ with S a locally spatial diamond.

Suppose X is a small v-stack proper and representable in spatial diamonds over a base S, and that X is equipped with an action by $\mathbb{G}_{m,S}^\diamondsuit$ satisfying the conditions [FS21, Hypothesis IV.6.1.]. One can consider the v-stacks

$$X^{\pm} = \mathrm{Hom}_{\mathbb{G}_m^\diamondsuit}\big((\mathbb{A}^1)^{\pm,\diamondsuit}, X\big) \tag{6.3}$$

which (by hypothesis) are represented by a finite partition of X into locally closed subsets. This also induces a partition of the fixed-point v-stack $X^0 = X^{\mathbb{G}_m^\diamondsuit}$ into closed and open subsets. We have inclusion maps $q^{\pm}\colon X^{\pm} \to X$ and projection maps $p^{\pm}\colon X^{\pm} \to X^0$, from that we obtain the hyperbolic localization functor

$$L_{X/S}\colon \mathrm{D}(X/\mathbb{G}_{m,S}^\diamondsuit,\Lambda) \to \mathrm{D}(X^0,\Lambda), \tag{6.4}$$

which can be expressed as $R(p^+)_!(q^+)^*$ or equivalently as $R(p^-)_*R(q^-)^!$ by [FS21, Theorem IV.6.5]. This functor enjoys many compatibilities, in analogy to [Ric19], which we will exploit to compute nearby cycles, see [FS21, Propositions IV.6.12, IV.6.13, IV.6.14].

6.2. **Over C.** We continue with the notation and denote by $F/\mathbb{Q}_p$ a complete discretely valued field with ring of integers O and perfect residue field k of characteristic $p > 0$. Also, we fix a complete algebraic closure C/F, and a connected reductive F-group G.

In this section, we recall the structure of the categories of monodromic sheaves with bounded support $\mathrm{D}(\mathrm{Hk}_{G,C}, \Lambda)^{\mathrm{bd}}$ and $\mathrm{D}(\mathrm{Gr}_{G,C}, \Lambda)^{\mathrm{mon,bd}}$ studied in [FS21, Section VI]. As in Section 5, for any cocharacter $\lambda\colon \mathbb{G}_m \to G_C$, we have the induced $\mathbb{G}_m^{\diamondsuit}$-action on $\mathrm{Gr}_{G,C}$, whose attractors only depend on the attracting parabolic $P \subset G_C$.

In particular, hyperbolic localization gives a constant terms functor

$$\mathrm{CT}_P\colon \mathrm{D}(\mathrm{Hk}_{G,C}, \Lambda)^{\mathrm{bd}} \to \mathrm{D}(\mathrm{Gr}_{G,C}, \Lambda)^{\mathrm{mon,bd}} \xrightarrow{L_{\mathrm{Gr}_G/C}} \mathrm{D}(\mathrm{Gr}_{M,C}, \Lambda) \tag{6.5}$$

providing the main tool to effectively study the category of derived étale sheaves on $\mathrm{Hk}_{G,C}$ as in [FS21, Corollary VI.3.5]. One of the crucial techniques is the following conservativity lemma [FS21, Proposition VI.4.2] whose proof we sketch for convenience.

Lemma 6.1. *Let $T \subset B \subset G_C$ be an arbitrary maximal torus and a Borel containing it. Then $A \in \mathrm{D}(\mathrm{Hk}_{G,C}, \Lambda)^{\mathrm{bd}}$ vanishes if and only if $\mathrm{CT}_B(A) \in \mathrm{D}(\mathrm{Gr}_{T,C}, \Lambda)$ does.*

Proof. The proof is done by considering a maximal strata where A is concentrated. This strata is of the form $[\mathrm{Spd}\, C/(L_C^+G)_\mu]$ for the stabilizer of $\mu \in X_*(T)_+$. The attractor of $\mathrm{Gr}_{G,C,\mu}$ at the anti-dominant coweight $-\mu$ with respect to B is an isolated point. Using the $R(p^+)_!(q^+)^*$-version of hyperbolic localization, we see that the fiber of CT_B over $\mu \in \mathrm{Gr}_T$ agrees with pullback to this point. □

This allows us to localize several properties of derived objects in $\mathrm{D}(\mathrm{Hk}_{G,C}, \Lambda)^{\mathrm{bd}}$. For instance, A is ULA if and only if $\mathrm{CT}_B(A)$ which, in turn, is equivalent to $[\mu]^*A$ being a perfect object for all maps $[\mu]\colon \mathrm{Spd}\, C \to \mathrm{Hk}_G$ with $\mu \in X_*(T)$, see [FS21, Propositions VI.6.4, VI.6.5].

Next, we move to the natural perverse t-structure on $\mathrm{D}(\mathrm{Hk}_{G,C,\mu}, \Lambda)^{\mathrm{bd}}$, see [FS21, Definition/Proposition VI.7.1]. This is given in terms of the following subcategory

$${}^p\mathrm{D}^{\leq 0}(\mathrm{Hk}_{G,C}, \Lambda)^{\mathrm{bd}} = \{A \in \mathrm{D}(\mathrm{Hk}_{G,C}, \Lambda)^{\mathrm{bd}} \colon j_\mu^* A \in \mathrm{D}^{\leq -\langle 2\rho, \mu\rangle}\}, \tag{6.6}$$

which determines ${}^p\mathrm{D}^{\geq 0}(\mathrm{Hk}_{G,C}, \Lambda)$ uniquely. Intersecting these two, we get the category of perverse sheaves $\mathrm{Perv}(\mathrm{Hk}_{G,C}, \Lambda)$.

Thanks to [FS21, Proposition VI.7.4], the t-structure is preserved under and detected by $\mathrm{CT}_B[\deg_B]$. Here, for any C-parabolic $B \subset P$ with Levi quotient M, the degree is the locally constant function on $\mathrm{Gr}_{P,C}$ induced by

$$\deg_P(\lambda) = \langle 2\rho_G - 2\rho_M, \lambda\rangle, \tag{6.7}$$

where $\lambda \in X_*(T)$ is a coweight and ρ_M is the half-sum of all B-positive M-roots. The main geometric fact used in the proof of [FS21, Proposition VI.7.4] is the equidimensionality of semi-infinite orbits.

When working with torsion coefficients, it is convenient to single out flat perverse sheaves, which are those objects A such that for every Λ-module M the complex $A \otimes_\Lambda^{\mathbb{L}} M$ is perverse.

Definition 6.2. The Satake category $\mathrm{Sat}(\mathrm{Hk}_{G,C}, \Lambda)$ is the full subcategory of flat ULA objects in $\mathrm{Perv}(\mathrm{Hk}_{G,C}, \Lambda)$.

The Satake category is endowed with a monoidal product

$$\star\colon \mathrm{Sat}(\mathrm{Hk}_{G,C}, \Lambda) \times \mathrm{Sat}(\mathrm{Hk}_{G,C}, \Lambda) \to \mathrm{Sat}(\mathrm{Hk}_{G,C}, \Lambda) \tag{6.8}$$

arising from the convolution Hecke stack $\mathrm{Hk}_{G,C}\widetilde{\times}\mathrm{Hk}_{G,C}$, see [FS21, Proposition VI.8.1]. Due to the fusion interpretation [FS21, Definition/Proposition VI.9.4], the monoidal structure is naturally symmetric monoidal.

Taking cohomology of the affine Grassmannian furnishes a fiber functor

$$F\colon \mathrm{Sat}(\mathrm{Hk}_{G,C},\Lambda) \to \mathrm{Rep}(\Lambda) \tag{6.9}$$

to the category of Λ-finite locally free modules and the Tannakian formalism gives us an interpretation of these categories in terms of a group of automorphisms.

Theorem 6.3 (Fargues–Scholze)**.** *Fix a compatible system of primitive prime-to-p-th roots of unity in C. Then, the automorphism group of the fiber functor F is naturally isomorphic to the Langlands dual group $\widehat{G}_\Lambda$ formed over Λ.*

One may regard the dual group $\widehat{G}_\Lambda$ as combinatorially defined in terms of root data. The cyclotomic twist in [FS21, Theorem VI.11.1] is trivialized using the compatible system of roots of unity in C.

6.3. **Over $\bar{k}$.** Let $\mathcal{G}$ be a parahoric O-model of G. In this section, we look at what happens with the geometric special fiber Hecke v-stack $\mathrm{Hk}_{\mathcal{G},\bar{k}} = L^+_{\bar{k}}\mathcal{G}^\diamondsuit\backslash L_{\bar{k}}G^\diamondsuit/L^+_{\bar{k}}\mathcal{G}^\diamondsuit$. We note that the categories of étale sheaves compare well to their scheme-theoretic companions, see Proposition A.5.

We continue with the notation and, in addition, fix a maximal $\breve{F}$-split F-torus $S\subset G$ containing a maximal F-split torus (see [BT84, Proposition 5.1.10]) with centralizer T, which is a maximal F-torus inside G, such that their connected Néron O-models $\mathcal{S}\subset\mathcal{T}$ embed into $\mathcal{G}$. Each parabolic subgroup $P\subset G_{\breve{F}}$ with Levi M containing $S_{\breve{F}}$ extends to a diagram of O-group schemes $\mathcal{M}\leftarrow\mathcal{P}\to\mathcal{G}_{\breve{O}}$ by taking flat closures. Again, choosing a cocharacter $\lambda\colon \mathbb{G}_m\to\mathcal{S}_{\breve{O}}\subset\mathcal{G}_{\breve{O}}$ with $M=M_\lambda$ and $P=P^+_\lambda$, the formalism of Section 5 applies to define $\mathbb{G}_m^\diamondsuit$-actions on the pro-smooth cover $\mathrm{Gr}_{\mathcal{G},\bar{k}}=\mathcal{F}\ell^\diamondsuit_{\mathcal{G},\bar{k}}$. This gives rise to constant term functors

$$\mathrm{CT}_{\mathcal{P}}\colon \mathrm{D}(\mathrm{Hk}_{\mathcal{G},\bar{k}},\Lambda)^{\mathrm{bd}} \to \mathrm{D}(\mathcal{F}\ell^{0,\diamondsuit}_{\mathcal{G},\bar{k}},\Lambda)^{\mathrm{bd}}, \tag{6.10}$$

not depending on the choice of λ such that $M=M_\lambda$ and $P=P^+_\lambda$. Here, the fixed points $\mathcal{F}\ell^0_{\mathcal{G},\bar{k}}$ contain $\mathcal{F}\ell_{\mathcal{M},\bar{k}}$ as an open and closed sub-ind-scheme by Theorem 5.2, but are strictly bigger unless $\mathcal{G}_{\breve{O}}$ is special parahoric.

As in the previous section, we analyse the key properties ULA, flatness and perversity. The crucial step is the following conservativity result.

Proposition 6.4. *An object $A\in\mathrm{D}(\mathrm{Hk}_{\mathcal{G},\bar{k}},\Lambda)^{\mathrm{bd}}$ vanishes if and only if $\mathrm{CT}_{\mathcal{B}}(A)$ does for every $\breve{F}$-Borel $S_{\breve{F}}\subset B\subset G_{\breve{F}}$.*

Proof. Just like in Lemma 6.1, we argue on a maximal strata of $\mathrm{Hk}_{\mathcal{G},\bar{k}}$ where A does not vanish, say one indexed by some w. In this case, by Lemma 5.3 there is a choice of $\breve{F}$-Borel B for which the associated attractor intersects $\mathcal{F}\ell_{\mathcal{G},\bar{k},w}$ in an isolated point. In this case, $\mathrm{CT}_{\mathcal{B}}$ agrees with pullback to this point. □

Definition 6.5. An object $A\in\mathrm{D}(\mathrm{Hk}_{\mathcal{G},\bar{k}},\Lambda)^{\mathrm{bd}}$ is ULA whenever its pullback to $\mathcal{F}\ell^\diamondsuit_{\mathcal{G},\bar{k},w}$ is ULA over $\mathrm{Spd}\,\bar{k}$.

A priori, this notion depends on the choice of left or right trivialization, but it follows a posteriori from Proposition 6.7 that it does not, see [FS21, Proposition VI.6.2]. The ULA property interacts very well with constant terms:

Proposition 6.6. *If $A\in\mathrm{D}(\mathrm{Hk}_{\mathcal{G},\bar{k}},\Lambda)^{\mathrm{bd}}$ is ULA, then so is $\mathrm{CT}_{\mathcal{P}}(A)$. Conversely, if $\mathrm{CT}_{\mathcal{B}}(A)$ is ULA for all Borel subgroups $S_{\breve{F}}\subset B\subset G_{\breve{F}}$, then so is A.*

Proof. Abusing notation, we also call A the pullback of this object to $\mathcal{F}\ell^{\diamondsuit}_{\mathcal{G},\bar{k}}$. By [FS21, Theorem IV.2.23], to prove that for $B = A$ or $B = \mathrm{CT}_{\mathcal{P}}(A)$, the object B is ULA it is enough to show that

$$p_1^*\mathbb{D}(B) \otimes p_2^* B \to R\mathscr{H}\mathrm{om}(p_1^* B, Rp_2^! B) \tag{6.11}$$

is an isomorphism. Now, for any pair of flat closures of parabolics $\mathcal{P}_1$ and $\mathcal{P}_2$ a direct computation (using properties of hyperbolic localization, cf. the proof of [FS21, Proposition VI.6.4]) shows that

$$\mathrm{CT}_{\mathcal{P}_1\times\mathcal{P}_2}(p_1^*\mathbb{D}(A) \otimes p_2^* A) = p_1^*\mathbb{D}(\mathrm{CT}_{\mathcal{P}_1^-}(A)) \otimes p_2^* \mathrm{CT}_{\mathcal{P}_2}(A) \tag{6.12}$$

and that

$$\mathrm{CT}_{\mathcal{P}_1\times\mathcal{P}_2}(R\mathscr{H}\mathrm{om}(p_1^* A, Rp_2^! A)) = R\mathscr{H}\mathrm{om}(p_1^*(\mathrm{CT}_{\mathcal{P}_1^-}(A), Rp_2^! \mathrm{CT}_{\mathcal{P}_2}(A))) \tag{6.13}$$

where $\mathcal{P}_1^-$ is opposite to $\mathcal{P}_1$. In the forward direction, it is enough to use this for $\mathcal{P}_1 = \mathcal{P}^-$ and $\mathcal{P}_2 = \mathcal{P}$. For the converse, we let K denote the cone of Equation (6.11). By the conservativity of Proposition 6.4, it is enough to prove $\mathrm{CT}_{\mathcal{B}_1,\mathcal{B}_2}(K) = 0$ for all $\mathcal{B}_1$, $\mathcal{B}_2$ since these exhaust the Borel subgroups of of $G_{\breve{F}} \times G_{\breve{F}}$. But this follows from the computation above, [FS21, Proposition IV.2.19] and the hypothesis that $\mathrm{CT}_{\mathcal{B}_1}(A)$ is ULA. □

We prove that ULA sheaves admit an easy description in terms of restrictions to Schubert strata:

Proposition 6.7. *The following are equivalent for an object $A \in \mathrm{D}(\mathrm{Hk}_{\mathcal{G},\bar{k}}, \Lambda)^{\mathrm{bd}}$:*

(1) *A is ULA.*
(2) *For all strata of A pullback along $[w]\colon \mathrm{Spd}\,\bar{k} \to \mathrm{Hk}_{\mathcal{G},\bar{k}}$ is a perfect complex*[5] *in*
$$\mathrm{D}(\mathrm{Spd}\,\bar{k}, \Lambda) = \mathrm{D}(\Lambda). \tag{6.14}$$
(3) *The pullback to $\mathcal{F}\ell^{\diamondsuit}_{\mathcal{G},\bar{k}}$ lies in*
$$\mathrm{D}_{\mathrm{cons}}(\mathcal{F}\ell_{\mathcal{G},\bar{k}}, \Lambda)^{\mathrm{bd}} \subset \mathrm{D}(\mathcal{F}\ell^{\diamondsuit}_{\mathcal{G},\bar{k}}, \Lambda)^{\mathrm{ula,bd}}, \tag{6.15}$$
where $\mathrm{D}_{\mathrm{cons}}(\mathcal{F}\ell_{\mathcal{G},\bar{k}}, \Lambda)^{\mathrm{bd}}$ is the category of perfect-constructible Λ-sheaves with bounded support on the ind-scheme $\mathcal{F}\ell_{\mathcal{G},\bar{k}}$ and the inclusion is the one constructed in [Sch17, Section 27].

Proof. The equivalence of (2) and (3) follows from Proposition A.5, and the fact that the equivalence in Proposition A.5 is compatible with pullback to $\mathcal{F}\ell_{\mathcal{G}}$, respectively to $\mathcal{F}\ell^{\diamondsuit}_{\mathcal{G}}$ or to strata.

Let j_w denote the inclusion of the strata in $\mathrm{Hk}_{\mathcal{G},\bar{k}}$ corresponding to w. For proving that (2) implies (1) one can, as in [FS21, Proposition VI.6.5], reduce to showing that $R(j_w)_!\Lambda$ is ULA (for each w). But their pullback to $\mathcal{F}\ell_{\mathcal{G},\bar{k}}$ are clearly algebraic and by Proposition A.1 they are also ULA, see also [FS21, Proposition IV.2.30]. Alternatively one can use the Demazure resolution, compare with [FS21, Proposition VI.5.7].

For the converse implication, we induct on the number of strata where A does not vanish and consider the cone of $R(j_w)_! j_w^* A \to A$ for a maximal strata w. Indeed, pullback by open immersion preserves being ULA by [FS21, Proposition VI.2.13.(i)]. Then we apply [FS21, Proposition VI.4.1] to see that $j_w^* A \in \mathrm{D}(\mathrm{Hk}_{\mathcal{G},\bar{k},\mu}, \Lambda)$ has a perfect stalk, and use that $R(j_w)_!(j_w^* A)$ is ULA, by the proven (2) implies (1). Then we can conclude by induction. □

Arguing as in [FS21, Definition/Proposition VI.7.1], we can define a perverse t-structure.

[5] If $\Lambda = \Lambda_0[\ell^{-1}]$ as in (6.2), then we require the pullback to arise as the ℓ localization of a perfect complex over Λ_0.

Definition 6.8. The perverse t-structure on $\mathrm{Hk}_{\mathcal{G},\bar{k}}$ is the only such that

$$^{p}\mathrm{D}^{\leq 0}(\mathrm{Hk}_{\mathcal{G},\bar{k}},\Lambda)=\{A\in \mathrm{D}(\mathrm{Hk}_{\mathcal{G},\bar{k}},\Lambda)\colon j_w^*A\in \mathrm{D}^{\leq -l(w)}(\mathrm{Hk}_{\mathcal{G},\bar{k}},\Lambda)\},\tag{6.16}$$

respectively

$$^{p}\mathrm{D}^{\geq 0}(\mathrm{Hk}_{\mathcal{G},\bar{k}},\Lambda)=\{A\in \mathrm{D}(\mathrm{Hk}_{\mathcal{G},\bar{k}},\Lambda)\colon Rj_w^!A\in \mathrm{D}^{\geq -l(w)}(\mathrm{Hk}_{\mathcal{G},\bar{k}},\Lambda)\}.\tag{6.17}$$

Perverse sheaves

$$\mathrm{Perv}(\mathrm{Hk}_{\mathcal{G},\bar{k}},\Lambda)={}^{p}\mathrm{D}^{\leq 0}(\mathrm{Hk}_{\mathcal{G},\bar{k}},\Lambda)\cap {}^{p}\mathrm{D}^{\geq 0}(\mathrm{Hk}_{\mathcal{G},\bar{k}},\Lambda)\tag{6.18}$$

are the heart of the t-structure. Such an A is flat perverse if in addition $A\otimes^{\mathbb{L}}_{\Lambda} M$ is in $\mathrm{Perv}(\mathrm{Hk}_{\mathcal{G},\bar{k}},\Lambda)$ for all Λ-modules M.

We note that, in general, there cannot be any degree shifts such that $\mathrm{CT}_{\mathcal{P}}[\deg_{\mathcal{P}}]$ preserves the perverse t-structure, due to lack of parity. But, we define

$$\deg_{\mathcal{P}}(\lambda_I)=\langle 2\rho_G-2\rho_M,\lambda\rangle\tag{6.19}$$

for translation elements $\lambda_I\in X_*(T)_I$. This is useful for the following result:

Proposition 6.9. *Assume that $\mathcal{G}_{\breve{O}}$ is special parahoric, and let $A\in \mathrm{D}(\mathrm{Hk}_{\mathcal{G},\bar{k}},\Lambda)$. Then A is perverse if and only if $\mathrm{CT}_{\mathcal{B}}(A)[\deg_{\mathcal{B}}]\in \mathrm{Perv}(\mathcal{F}\ell^{\diamondsuit}_{\mathcal{T},\bar{k}},\Lambda)$ for all Borel subgroups $S_{\breve{F}}\subset B\subset G_{\breve{F}}$. The same applies to the flat objects.*

Proof. It suffices to follow the proof of [FS21, Proposition VI.7.4]. For preserving the t-structure, we use the fact that the non-empty intersections $\mathcal{S}_{\bar{k},\lambda_I}\cap\mathcal{F}\ell_{\mathcal{G},\bar{k},\nu_I}$ are equidimensional of dimension $\langle 2\rho_G,\lambda+\nu\rangle$, see Lemma 5.5. The converse then follows from Proposition 6.4. ☐

In particular, it is now permitted to introduce the Satake category at special level. Notice that there is no hope of such a well-behaved class of objects to exist at arbitrary level, because the quotient $\tilde{W}/W_{\mathcal{G}}$ carries no natural abelian structure.

Definition 6.10. Let $\mathcal{G}_{\breve{O}}$ be special parahoric. Then the Satake category $\mathrm{Sat}(\mathrm{Hk}_{\mathcal{G},\bar{k}},\Lambda)$ is the full subcategory of $\mathrm{Perv}(\mathrm{Hk}_{\mathcal{G},\bar{k}},\Lambda)$ comprised of flat ULA objects.

This category lies within the category of perverse sheaves $\mathrm{Perv}(\mathrm{Hk}^{\mathrm{sch}}_{\mathcal{G},\bar{k}},\Lambda)$ on the schematic Hecke stack $\mathrm{Hk}^{\mathrm{sch}}_{\mathcal{G},\bar{k}}=L^+_{\bar{k}}\mathcal{G}\backslash L_{\bar{k}}G/L^+_{\bar{k}}\mathcal{G}$ by Proposition 6.7 and Section A for the comparison with sheaves on schematic v-stacks. It carries moreover a monoidal structure given by convolution $\star$.

6.4. **Over O_C.** Let $f\colon X\to \mathrm{Spec}(O_C)$ be a scheme of finite presentation over O_C and denote by j the inclusion $X_\eta\hookrightarrow X$ of the generic fiber. In [HS21, Theorem 1.7], Hansen and Scholze prove that the pullback functor

$$j^*\colon \mathrm{D}(X,\Lambda)\to \mathrm{D}(X_\eta,\Lambda)\tag{6.20}$$

restricts to an equivalence between f-ULA and f_η-ULA objects. In the setup of diamonds, the argument for full faithfulness is the same as was explained to us by Scholze, and it consists of proving the adjunction map $A\to Rj_*j^*A$ is an isomorphism.

Lemma 6.11. *Let X be a small v-sheaf over $\mathrm{Spd}(O_C)$ representable in locally spatial diamonds, compactifiable and of finite trascendence degree. Let $A\in \mathrm{D}(X,\Lambda)$ be ULA for the structure map to $\mathrm{Spd}(O_C)$. Then $A\to Rj'_*j'^*A$ is a isomorphism, where $j'\colon X_\eta\to X$ and $j\colon \mathrm{Spd}(C)\to \mathrm{Spd}(O_C)$ denote the inclusion of generic fibers.*

Proof. By hypothesis j'^*A is ULA with respect to $\mathrm{Spd}(C)$. In particular, by [FS21, Proposition IV.2.19] the map

$$j'^*A \otimes^{\mathbb{L}}_{\Lambda} \Lambda \cong R\mathscr{H}\mathrm{om}(\mathbb{D}_{X_\eta/\operatorname{Spd}(C)}(j'^*A), Rf_\eta^!\Lambda) \tag{6.21}$$

is an isomorphism. Since j' is an open immersion $j'^* = Rj'^!$ and $\mathbb{D}_{X_\eta/\operatorname{Spd}(C)}(j'^*A) = j'^*\mathbb{D}_{X/\operatorname{Spd}(O_C)}(A)$ as follows from [Sch17, Theorem 1.8.(v)]. We get

$$\begin{aligned} Rj'_*j'^*A \cong{} & Rj'_* R\mathscr{H}\mathrm{om}(j'^*\mathbb{D}_{X/\operatorname{Spd}(O_C)}(A), Rf_\eta^!\Lambda) \\ \cong{} & R\mathscr{H}\mathrm{om}(\mathbb{D}_{X/\operatorname{Spd}(O_C)}(A), Rj'_*Rf_\eta^!\Lambda) \\ \cong{} & R\mathscr{H}\mathrm{om}(\mathbb{D}_{X/\operatorname{Spd}(O_C)}(A), Rf^!Rj_*\Lambda) \end{aligned} \tag{6.22}$$

the result now follows from the identity $\Lambda = Rj_*\Lambda$ and double duality for ULA sheaves. □

In particular, a f_η-ULA object A comes from a f-ULA object if and only if Rj_*j^*A is f-ULA.

Below, we prove essential surjectivity for $\mathrm{Hk}_{\mathcal{G},O_C}$, the Hecke stack over $\operatorname{Spd} O_C$. For hyperspecial parahoric $\mathcal{G}$, this is [FS21, Corollary VI.6.7]. Before doing this, recall that hyperbolic localization allows us to define again a constant term functors

$$\mathrm{CT}_{\mathcal{P}} \colon \mathrm{D}(\mathrm{Hk}_{\mathcal{G},O_C}, \Lambda)^{\mathrm{bd}} \to \mathrm{D}(\mathrm{Gr}^0_{\mathcal{G},O_C}, \Lambda)^{\mathrm{bd}}. \tag{6.23}$$

By [FS21, Proposition IV.6.12], there is a natural equivalence

$$\mathrm{CT}_{\mathcal{P}} \circ Rj_{\mathcal{G},*} \cong Rj_{\mathcal{M},*} \circ \mathrm{CT}_P, \tag{6.24}$$

with $j_{\mathcal{G}}, j_{\mathcal{M}}$ denoting the inclusion of the respective generic fibers. Now, we can probe integral ULA objects.

Proposition 6.12. *Consider the inclusion of Hecke stacks $j\colon \mathrm{Hk}_{G,C} \to \mathrm{Hk}_{\mathcal{G},O_C}$. There is an equivalence*

$$j^* \colon \mathrm{D}(\mathrm{Hk}_{\mathcal{G},O_C}, \Lambda)^{\mathrm{bd,ula}} \to \mathrm{D}(\mathrm{Hk}_{G,C}, \Lambda)^{\mathrm{bd,ula}}, \tag{6.25}$$

whose inverse functor is Rj_*.

Proof. Suppose $A \in \mathrm{D}(\mathrm{Hk}_{G,C}, \Lambda)^{\mathrm{bd,ula}}$, it suffices to prove $Rj_*A \in \mathrm{D}(\mathrm{Hk}_{\mathcal{G},O_C}, \Lambda)^{\mathrm{bd,ula}}$. Let B denote the pullback of A to $\mathrm{Gr}_{\mathcal{G},O_C}$, which by definition is ULA. By smooth base change, Rj_*A pulls back to Rj_*B (here we implicitly use [FS21, Proposition VI.4.1]). By [FS21, Theorem IV.2.23], we must show

$$p_1^*\mathbb{D}(Rj_*B) \otimes p_2^*Rj_*B \to R\mathscr{H}\mathrm{om}(p_1^*Rj_*B, Rp_2^!Rj_*B) \tag{6.26}$$

is an isomorphism. Let K denote the cone of this map. By assumption, and since $j^* = Rj^!$, this map is an isomorphism on the generic fiber. Consequently, $K = i_*L$ for some $L \in D(\mathrm{Hk}_{\mathcal{G},\bar{k}}, \Lambda)^{\mathrm{bd}}$ and the inclusion $i\colon \mathrm{Hk}_{\mathcal{G},\bar{k}} \to \mathrm{Hk}_{\mathcal{G},O_C}$. We may use the conservativity result Proposition 6.4 to prove $L = 0$. This reduces us to proving that $\mathrm{CT}_{\mathcal{B}}(Rj_*A) = Rj_*\,\mathrm{CT}_{\mathcal{B}}(A)$ is ULA for all Borel subgroups $S_{\breve{F}} \subset B \subset G_{\breve{F}}$. Now, the fixed-point locus $\mathrm{Gr}^0_{\mathcal{G},O_C}$ of the action induced by $\mathcal{B}$ is ind-representable by a locally finite type scheme over O_C of relative dimension 0. We call this scheme X and let $h\colon X_\eta \to X$ the inclusion of generic fibers. By inspection, $\mathrm{D}(\mathrm{Gr}^0_{\mathcal{G},O_C}, \Lambda) \cong \mathrm{D}(X, \Lambda)$ and $\mathrm{D}(\mathrm{Gr}^0_{\mathcal{G},C}, \Lambda) \cong \mathrm{D}(X_\eta, \Lambda)$. In particular $Rj_* \cong c_X^* Rh_* Rc_{X_\eta,*}$ with notation as in [Sch17, Section 27]. By [HS21, Theorem 1.7], Rh_* preserves ULA objects, which allows us to conclude the same holds for Rj_*. □

6.5. **Nearby cycles.** We can now look at the nearby cycles functor

$$\Psi_{\mathcal{G}} := i^* Rj_* \colon \mathrm{D}(\mathrm{Hk}_{G,C}, \Lambda)^{\mathrm{bd}} \to \mathrm{D}(\mathrm{Hk}_{\mathcal{G},\bar{k}}, \Lambda)^{\mathrm{bd}}, \tag{6.27}$$

Arising from the diagram

$$\mathrm{Hk}_{G,C} \xrightarrow{j} \mathrm{Hk}_{\mathcal{G},O_C} \xleftarrow{i} \mathrm{Hk}_{\mathcal{G},\bar{k}} \tag{6.28}$$

of geometric fibers inclusions of the integral Hecke stack.

Proposition 6.13. *The functor of nearby cycles lies in a natural equivalence*

$$\mathrm{CT}_{\mathcal{P}}[\deg_{\mathcal{P}}] \circ \Psi_{\mathcal{G}} \cong \Psi_{\mathcal{M}} \circ \mathrm{CT}_P[\deg_P], \tag{6.29}$$

that is, it commutes with shifted constant term functors.

Proof. Without the shift, this is a direct consequence of [FS21, Proposition IV.6.12]. Using Theorem 5.2, this also shows that $\mathrm{CT}_{\mathcal{P}} \circ \Psi_{\mathcal{G}}$ is supported on the open and closed sub-v-sheaf $\mathcal{F}\ell_{\mathcal{M},\bar{k}} \subset \mathcal{F}\ell^0_{\mathcal{G},\bar{k}}$. So the shifts agree by definition. □

Surprisingly, this commutativity property delivers us a lot of control on the values assumed by $\Psi_{\mathcal{G}}$ on the Satake category.

Corollary 6.14. *Nearby cycles $\Psi_{\mathcal{G}}$ restrict to a functor*

$$\mathrm{D}(\mathrm{Hk}_{G,C}, \Lambda)^{\mathrm{bd,ula}} \to \mathrm{D}(\mathrm{Hk}_{\mathcal{G},\bar{k}}, \Lambda)^{\mathrm{bd,ula}} \tag{6.30}$$

and, if $\mathcal{G}_{\breve{O}}$ is furthermore special parahoric, then it even restricts to

$$\mathrm{Sat}(\mathrm{Hk}_{G,C}, \Lambda) \to \mathrm{Sat}(\mathrm{Hk}_{\mathcal{G},\bar{k}}, \Lambda) \tag{6.31}$$

between the Satake categories.

Proof. This follows from Proposition 6.6 and Proposition 6.9. □

Let us examine the nearby cycles $\Psi_{\mathcal{G}}(\mathrm{Sat}(V))$ applied to a Satake object $\mathrm{Sat}(V) \in \mathrm{Sat}(\mathrm{Hk}_{G,C}, \Lambda)$ corresponding to a $\widehat{G}_\Lambda$-representation V with μ as its highest weight. Given an F-Borel $B \subset G$, the commutativity of Proposition 6.13 yields

$$\mathrm{CT}_{\mathcal{B}}[\deg_{\mathcal{B}}]\big(\Psi_{\mathcal{G}}(\mathrm{Sat}(V))\big) = \bigoplus_{\lambda_I} V(\lambda_I) \cdot \lambda_I, \tag{6.32}$$

where now the $\widehat{G}_\Lambda$-representation is regarded as a $\widehat{T}^I_\Lambda$-representation by restriction. Here, we use that (by construction) the constant term functor corresponds via geometric Satake to restriction of representations, see [FS21, Section VI.11]. In particular, we get:

Corollary 6.15. *For a $\widehat{G}_\Lambda$-representation V with highest weight μ, the compactly supported cohomology groups*

$$\mathrm{H}^l_c\big(\mathcal{S}_{\bar{k},w}, \Psi_{\mathcal{G}}(\mathrm{Sat}(V))\big) \tag{6.33}$$

vanish for all $l \in \mathbb{Z}$ unless $\mathcal{F}\ell_{\mathcal{G},\bar{k},w} \subset \mathcal{A}_{\mathcal{G},\bar{k},\mu}$.

Proof. This follows from Lemma 3.12. □

We are now finally ready to compute the special fiber of the local model.

Theorem 6.16. *There is an equality $\mathcal{A}^\diamondsuit_{\mathcal{G},\mu} = \mathcal{M}_{\mathcal{G},\mu,k_E}$ as sub-v-sheaves of $\mathcal{F}\ell^\diamondsuit_{\mathcal{G},k_E}$.*

Proof. By specializing the orbit of μ under the finite Weyl group, it is easy to see that $\mathcal{A}^{\diamondsuit}_{\mathcal{G},\mu}$ is contained in the special fiber of $\mathcal{M}_{\mathcal{G},\mu}$. By Corollary 6.15, it is thus enough to prove that for a maximal stratum in $\mathcal{M}_{\mathcal{G},\bar{k},\mu}$ enumerated by w, we have $\mathrm{H}^l_c(\mathcal{S}_{\bar{k},w}, A) \neq 0$ for some $l \in \mathbb{Z}$ and for $A := \Psi_{\mathcal{G}}(\mathrm{Sat}(V))$ for some $\widehat{G}_\Lambda$-representation V with highest weight μ. Using induction on μ, we may and do assume that w lies in the open complement of the closed union of $\mathcal{M}_{\mathcal{G},O_C,\lambda}$ for all $\lambda < \mu$. Indeed, if $\lambda < \mu$, then $\mathcal{A}^{\diamondsuit}_{\mathcal{G},\lambda} \subset \mathcal{A}^{\diamondsuit}_{\mathcal{G},\mu}$. By Lemma 5.3, our Borel subgroup $S_{\breve{F}} \subset B \subset G_{\breve{F}}$ can always be chosen such that w is an isolated point of the attractor $\mathcal{F}\ell^+_{\mathcal{G},\bar{k},w}$. Since w enumerates a maximal stratum, we also see that w is an isolated point of $\mathcal{M}^+_{\mathcal{G},\bar{k},\mu}$, so that $\mathrm{H}^*_c(\mathcal{S}_{\bar{k},w}, A) = \mathrm{H}^*(\{w\}, A) =: A_w$ is the stalk of A at w.

Consider $X = \mathcal{M}_{\mathcal{G},O_C,\mu} \times_{\mathrm{Spd}\, O_C} U$ where U denotes the analytic locus of the open unit ball $\mathbb{D}^{\diamondsuit}_{O_C}$. Let $g\colon X_C \hookrightarrow X$ be the inclusion of the generic fiber. Let K denote a completed algebraic closure of $k(\!(t)\!)$. We may choose a $\mathrm{Spd}(K)$-valued point $\overline{w}$ of X that lies over w. It suffices to prove $A_{\overline{w}}$ is not identically 0. Since U is smooth over $\mathrm{Spd}\, O_C$, the smooth base-change theorem and our inductive assumption on w allows us to compute

$$A_{\overline{w}} = (Rg_*\Lambda[\langle 2\rho, \mu\rangle])_{\overline{w}}, \tag{6.34}$$

where V is chosen to have weight multiplicity 1 at μ. Since X_C, X and $K^{\diamondsuit}$ are locally spatial diamonds, we may compute the right-side term site-theoretically. Letting $l := -\langle 2\rho, \mu\rangle$, we have

$$\mathrm{H}^l(A_{\overline{w}}) = \lim\nolimits_W \mathrm{H}^0(W_C, \Lambda) \tag{6.35}$$

where W ranges over étale neighborhoods of $\overline{w}$ in X. By Remark 4.15 and openness of $X \to \mathcal{M}_{\mathcal{G},O_C,\mu}$, the generic fiber X_C is dense in X which proves that the above expression does not vanish. $\square$

6.6. **Centrality of nearby cycles.** In the classical theory, say, over function fields, it is known that nearby cycles on Hecke stacks give central perverse sheaves on partial affine flag varieties, see [Gai01]. Centrality holds true in our context as well:

Proposition 6.17. *For every $A \in \mathrm{D}(\mathrm{Hk}_{G,C}, \Lambda)^{\mathrm{bd,ula}}$ and $B \in \mathrm{D}(\mathrm{Hk}_{\mathcal{G},\bar{k}}, \Lambda)^{\mathrm{bd,ula}}$, there is a canonical isomorphism*

$$\Psi_{\mathcal{G}}(A) \star B \cong B \star \Psi_{\mathcal{G}}(A) \tag{6.36}$$

in $\mathrm{D}(\mathrm{Hk}_{\mathcal{G},\bar{k}}, \Lambda)$.

Proof. We can repeat the proof of [Zhu14, Proposition 7.4] in our context:

Similar to [FS21, Definition/Proposition VI.9.4], we work with the convolution integral Hecke stack

$$\mathrm{Hk}_{\mathcal{G}}^{I;I_1,\ldots,I_k} \to (\mathrm{Spd}\, O)^I, \tag{6.37}$$

where $I = I_1 \sqcup \ldots \sqcup I_k$ is a finite partitioned index set. It parametrizes $\mathcal{G}$-bundles $\mathcal{E}_0, \ldots, \mathcal{E}_k$ over B^+_{dR} together with isomorphisms of $\mathcal{E}_{j-1}$ and $\mathcal{E}_j$ outside the union of the divisors ξ_i for all $i \in I_j$. We fix $I := \{1,2\}$ and drop it from the notation. There are three ordered partitions $\{1\} \sqcup \{2\}$, $\{2\} \sqcup \{1\}$ and $\{1,2\}$, leading to the diagram of v-sheaves:

$$\begin{tikzcd} \mathrm{Hk}_{\mathcal{G}}^{\{1\},\{2\}}|_{\mathrm{Spd}\, O_C} \arrow[r, "m"] \arrow[dr, "p"'] & \mathrm{Hk}_{\mathcal{G}}^{\{1,2\}}|_{\mathrm{Spd}\, O_C} & \mathrm{Hk}_{\mathcal{G}}^{\{2\},\{1\}}|_{\mathrm{Spd}\, O_C} \arrow[l, "n"'] \arrow[dl, "q"] \\ & \mathrm{Hk}_{\mathcal{G},O_C} \times \mathrm{Hk}_{\mathcal{G},\bar{k}} & \end{tikzcd} \tag{6.38}$$

The diagram arises by base change along the map $\mathrm{Spd}\, O_C \to (\mathrm{Spd}\, O)^2$ induced by the divisor $\pi = 0$ in the second coordinate. The maps m, n are the natural projections given by remembering $\mathcal{E}_0$ and $\mathcal{E}_2$, and are ind-proper, as one sees by pulling back to the convolution affine Grassmannian, combine Theorem 4.9 with the proof of [SW20, Proposition 20.4.1]. The maps p, q are given by sending $(\mathcal{E}_0, \mathcal{E}_1, \mathcal{E}_2)$ to the ordered pair $((\mathcal{E}_0, \mathcal{E}_1), (\mathcal{E}_1, \mathcal{E}_2))$, respectively $((\mathcal{E}_1, \mathcal{E}_2), (\mathcal{E}_0, \mathcal{E}_1))$, and are pro-(cohomologically smooth) because $L_O^+\mathcal{G} \to \mathrm{Spd}\, O$ is so. More precisely, one passes to a bounded part and factors the action of $L_O^+\mathcal{G}$ through a congruence quotient.

Furthermore, the maps m, n are convolution maps in the special fiber and induce isomorphisms over the generic fiber such that (6.38) commutes. The commutativity yields a canonical isomorphism by using adjunctions

$$Rm_{\eta,*}p_\eta^*(A \boxtimes B) \cong Rn_{\eta,*}q_\eta^*(A \boxtimes B), \tag{6.39}$$

which will induce the desired isomorphism (6.36) upon applying the nearby cycles for the family $\mathrm{Hk}_{\mathcal{G}}^{\{1,2\}}|_{\mathrm{Spd}\, O_C}$. Indeed, since $(Rj_*A) \boxtimes B$ is ULA by Proposition 6.12 and [FS21, Corollary IV.2.25] for outer tensor products, it is still ULA after cohomologically smooth pullback along p, q and proper pushforward along m, n (here, we use that the support of A, B is bounded). Thus, (6.39) canonically extends integrally yielding (6.36) after restriction to the special fiber. □

The following would be a natural reinforcement of the previous proposition to also preserving perversity. For schemes, nearby cycles always preserve perversity [HS21, Lemma 6.3]. In our setting, this is not immediate and would transport Gaitsgory's central functor [Gai01] to the p-adic context.

Conjecture 6.18. *For every $A \in \mathrm{D}(\mathrm{Hk}_{G,C}, \Lambda)$, the Λ-flat central sheaf $\Psi_{\mathcal{G}}(A) \in \mathrm{D}(\mathrm{Hk}_{\mathcal{G},\bar{k}}, \Lambda)^{\mathrm{bd,ula}}$ is perverse.*

By a combination of Corollary 6.14 and Proposition 6.17, we know that Conjecture 6.18 holds true whenever $\mathcal{G}_{\breve{O}}$ is special parahoric. Also, using the representability in Theorem 1.2 and a comparison with schematic nearby cycles (see [Sch17, Proposition 27.6.]), the conjecture holds true whenever μ is minuscule. In general, we lack tools to verify Conjecture 6.18 – shifted constant terms appear to be insufficient – but one still expects some form of Artin vanishing to hold in this very particular context of the Hecke stack.

Remark 6.19. After the first version of this paper was written, Conjecture 6.18 was proved in [ALWY23, Theorem 4.17]. We remark that Conjecture 6.18 plays a role in the proof of unibranchness of local models in [GL24, Theorem 1.3], and thereby in the proof of Conjecture 1.1 for $p = 2, 3$ as explained in the text following Theorem 1.2.

7. Minuscule implies representable

Our goal in this section is to prove the Scholze–Weinstein conjecture on minuscule local models [SW20, Conjecture 21.4.1] as stated in Theorem 1.2. This is sharp, see Proposition 4.8. We verify the representability part without any assumption on the prime p or the pair $(\mathcal{G}, \mu)$, thereby showing the existence of weakly normal projective O_E-schemes $\mathcal{M}_{\mathcal{G},\mu}^{\mathrm{sch}}$ with natural, equivariant isomorphisms $(\mathcal{M}_{\mathcal{G},\mu}^{\mathrm{sch}})^\diamond \cong \mathcal{M}_{\mathcal{G},\mu}$ in all cases.

As for their geometry, we show under Assumption 1.9 and Assumption 1.13 that the special fiber is given by $\mathcal{A}_{\mathcal{G},\mu}^{\mathrm{can}}$, in particular reduced and even weakly normal. This implies the geometry part of the Scholze–Weinstein conjecture under those assumptions, see the discussion after Conjecture 1.1.

Recall that our strategy for representability involves specializations triples. Since explicitly calculating the specialization map seems very hard, we need to consider convolutions of local models, so as to partially resolve $\mathcal{M}_{\mathcal{G},\mu}$ and understand their integral sections better.

7.1. **Convolution.** We continue to denote by $F/\mathbb{Q}_p$ a complete discretely valued field with ring of integers O and perfect residue field k of characteristic $p > 0$. Fix a completed algebraic closure C/F, and a connected reductive F-group G with parahoric O-model $\mathcal{G}$. Also, we fix an auxiliary maximal $\breve{F}$-split F-torus $S \subset G$ whose connected Néron model $\mathcal{S}$ embeds in $\mathcal{G}$, see [BT84, Proposition 5.10], and denote by T its centralizer with connected Néron model $\mathcal{T} \subset \mathcal{G}$. Additionally, we fix an auxiliary $\breve{F}$-Borel $T_{\breve{F}} \subset B \subset G_{\breve{F}}$.

When proving the representability of the v-sheaf local models, it is not difficult to reduce to the case that G is the Weil restriction of a split group, see proof of Theorem 7.21. In this case, it will be helpful to partially resolve the local model via convolution. We recall that the Beilinson–Drinfeld Grassmannian admits the following convolution variant

$$\mathrm{Gr}_{\mathcal{G}} \widetilde{\times} \dots \widetilde{\times} \mathrm{Gr}_{\mathcal{G}} := L_O\mathcal{G} \times^{L_O^+\mathcal{G}} \dots \times^{L_O^+\mathcal{G}} \mathrm{Gr}_{\mathcal{G}} \to \mathrm{Spd}\, O, \tag{7.1}$$

which, in terms of torsors, parametrizes successive modifications of $\mathcal{G}$-torsors together with a generic trivialization of the last. It admits natural closed sub-v-sheaves

$$\mathcal{M}_{\mathcal{G},\mu_\bullet} := \mathcal{M}_{\mathcal{G},\mu_1} \widetilde{\times} \dots \widetilde{\times} \mathcal{M}_{\mathcal{G},\mu_n}, \tag{7.2}$$

for any sequence $\mu_\bullet = (\mu_1, \dots, \mu_n)$ of B_C-dominant coweights μ_i of T_C, after base change to $\mathrm{Spd}\, O_E$, where E is the reflex field of $\mu_\bullet$. We will still call them (convolution) local models for simplicity. More precisely, denote by $\widetilde{\mathcal{M}_{\mathcal{G},O_C,\mu_i}}$ the preimage in $L_{O_C}\mathcal{G}$ of $\mathcal{M}_{\mathcal{G},O_C,\mu_i} \subset \mathrm{Gr}_{\mathcal{G},O_C}$, which is an $L^+_{O_C}\mathcal{G}$-torsor over $\mathcal{M}_{\mathcal{G},O_C,\mu_i}$. Then

$$\mathcal{M}_{\mathcal{G},O_C,\mu_\bullet} = \widetilde{\mathcal{M}_{\mathcal{G},O_C,\mu_1}} \times^{L^+_{O_C}\mathcal{G}} \dots \times^{L^+_{O_C}\mathcal{G}} \widetilde{\mathcal{M}_{\mathcal{G},O_C,\mu_2}} \times^{L^+_{O_C}\mathcal{G}} \mathcal{M}_{\mathcal{G},\mu_n}. \tag{7.3}$$

This presentation is not "minimal" in the following sense: Namely, given a contracted product $X \times^H Y$ in any topos and a normal subgroup $N \subset H$ acting trivially on Y, then the natural map

$$X \times^H Y \to X/N \times^{H/N} Y$$

is an isomorphism. Hence, in (7.3) we may replace $L^+_{O_C}\mathcal{G}$ by some sufficiently large congruence quotient, and accordingly the torsors $\widetilde{\mathcal{M}_{\mathcal{G},O_C,\mu_i}}$ by their pushforwards to these congruence quotients. Let us note that the multiplication

$$\mathcal{M}_{\mathcal{G},O_C,\mu_\bullet} \to \mathrm{Gr}_{\mathcal{G},O_C}$$

has image $\mathcal{M}_{\mathcal{G},O_C,|\mu_\bullet|}$ with $|\mu_\bullet| := \mu_1 + \dots + \mu_n$, and can therefore be regarded as a (partial) resolution of the latter. Regarding the structure of the convolution local models, we can record the following.

Lemma 7.1. *The convolution local model $\mathcal{M}_{\mathcal{G},\mu_\bullet}$ is a proper, flat π-adic kimberlite over* $\mathrm{Spd}\, O_E$ *with topologically dense generic fiber.*

Proof. In order to prove that $\mathcal{M}_{\mathcal{G},\mu_\bullet}$ is a proper flat π-adic kimberlite, we must first show that this proper v-sheaf is v-formalizing. Replace the universal $L^+_{O_C}\mathcal{G}$-torsors by the corresponding $W^+_{O_C}\mathcal{G}$-torsors, see [Gle24, Definition 5.7] and denote by $\mathcal{M}^{\mathrm{for}}_{\mathcal{G},O_C,\mu_\bullet}$ the corresponding convolution. The natural map

$$\mathcal{M}^{\mathrm{for}}_{\mathcal{G},O_C,\mu_\bullet} \to \mathcal{M}_{\mathcal{G},O_C,\mu_\bullet} \tag{7.4}$$

is an isomorphism because both v-sheaves are qcqs and have the same geometric points, compare with [Gle24, Proposition 5.19]. It is now easy to show that it is formally separated and formally adic, with representable special fiber, see Proposition 7.2. Via projection to the first factor we have a map

$$\mathcal{M}_{\mathcal{G},O_C,\mu_\bullet} \to \mathcal{M}_{\mathcal{G},O_C,\mu_1}, \tag{7.5}$$

which splits after pullback to $\widetilde{\mathcal{M}_{\mathcal{G},O_C,\mu_1}}$ into the projection of the product $\widetilde{\mathcal{M}_{\mathcal{G},O_C,\mu_1}} \times_{\mathrm{Spd}\, O_C} \mathcal{M}_{\mathcal{G},O_C,(\mu_2,\dots,\mu_n)}$. Hence, after replacing $L^+_{O_C}\mathcal{G}$ (and hence $\widetilde{\mathcal{M}_{\mathcal{G},O_C,\mu_1}}$) by a sufficiently large congruence quotient, we can deduce that $\mathcal{M}_{\mathcal{G},O_C,\mu_\bullet}$ has dense generic fiber by induction on n, Proposition 4.14 and preservation of closures under open maps. $\square$

From now on, we will always consider sequences of minuscule dominant coweights $\mu_\bullet = (\mu_1, \dots, \mu_n)$ whose sum

$$|\mu_\bullet| = \mu_1 + \cdots + \mu_n \tag{7.6}$$

is still minuscule. Basically, this means that the support of each μ_i lies in disjoint irreducible components of the Dynkin diagram. We say that a coweight is *tiny* if it is minuscule and its support is contained in at most one irreducible component.

Proposition 7.2. *Both fibers of $\mathcal{M}_{\mathcal{G},O_C,\mu_\bullet} \to \operatorname{Spd} O_C$ are representable. More precisely, we have isomorphisms*

$$\mathcal{M}_{\mathcal{G},O_C,\mu_\bullet}|_{\operatorname{Spd} C} \cong \mathcal{F}^\diamondsuit_{G,C,\mu_\bullet} \cong \mathcal{F}^\diamondsuit_{G,C,\mu_1} \times \cdots \times \mathcal{F}^\diamondsuit_{G,C,\mu_n}, \tag{7.7}$$

and also

$$\mathcal{M}_{\mathcal{G},O_C,\mu_\bullet}|_{\operatorname{Spd} \bar{k}} \cong \mathcal{A}^\diamondsuit_{\mathcal{G},\mu_\bullet}, \tag{7.8}$$

where on the right we mean the convolution $\mathcal{A}_{\mathcal{G},\mu_1} \widetilde{\times} \dots \widetilde{\times} \mathcal{A}_{\mathcal{G},\mu_n}$.

Proof. The description of the generic fiber via convolution is formal, and it is formal (using Theorem 6.16) that $\mathcal{M}_{\mathcal{G},O_C,\mu_\bullet}|_{\operatorname{Spd} k}$ is the convolution of the $\mathcal{A}^\diamondsuit_{\mathcal{G},\mu_i}$. Using Lemma A.2 this convolution identifies with $\mathcal{A}^\diamondsuit_{\mathcal{G},\mu_\bullet}$. □

We are aiming to carefully write down certain "minimal" $\mathcal{G}^{>i,\diamondsuit}_{\mathrm{ad},O_C}$-torsors $\mathcal{M}^{\mathrm{tor}}_{\mathcal{G},O_C,\mu_i} \to \mathcal{M}_{\mathcal{G},O_C,\mu_i}$ for some associated smooth connected groups $\mathcal{G}^{>i}_{\mathrm{ad},O_C}$, such that

$$\mathcal{M}^{\mathrm{tor}}_{\mathcal{G},O_C,\mu_1} \times^{\mathcal{G}^{>1,\diamondsuit}_{\mathrm{ad},O_C}} \dots \times^{\mathcal{G}^{>i-1,\diamondsuit}_{\mathrm{ad},O_C}} \mathcal{M}^{\mathrm{tor}}_{\mathcal{G},O_C,\mu_i} \times^{\mathcal{G}^{>i,\diamondsuit}_{\mathrm{ad},O_C}} \dots \times^{\mathcal{G}^{>n-1,\diamondsuit}_{\mathrm{ad},O_C}} \mathcal{M}^{\mathrm{tor}}_{\mathcal{G},O_C,\mu_n} \tag{7.9}$$

recovers the convolution local model $\mathcal{M}_{\mathcal{G},O_C,\mu_\bullet}$. We begin by introducing the group schemes $\mathcal{G}^{>i}_{\mathrm{ad},O_C}$.

Lemma 7.3. *Let $\mu_{>i} = \mu_{i+1} + \cdots + \mu_n$ and let $G^{>i}_{\mathrm{ad},C}$ be the quotient of G_C by the intersection of all conjugates of $P^-_{\mu_{>i}}$. Then, $G^{>i}_{\mathrm{ad},C}$ acts faithfully on $\mathcal{F}_{G,C,\mu_{>i}}$.*

Furthermore, if we let $\mathcal{G}^{>i}_{\mathrm{ad},O_C}$ be the unique fppf quotient[6] of $\mathcal{G}_{\mathrm{ad},O_C}$ with generic fiber $G^{>i}_{\mathrm{ad},C}$, then $\mathcal{G}^{>i}_{\mathrm{ad},O_C}$ acts on $\mathcal{M}_{\mathcal{G},O_C,\mu_{>i}}$, and its fibers are smooth, affine, connected with trivial center.

Proof. The claims on $G^{>i}_{\mathrm{ad},C}$ follow from Lemma 7.1. The smooth group scheme quotient with the asserted properties obviously exist, due to [BT84, Proposition 1.7.6], and it clearly inherits connected fibers. The generic fiber is clearly adjoint: apply semi-simplicity of G_{ad}. However, it is more delicate to show that the special fiber is adjoint.

Assume without loss of generality (Proposition 4.16) that $G = \operatorname{Res}_{F'/F} G'$ is an adjoint F-simple group with G' absolutely simple, and F' a finite field extension of F. Let $\mathcal{G}'$ be that parahoric over F' associated with $\mathcal{G}$. Then, a simple calculation reveals that

$$\mathcal{G}^{>i} = \operatorname{Res}_{A_i/O_C} \mathcal{G}'_{A_i}, \tag{7.10}$$

where A_i is the finite O_C-algebra obtained as the image of $O_{F'} \otimes_O O_C$ in the product of those copies of C indexed by the support of $\mu_{>i}$. Indeed, this smooth connected group scheme has the desired universal property and its special fiber is adjoint by [CGP15, Proposition A.5.15 (1)]. (Notice that the reducedness hypothesis is superfluous for calculating the center.) □

Now, we come to the definition of the torsors.

Definition 7.4. The v-sheaf $\mathcal{M}^{\mathrm{tor}}_{\mathcal{G},O_C,\mu_i}$ is defined as the pushforward to $(\mathcal{G}^{>i}_{\mathrm{ad},O_C})^\diamondsuit$ of the natural $L^+_{O_C}\mathcal{G}$-torsor $\widetilde{\mathcal{M}_{\mathcal{G},O_C,\mu_i}}$ over $\mathcal{M}_{\mathcal{G},O_C,\mu_i}$.

[6]Beware that the morphism of parahoric group schemes $\mathcal{G} \to \mathcal{G}_{\mathrm{ad}}$ is not always an fppf surjection.

One might expect, just like in Theorem 7.21, that these torsors have natural algebraic models over the ring of integers of some reflex field and this will be an integral part of our strategy. Since the fibers are relatively easy to understand by the Demazure resolution, we now focus on the behavior over a certain $\mathcal{G}_{O_C}$-semi-orbit.

Recall that for any $\lambda \in W_0 \cdot \mu$, where W_0 is the Weyl group of (G, S), the induced point $[\lambda]\colon \operatorname{Spd} C \to \operatorname{Gr}_{G,C,\mu}$ uniquely extends to a point $[\lambda]\colon \operatorname{Spd} O_C \to \mathcal{M}_{\mathcal{G},O_C,\mu}$ by properness of v-sheaf local models.

Definition 7.5. Let $\mathcal{M}^\circ_{\mathcal{G},\mu} \subset \mathcal{M}_{\mathcal{G},\mu}$ be the unique sub-v-sheaf whose base change $\mathcal{M}^\circ_{\mathcal{G},O_C,\mu} \subset \mathcal{M}_{\mathcal{G},O_C,\mu}$ to $\operatorname{Spd} O_C$ is given by the finite (non-disjoint) union

$$\mathcal{M}^\circ_{\mathcal{G},O_C,\mu} = \bigcup_{\lambda \in W_0 \cdot \mu} \mathcal{G}^\diamondsuit_{O_C} \cdot [\lambda]. \tag{7.11}$$

We recall that the elements $\lambda \in W_0 \cdot \mu$ are the rational conjugates of μ in $X_*(T)$ and correspond to the open Schubert orbits in the μ-admissible locus, see the discussion after Definition 3.11. Also, it is easy to see and left to the reader that the definition of $\mathcal{M}^\circ_{\mathcal{G},\mu}$ does not depend on the choice of the auxiliary maximal $\breve{F}$-split F-torus $S \subset G$ whose connected Néron model $\mathcal{S}$ embeds in $\mathcal{G}$. As we will see in the following lemma, the stabilizer of rational conjugates $[\lambda]$ is actually representable and well behaved.

Lemma 7.6. *Let $\mathcal{P}^-_\lambda$ be the flat closure in $\mathcal{G}_{O_C}$ of the repeller parabolic $P^-_\lambda \subset G_C$ defined by λ. Then*

(1) *$\mathcal{P}^{-,\diamondsuit}_\lambda$ is the $\mathcal{G}^\diamondsuit_{O_C}$-fixer of λ inside $\mathcal{M}_{\mathcal{G},O_C,\mu}$.*
(2) *$\mathcal{P}^-_\lambda \to \operatorname{Spec}(O_C)$ is smooth affine with connected fibers.*

Proof. By topological flatness, it is clear that $\mathcal{P}^{-,\diamondsuit}_\lambda$ fixes λ. For dimension reasons, the special fiber of $\mathcal{P}^-_\lambda$ is equal to the fixer in $\mathcal{G}_{O_C}$ of λ_I in the affine flag variety $\mathcal{F}\ell_{\mathcal{G},\bar{k}}$, which in particular shows that the special fiber of $\mathcal{P}^-_\lambda$ is connected. Having described the special fiber of $\mathcal{P}^-_\lambda$, we see that all (K, K^+)-valued points of the $\mathcal{G}^\diamondsuit_{O_C}$-fixer of λ inside $\mathcal{M}_{\mathcal{G},O_C,\mu}$ actually belong to the closed subgroup $\mathcal{P}^{-,\diamondsuit}_\lambda$, so it necessarily lies in that closed subgroup.

It suffices now to verify smoothness of $\mathcal{P}^-_\lambda$. (Observe that the conjugation action of λ on G_C does not always extend to $\mathcal{G}_{O_C}$, so $\mathcal{P}^-_\lambda$ is not a repeller subgroup.) By [BT84, Corollaire 2.2.5], we can do this by restricting to the flat closures of a-root groups of P^-_λ with respect to the (non-maximal) split torus S_C. Using the structure of $\mathcal{U}_a \subset \mathcal{G}$ defined over O, we see that this amounts to check that the morphism

$$\operatorname{Res}_{B/A}\mathbb{A}^1 \to \operatorname{Res}_{C/A}\mathbb{A}^1 \tag{7.12}$$

induced by a surjection $B \to C$ of finite free A-algebras is a smooth cover. Indeed, the a-root group of $\mathcal{P}^-_\lambda$ decomposes scheme-theoretically as a product of fibers of such morphisms. □

We want to uniquely characterize the left $\mathcal{G}^\diamondsuit_{O_C}$-equivariant right $\mathcal{G}^{>i,\diamondsuit}_{O_C}$-torsor

$$\mathcal{M}^{\mathrm{tor},\circ}_{\mathcal{G},O_C,\mu_i} := \mathcal{M}^{\mathrm{tor}}_{\mathcal{G},O_C,\mu_i} \times_{\mathcal{M}_{\mathcal{G},O_C,\mu_i}} \mathcal{M}^\circ_{\mathcal{G},O_C,\mu_i}. \tag{7.13}$$

For this, we use the following abstract statement.

Lemma 7.7. *Let $\mathfrak{X}$ be any topos, and let $J, A \in \mathfrak{X}$ be group objects. Let $Z := J/P$ be an orbit for J.*

(1) *The groupoid of left J-equivariant right A-torsors $\mathcal{T}$ over Z is equivalent to the groupoid of right A-torsors $\mathcal{S}$ over a terminal object equipped with a morphism of groups $\varphi_{\mathcal{T}}\colon P \to \operatorname{Aut}_A(\mathcal{S})$, with equivalence given by sending a $\mathcal{T}$ to the fiber $\mathcal{S} := \mathcal{T}_{1\cdot P}$ of $1 \cdot P \in Z$ with its action by P.*

(2) *If P is self-normalizing, then for each left J-equivariant right A-torsor $f\colon \mathcal{T}\to Z$ each morphism $\sigma\colon \mathcal{T}\to\mathcal{T}$, which is equivariant for J and A, is automatically a morphism of left J-equivariant right A-torsors, that is, $f\circ\sigma = f$. If furthermore $\varphi_{\mathcal{T}}$ has trivial centralizer, then $\sigma = \mathrm{id}$.*

Note that if $\mathcal{S}$ is trivial, then $\mathrm{Aut}_A(\mathcal{S})\cong A$.

Proof. For (1), it suffices to note that an inverse is given by sending a pair $(\mathcal{S}, P\to\mathrm{Aut}_A(\mathcal{S}))$ to the contracted product $J\times^P\mathcal{S}$. For (2), one notes that by A-equivariance σ descents to an J-equivariant morphism of Z. As P is self-normalizing, this morphism must be the identity. □

We can therefore conclude that the left $\mathcal{G}^{\diamondsuit}_{O_C}$-equivariant $\mathcal{G}^{>i,\diamondsuit}_{O_C}$-right torsor $\mathcal{M}^{\mathrm{tor},\circ}_{\mathcal{G},O_C,\mu_i}$ is governed by certain morphisms of groups

$$\mathcal{P}^{\diamondsuit}_{\lambda_i}\to(\mathcal{G}^{>i}_{\mathrm{ad},O_C})^{\diamondsuit} \tag{7.14}$$

for all rational conjugates λ_i of μ_i by applying Lemma 7.7. In our situation, the generic fiber of $\mathcal{M}^{\mathrm{tor},\circ}_{\mathcal{G},O_C,\mu_i}$ is quite well-understood, and thus we will apply the following general result describing extensions of a given left equivariant right torsor.

Lemma 7.8. *We use the notation of Lemma 7.7. Furthermore, let $Y\in\mathfrak{X}$ be any object, and denote by a subscript $(\text{-})_Y$ the base change to Y. Assume that $\widetilde{\mathcal{T}}\to Z_Y$ is a left J_Y-equivariant right A_Y-torsor with associated tuple $(\widetilde{\mathcal{S}},\varphi_{\widetilde{\mathcal{T}}}\colon P_Y\to\mathrm{Aut}_{A_Y}(\widetilde{\mathcal{S}}))$. Then the groupoid of pairs of a left J-equivariant right A-torsors $\mathcal{T}$ over Z with an isomorphism $\mathcal{T}_Y\cong\widetilde{\mathcal{T}}$ identifies with the groupoid of following data: $\mathcal{S}$ an A-torsor over a terminal object of $\mathfrak{X}$, an identification $\gamma\colon\mathcal{S}_Y\cong\widetilde{\mathcal{S}}$, and a morphism of groups $\varphi\colon P\to\mathrm{Aut}_A(\mathcal{S})$ such that φ_Y agrees with $\varphi_{\widetilde{\mathcal{T}}}$ under the identification $\mathrm{Aut}_{A_Y}(\mathcal{S}_Y)\cong\mathrm{Aut}_{A_Y}(\widetilde{\mathcal{S}})$ induced by γ.*

Proof. This follows from Lemma 7.7. □

In our case, the A-torsors $\widetilde{\mathcal{S}}\cong A_{Y\times Z},\mathcal{S}\cong A_Z$ we are interested in are trivial, and the morphism $A(\mathfrak{X})\to A(Y)$ is injective. Then φ is determined by γ (and $\varphi_{\widetilde{\mathcal{T}}}$), and after fixing a section $z\in\widetilde{\mathcal{S}}(Y)\subset\widetilde{\mathcal{T}}(Y)$ we get thus an injection from isomorphism classes of pairs $(\mathcal{T},\mathcal{T}_Y\cong\widetilde{\mathcal{T}})$ to $A(Y)/A(\mathfrak{X})$ by sending $(\mathcal{T},\mathcal{T}_Y\cong\widetilde{\mathcal{T}})$ to the class of $z^{-1}y_Y\in A(Y)/A(\mathfrak{X})$, where $y\in\mathcal{S}(\mathfrak{X})\subset\mathcal{T}(Y)$ is any section. If $\varphi_{\widetilde{\mathcal{T}}}$ has trivial centralizer, the groupoid of such pairs is equivalent to the groupoid of left J-equivariant right A-torsors $\mathcal{T}$ over Z, which are isomorphic to $\widetilde{\mathcal{T}}$ over Y (but we do not fix such an isomorphism).

Applying these considerations to the orbits in $\mathcal{M}^{\circ}_{\mathcal{G},O_C,\mu_i}$ with its left $\mathcal{G}^{\diamondsuit}_{O_C}$-equivariant right $\mathcal{G}^{>i}_{\mathrm{ad},O_C}$-torsor $\mathcal{M}^{\mathrm{tor},\circ}_{\mathcal{G},\mu_i}$ having generic fiber $\mathcal{F}^{\mathrm{tor}}_{G,\mu_i,C}$, we need therefore to

- fix base points over O_C in the orbits in $\mathcal{M}^{\circ}_{\mathcal{G},O_C,\mu_i}$,
- fix C-points $\tilde{\lambda}_j\in\mathcal{F}^{\mathrm{tor},\diamondsuit}_{G,C,\mu_i}$ lying over the generic fibers of the chosen base points in $\mathcal{M}^{\circ}_{\mathcal{G},O_C,\mu_i}$,
- find sections $y_j\in\mathcal{M}^{\circ,\mathrm{tor}}_{\mathcal{G},O_C,\mu_i}(O_C)$ lying over the chosen base points in $\mathcal{M}^{\circ}_{\mathcal{G},O_C,\mu_i}$,
- calculate the difference of $y_j^{-1}\tilde{\lambda}_j\in G^{>i}_{\mathrm{ad},C}(C)$, which yields the desired n classes modulo $\mathcal{G}^{>i}_{\mathrm{ad},O_C}(O_C)$.

The first point is easy as we can take the $[\lambda]\colon\operatorname{Spd}O_C\to\mathcal{M}^{\circ}_{\mathcal{G},O_C,\mu_i}$ with λ running through the rational Weyl group conjugates of μ_i. For the second point, we can fix a (suitable) uniformizer $\xi\in B^+_{\mathrm{dR}}(C)$ and consider the images of the $\lambda(\xi)\in LG(C)$ in $\mathcal{F}^{\mathrm{tor},\diamondsuit}_{\mathcal{G},\mu_i,C}$. To state the outcome, we have to make the following definition.

Definition 7.9. Let $\nu \in X_*(T)$. The different $\delta_G(\nu)$ is the class in $T(C)/\mathcal{T}(O_C)$ of

$$\prod_{\sigma \neq 1} \nu^\sigma(\pi_E^\sigma - \pi_E), \tag{7.15}$$

where $F \subset E \subset C$ is the reflex field of ν, $\pi_E \in O_E$ some uniformizer and σ varies over the non-trivial cosets in the quotient $\mathrm{Gal}_F/\mathrm{Gal}_E$ of the absolute Galois groups.

For a uniformizer $\pi_E \in O_E$ for a finite field extension E/F, contained in C, we denote by $\pi_E^\flat \in O_C^\flat$ a chosen sequence of compatible p^n-roots of π_E. Recall that $\xi_E := \pi_E - [\pi_E^\flat] \in W_{O_E}(O_C^\flat)$ maps to a uniformizer of $B_{\mathrm{dR}}^+(C)$. We can now define the $\tilde{\lambda}$ as the images of $\lambda(\xi_F) \in \mathcal{F}^{\mathrm{tor},\diamondsuit}_{\mathcal{G},C,\mu_i}(C)$ for any $\lambda \in W_0 \cdot \mu_i$.

Proposition 7.10. *The v-sheaf $\mathcal{M}^{\mathrm{tor},\circ}_{\mathcal{G},,O_C,\mu_i}$ is the unique left $\mathcal{G}_{O_C}$-equivariant right $\mathcal{G}^{>i}_{\mathrm{ad},O_C}$-torsor over $\mathcal{M}^{\circ}_{\mathcal{G},O_C,\mu_i}$ with generic fiber isomorphic to $\mathcal{F}^{\mathrm{tor},\diamondsuit}_{\mathcal{G},C,\mu_i}$ determined by the images of $\delta_G(\lambda)$ in $G^{>i}_{\mathrm{ad},C}(C)/\mathcal{G}^{>i}_{\mathrm{ad},O_C}(O_C)$ for $\lambda \in W_0 \cdot \mu_i$ (and the above choices for the $\tilde{\lambda}$'s).*

Proof. Let us fix some $\lambda \in W_0 \cdot \mu_i$. Consider the morphism

$$T' := \mathrm{Res}_{E/F}\mathbb{G}_m \to G \tag{7.16}$$

of algebraic groups induced by λ as follows: compose its Weil restriction $T' := \mathrm{Res}_{E/F}\mathbb{G}_m \to \mathrm{Res}_{E/F}T_E$ with the norm map $\mathrm{Res}_{E/F}T_E \to T$. Note that $\lambda\colon \mathbb{G}_{m,E} \to G_E$ can be reconstructed from this composition by restricting its base change to E to the first factor. Set $\mathcal{T}' := \mathrm{Res}_{O_E/O}\mathbb{G}_m$. We will now construct a section $y \in L_O\mathcal{T}'(O_C)$, whose image in $\mathrm{Gr}_{\mathcal{G}}$ is the section $[\lambda]$, and then calculate $y^{-1}\tilde{\lambda} \in LT(C)$ as necessary.

For this we claim that the element $\xi_E = \pi_E - [\pi_E^\flat]$ becomes a unit after inverting $\xi_F = \pi_F - [\pi_F^\flat]$, thus giving rise to an element $y \in \mathbb{G}_m(W_{O_E}(O_C^\flat)) \subset L_O\mathcal{T}'(O_C)$. Indeed, let $P(X) = X^d + a_1X^{d-1} + \ldots + a_d$ be the minimal polynomial of π_E over F, which is Eisenstein as E/F is totally ramified. Then the norm of ξ_E in $W_{O_F}(O_C^\flat)$ equals

$$P([\pi_E^\flat]) = [\pi_E^\flat]^d + a_1[\pi_E^\flat]^{d-1} + \cdots + a_d. \tag{7.17}$$

Reducing modulo ξ_F, this element certainly vanishes because $[\pi_E^\flat] \equiv [\pi_E^\flat]^\sharp = \pi_E$ modulo ξ_F. On the other hand, $P([\pi_E^\flat])$ is clearly a primitive element of degree 1 inside $W_O(O_C^\flat)$, as $a_d \in \pi_F O^\times$. Hence, $P([\pi_E^\flat])$ and ξ_F generate the same principal ideal, see [BS19, Lemma 2.24].

Let us now consider the generic fiber of the point y. For this we must pass to

$$B_{\mathrm{dR}}(C) \otimes_F E = \prod_\sigma B^\sigma_{\mathrm{dR}}(C). \tag{7.18}$$

Here, notice that we are conjugating the natural E-structure on the corresponding factor of the right side by σ. Then the coordinate of ξ_E for $\sigma \neq 1$ is a unit in $B^+_{\mathrm{dR}}(C)$ which reduces modulo the uniformizer ξ_F to $\pi_E^\sigma - \pi_E$. As for its coordinate on the $\sigma = 1$ factor, it must be a prime element, because so is the norm in $B_{\mathrm{dR}}(C)$, as seen in the previous calculation. We can conclude that the generic fiber of the section y maps to λ, and that $\tilde{\lambda}^{-1}y$ is $\delta_G(\lambda)$. □

The remark below will not be needed in the continuation.

Remark 7.11. Inspecting Proposition 7.10, one sees that $\mathcal{M}^{\mathrm{tor},\circ}_{\mathcal{G},O_C,\mu_i}$ is representable by a smooth O_E-scheme. Indeed, the defining maps $\varphi_\lambda\colon \mathcal{P}^{-,\diamondsuit}_\lambda \to \mathcal{G}^{>i,\diamondsuit}_{\mathrm{ad},O_C}$ are naturally algebraic over a finite unramified extension of E and so are their integral models by [BT84, Proposition 1.7.6]. The considerations after Lemma 7.8 then furnishes an algebraic space, and it is a scheme because it is a torsor over a scheme under an affine group scheme.

7.2. **Specialization maps.** The aim of this subsection is to characterize the specialization map. We are going to see that it is already determined by the semi-orbit. In the following, we consider all pairs $(\mathcal{G}, \mu_I)$ where $\mathcal{G}$ is a parahoric O-model of some reductive F-group G, I some finite index set and $\mu_I = (\mu_i)_{i\in I}$ is a sequence of minuscule coweights in G_C such that $\sum_{i\in I} \mu_i$ is still minuscule (and so are all subsums). Morphisms of such pairs $(\mathcal{G}, \mu_I) \to (\mathcal{G}', \mu'_J)$ are given by morphisms of O-group schemes $\mathcal{G} \to \mathcal{G}'$, surjections of sets $\mathrm{pr}\colon I \twoheadrightarrow J$ such that, for all $j \in J$, the image of $\sum_{i\in \mathrm{pr}^{-1}(j)} \mu_i$ in G'_C lies in the conjugacy class of μ'_j. This generalizes the functoriality considered in Proposition 4.16. We denote $(\mathcal{G}, \mu_I)$ also by $(\mathcal{G}, \mu_\bullet)$ if the index set I is understood. By Lemma 7.1 the convolution local model $\mathcal{M}_{\mathcal{G},\mu_\bullet}$ admits a specialization map, see Section 2.3.

Theorem 7.12. *The specialization maps for all pairs $(\mathcal{G}, \mu_\bullet)$ as above*

$$\mathrm{sp}_{\mathcal{G},\mu_\bullet}\colon |\mathcal{F}_{G,C,\mu_\bullet}| \to |\mathcal{A}_{\mathcal{G},\bar{k},\mu_\bullet}| \tag{7.19}$$

are the only functorial collection of continuous and spectral maps, whose restrictions to $\mathcal{M}^\circ_{\mathcal{G},O_C,\mu_\bullet}(O_C)$ agree with the reduction maps induced by $O_C \to \bar{k}$.

Proof. We are going to uniquely determine the values taken by $\mathrm{sp}_{\mathcal{G},\mu_\bullet}$ on the subset $\mathcal{F}_{G,\mu_\bullet}(K)$ for a cofinal set of finite extensions K/F given by those Galois extensions that split G. This characterizes the map $\mathrm{sp}_{\mathcal{G},\mu_\bullet}$ by continuity with respect to the constructible topology. Indeed, $\mathcal{F}_{G,\mu_\bullet}$ is a smooth rigid space defined over E, see [Gle24, Theorem 4.47]. (If μ is not minuscule it is not true that the $\bar{\mathbb{Q}}_p$-points of $\mathrm{Gr}_{G,\mu}$ are dense for the constructible topology because Białynicki-Birula maps give a bijection between the $\bar{\mathbb{Q}}_p$-points.)

Having fixed a Galois extension K/F splitting G, we are however allowed to enlarge the parahoric group $\mathcal{G}$ in order to compute these values. In particular, we may and do assume that $G = \mathrm{Res}_{K/F} H$, where $H = G_K$ is a split reductive group. Refining $\mu_\bullet$ so that every element in the sequence is tiny allows us to conclude that every K-valued point of $\mathcal{F}_{G,\mu_\bullet}$ extends to an O_K-valued point of the semi-homogeneous $\mathcal{M}^\circ_{\mathcal{G},\mu_\bullet}$ by Lemma 7.13 below. ☐

Lemma 7.13. *Suppose K/F is Galois, $G = \mathrm{Res}_{K/F} H$ and all $\mu_i \in \mu_\bullet$ are tiny. Then, we have an equality*

$$\mathcal{M}^\circ_{\mathcal{G},\mu_\bullet}(O_K) = \mathcal{F}_{G,\mu_\bullet}(K) \tag{7.20}$$

of sets induced by the natural morphism.

Proof. Let us assume first that $K = F$. Then $\mathcal{G} = \mathcal{H}$ is a parahoric model of a split reductive F-group and the right side can be given by the Iwasawa decomposition

$$\mathcal{F}_{G,\mu}(K) = \bigcup_\lambda \mathcal{H}(O_K) \cdot \lambda, \tag{7.21}$$

see [BT72, Proposition 4.4.3]. Now, obviously the points of the form λ extend to integral points of $\mathcal{M}^\circ_{\mathcal{H},\mu}$, due to the splitness assumption, and thus the same holds for its $\mathcal{H}(O_K)$-orbits.

Now, consider an arbitrary finite Galois extension K/F. We get the result immediately for tiny coweights, as the $\mathcal{G}(O_K)$-action on $\mathcal{M}_{\mathcal{G},\mu}$ is via the parahoric subgroup $\mathcal{H}(O_K) \subset H(K)$. In general, we use this to show the claim by an inductive procedure. Suppose one is given an element $(x_1, \dots, x_n) \in \mathcal{F}_{G,\mu_\bullet}(K)$, such that each of the representatives $x_j \in \mathcal{F}^{\mathrm{tor}}_{G,\mu_j}(K)$ lies in $\mathcal{M}^{\mathrm{tor},\circ}_{\mathcal{G},\mu_j}(O_K)$ for $j < i$. Now note that x_i is in the $\mathcal{G}^{>i}(O_K)$-orbit of $\mathcal{M}^{\mathrm{tor},\circ}_{\mathcal{G},\mu_i}(O_K)$, due to the $n = 1$ case and the fact that the O_C-section of Proposition 7.10 descends to O_K, see also Remark 7.11. So we may replace it in the expression, and now the assumption holds for all $j < i + 1$. After finitely many steps of this iteration, we get the claim. ☐

Remark 7.14. It follows by inspecting the proof of Theorem 7.12 that in order to compute the specialization mapping, it is enough to consider Weil restrictions of split groups and their Iwahori models, see Assumption 7.16.

Remark 7.15. He–Pappas–Rapoport [HPR20, Conjecture 2.12] conjecture that, for any fixed pair $(\mathcal{G}, \mu)$, there is at most one flat projective O_E-scheme equipped with an $\mathcal{G}_{O_E}$-action having the correct fibers, identified $\mathcal{G}$-equivariantly. This is much stronger than Theorem 7.12 above, since it makes no reference to convolution or functoriality. Our approach is inspired by their conjecture in applying equivariant methods to pin down the specialization map.

7.3. **Comparison isomorphisms.** In this subsection, we use our work from the previous ones to establish certain comparison isomorphisms between (at least some of) our local models and those that have appeared elsewhere, see [PZ13, Lev16, Lou23a, FHLR22]. During this subsection, we shall work under the following:

Assumption 7.16. Given a pinned split simple adjoint group (H, T_H, B_H, e_H), let $G = \mathrm{Res}_{K/F} H$ with K/F an arbitrary finite extension, with K_0/F being the maximal unramified subextension. Also let $\mathcal{I}$ be the standard Iwahori model with respect to the chosen pinning.

In order to prove Theorem 7.21, we need to compare $\mathcal{M}_{\mathcal{I},\mu}$ to certain candidates $\mathcal{N}^{\mathrm{sch}}_{\underline{\mathcal{I}},\mu}$ constructed in [FHLR22, Definition 5.11] and denoted $\widetilde{M}_{\mathcal{G},\mu}$ there, which are variations on the work of Levin [Lev16]. Here, $\underline{\mathcal{I}}$ is a $O[\![t]\!]$-lift of $\mathcal{I}$ along $t \mapsto \pi$, obtained by taking restriction of scalars along an ad hoc lift $O[\![t]\!] \to O_0[\![u]\!]$ of $O \to O_K$ of the dilatation of $H \otimes O_0[\![u]\!]$ along $B_H \otimes O_0$ concentrated in the u-divisor, see [PZ13, Theorem 4.1] and [MRR20, Definition 2.1, Example 3.3]. The various lifts $O[\![t]\!] \to O_0[\![u]\!]$ defined in [FHLR22, Subsection 2.2] are given by choosing uniformizers and lifting Eisenstein polynomials over O_0 in such a way that they remain separable Eisenstein over both $k_0[\![t]\!]$ and $K_0[\![t]\!]$.

One has a schematic Beilinson–Drinfeld Grassmannian $\mathrm{Gr}^{\mathrm{sch}}_{\underline{\mathcal{I}}}$ defined in terms of power series rings, classifying $\underline{\mathcal{I}}$-torsors over $R[\![t-\pi]\!]$ trivialized over $R(\!(t-\pi)\!)$, and admitting uniformization via loop groups $L^{\mathrm{sch}}_O \underline{\mathcal{I}}/L^{\mathrm{sch},+}_O \underline{\mathcal{I}}$. The generic fiber is equivariantly isomorphic to the schematic affine Grassmannian $\mathrm{Gr}^{\mathrm{sch}}_G$ over F, see [FHLR22]. So we get an embedding $\mathcal{F}_{G,\mu} \subset \mathrm{Gr}^{\mathrm{sch}}_G|_{\mathrm{Spec}\, E}$ for a minuscule coweight μ.

Definition 7.17. The O_E-scheme $\mathcal{N}^{\mathrm{sch}}_{\underline{\mathcal{I}},\mu}$ is defined as the seminormalization of the flat closure of $\mathcal{F}_{G,\mu}$ inside $\mathrm{Gr}^{\mathrm{sch}}_{\underline{\mathcal{I}},O_E}$. For a minuscule sequence $\mu_\bullet$ of dominant coweights, we set $\mathcal{N}^{\mathrm{sch}}_{\underline{\mathcal{I}},O_E,\mu_\bullet}$ as the convolution product of the $\mathcal{N}^{\mathrm{sch}}_{\underline{\mathcal{I}},O_E,\mu_i}$. We define the $\mathcal{I}^{>i}_{O_C}$-torsor $\mathcal{N}^{\mathrm{sch,tor}}_{\underline{\mathcal{I}},O_C,\mu_i}$ by pushing forward the universal $L^{\mathrm{sch},+}_{O_C}\underline{\mathcal{I}}$-torsor under the natural projection.

The $\mathcal{N}^{\mathrm{sch}}_{\underline{\mathcal{I}},\mu}$ are normal O_E-schemes by [FHLR22, Theorem 5.14], so their formation commutes with base change to O_C. They also come with transition morphisms

$$\mathcal{N}^{\mathrm{sch}}_{\underline{\mathcal{I}},O_C,\mu} \to \mathcal{N}^{\mathrm{sch}}_{\underline{\tilde{\mathcal{I}}},O_C,\tilde{\mu}}, \tag{7.22}$$

which are closed immersions, where $\underline{\tilde{\mathcal{I}}}$ is a further $O[\![t]\!]$-lift defined with respect to the extension $\tilde{K}/K/F$, see [FHLR22, Section 5.3], and $\tilde{\mu}$ is the image of μ in $\tilde{G}$. In what follows, we shall simply say they are functorial in $(\underline{\mathcal{I}}, \mu)$.

Our next goal is to compare the v-sheaves associated with $\mathcal{N}^{\mathrm{sch}}_{\underline{\mathcal{I}},O_C,\mu_\bullet}$ to $\mathcal{M}_{\mathcal{G},O_C,\mu_\bullet}$. We start by recording what happens in the generic fiber.

Lemma 7.18. *There are unique equivariant isomorphisms*

$$\mathcal{N}^{\mathrm{sch,tor}}_{\underline{\mathcal{I}},\mu}|_{\mathrm{Spec}\, C} \cong \mathcal{F}^{\mathrm{tor}}_{G,C,\mu_i} \tag{7.23}$$

for each term μ_i of the sequence $\mu_\bullet$. They yield canonical equivariant isomorphisms

$$\mathcal{N}^{\mathrm{sch}}_{\underline{\mathcal{I}},\mu_\bullet}|_{\mathrm{Spec}\, C} \cong \mathcal{F}_{G,C,\mu_\bullet} \tag{7.24}$$

functorially in $(\underline{\mathcal{I}}, \mu_\bullet)$.

Proof. This follows by definition and uniqueness is ensured by Lemma 7.7. □

Next, we need to take care of the special fiber:

Proposition 7.19. *There are unique equivariant isomorphisms*

$$\left(\mathcal{N}^{\mathrm{sch,tor}}_{\underline{\mathcal{I}},\mu_i}|_{\operatorname{Spec}\bar{k}}\right)^{\mathrm{perf}} \cong \mathcal{A}^{\mathrm{tor}}_{\mathcal{I},\mu_i,\bar{k}} \tag{7.25}$$

for each term μ_i of the sequence $\mu_\bullet$. They yield canonical equivariant isomorphisms

$$\left(\mathcal{N}^{\mathrm{sch}}_{\underline{\mathcal{I}},\mu_\bullet}|_{\operatorname{Spec}\bar{k}}\right)^{\mathrm{perf}} \cong \mathcal{A}_{\mathcal{I},\mu_\bullet,\bar{k}} \tag{7.26}$$

functorially in $(\underline{\mathcal{I}},\mu_\bullet)$.

Proof. Set $\mathcal{I}'=\underline{\mathcal{I}}\otimes k[\![t]\!]$, a standard Iwahori model of the connected reductive group $G'=\operatorname{Res}_{k_0(\!(u)\!)/k(\!(t)\!)}H$. By [FHLR22, Theorem 5.14], we have $\mathcal{N}^{\mathrm{sch,perf}}_{\underline{\mathcal{I}},\mu,\bar{k}}=\mathcal{A}_{\mathcal{I}',\mu',\bar{k}}$. Hence, the statement above is just a generalization of Lemma 3.15 to convolution products.

Let $w\in\tilde{W}$ be an element such that $\mathcal{F}\ell_{\mathcal{I},\bar{k},w}\subset\mathcal{A}_{\mathcal{I},\mu_i,\bar{k}}$ and choose a Demazure resolution $\pi_{\dot{w}}\colon\mathcal{D}_{\mathcal{I},\bar{k},\dot{w}}\to\mathcal{F}\ell_{\mathcal{I},\bar{k},w}$. We have to compare the following pullback square

$$\begin{array}{ccc}
\mathcal{D}^{\mathrm{tor}}_{\mathcal{I},\bar{k},\dot{w}} & \xrightarrow{\pi^{\mathrm{tor}}_{\dot{w}}} & \mathcal{F}\ell^{\mathrm{tor}}_{\mathcal{I},\bar{k},w} \\
\downarrow{\scriptstyle p_{\mathcal{D}}} & & \downarrow{\scriptstyle p_{\mathcal{F}\ell}} \\
\mathcal{D}_{\mathcal{I},\bar{k},\dot{w}} & \xrightarrow{\pi_{\dot{w}}} & \mathcal{F}\ell_{\mathcal{I},\bar{k},w}
\end{array} \tag{7.27}$$

with its counterpart in the equicharacteristic setting. The bottow arrow was dealt with in Lemma 3.15 and the second paragraph there applies verbatim to comparing the left arrow. We claim that these suffice to recover the remainder of the diagram.

Note that the vertical arrows affine morphisms, so they can be written via relative spectra, that is, we have $\mathcal{D}^{\mathrm{tor}}_{\mathcal{I},\bar{k},\dot{w}}=\operatorname{Spec}(p_{\mathcal{D},*}\mathcal{O}_{\mathcal{D}^{\mathrm{tor}}})$ and $\mathcal{F}\ell^{\mathrm{tor}}_{\mathcal{I},\bar{k},w}=\operatorname{Spec}(p_{\mathcal{F}\ell,*}\mathcal{O}_{\mathcal{F}\ell^{\mathrm{tor}}})$. On the other hand, we know that $\pi_{\dot{w},*}\mathcal{O}_{\mathcal{D}}=\mathcal{O}_{\mathcal{F}\ell}$, so the same equality holds for $\pi^{\mathrm{tor}}_{\dot{w}}$ by flat base change, as the vertical arrows are perfectly smooth. But this means $p_{\mathcal{F}\ell,*}\mathcal{O}_{\mathcal{F}\ell^{\mathrm{tor}}}=\pi_{\dot{w},*}p_{\mathcal{D},*}\mathcal{O}_{\mathcal{D}^{\mathrm{tor}}}$, just as asserted.

Next, we show that the isomorphisms constructed above are unique. Without torsors, this has been verified in Proposition 3.10, so any automorphism respects the orbit part $\mathcal{F}\ell^{\circ,\mathrm{tor}}_{\mathcal{I},w}$. So we only have to verify the conditions of Lemma 7.7 on centralizers of the transfer homomorphism φ_w. In this case, it is given by

$$\operatorname{int}(w^{-1})\colon L^+\mathcal{I}\cap wL^+\mathcal{I}w^{-1}\mapsto w^{-1}L^+\mathcal{I}w\cap L^+\mathcal{I}. \tag{7.28}$$

We see that the image of the right side in $(\mathcal{I}^{>i}_{\bar{k}})^{\mathrm{perf}}$ contains the image of $\mathcal{B}^{\mathrm{perf}}_{\bar{k}}$, where $\mathcal{B}\subset\mathcal{G}$ is the flat closure of some Borel $B\subset G$. However, inspecting the description of $\mathcal{I}^{>i}_{\bar{k}}$ given in Lemma 7.3, we see that the centralizer must be trivial: indeed, it is contained in $\mathcal{T}^{>i}_{\bar{k}}$, which itself decomposes as a product of groups indexed by positive simple $\mathcal{S}^{>i}_{\bar{k}}$-roots a, acting faithfully on the corresponding a-root groups.

Finally, we must show that the isomorphisms just constructed are functorial with respect to $(\underline{\mathcal{I}},\mu_\bullet)$. This is easy for $\underline{\mathcal{I}}$, by uniqueness of equivariant automorphisms. As for $\mu_\bullet$, we appeal to Proposition 3.2 and the calculation of Picard groups in Theorem 3.8 and Remark 3.9 to recover the Stein factorization of the proper surjection

$$\mathcal{A}_{\mathcal{I},\mu_\bullet}\to\mathcal{A}_{\mathcal{I},\mu}. \tag{7.29}$$

In the equicharecteristic setting, the Stein factorization is already $\mathcal{A}_{\mathcal{I}',\mu'}$ due to Zariski's connectedness theorem applied to $\mathcal{N}^{\mathrm{sch}}_{\underline{\mathcal{I}},O_C,\mu}$. Therefore, we get a new equivariant surjection $\mathcal{A}_{\mathcal{I}',\mu'}\to$

$\mathcal{A}_{\mathcal{I},\mu}$, which becomes the identity after composing with the isomorphism $\mathcal{A}_{\mathcal{I},\mu} \cong \mathcal{A}_{\mathcal{I}',\mu'}$ of Lemma 3.15, by the uniqueness proved in Proposition 3.10. ☐

The last comparison involves the semi-orbits.

Proposition 7.20. *There are unique equivariant isomorphisms*

$$(\mathcal{N}^{\mathrm{sch},\circ,\mathrm{tor}}_{\mathcal{I},O_C,\mu_i})^{\diamondsuit} \simeq \mathcal{M}^{\circ,\mathrm{tor}}_{\mathcal{I},O_C,\mu_i} \tag{7.30}$$

for each term μ_i of the sequence $\mu_\bullet$. They yield canonical equivariant isomorphisms

$$(\mathcal{N}^{\mathrm{sch},\circ}_{\mathcal{I},O_C,\mu_\bullet})^{\diamondsuit} \simeq \mathcal{M}^{\circ}_{\mathcal{I},O_C,\mu_\bullet}, \tag{7.31}$$

compatibly with those of Lemma 7.18 and Proposition 7.19 in the obvious sense.

Proof. We have already identified the generic fibers of these v-sheaves, see Lemma 7.18. By Lemma 7.8, we reduce to calculating $\mathrm{Spd}\, O_C$-valued points of the left side torsor and compare their residue to that of Proposition 7.10. The resulting isomorphism will then reduce to the expected isomorphisms over $\mathrm{Spd}\, k$ obtained in Proposition 7.19, by uniqueness of equivariant automorphisms.

Now, we repeat the same calculation of Proposition 7.10, that goes back to Zhu [Zhu14], see [Lev16, Proposition 4.2.8]. Here, we work with the power series loop group $L^{\mathrm{sch}}_{O_C}\underline{\mathcal{T}}$ in the setting of [FHLR22, Section 2.2]. After refining $\mu_\bullet$, we may and do assume that each term $\mu_i \in \mu_\bullet$ is concentrated in a single component of the Dynkin diagram of G. There is a natural map

$$\mathrm{Res}_{O[\![u]\!]/O[\![t]\!]}\mathbb{G}_m \to \underline{\mathcal{T}} \tag{7.32}$$

induced by λ_i via taking the norm of restriction of scalars. Hence, we may and do assume that $\underline{\mathcal{T}} = \mathrm{Res}_{O[\![u]\!]/O[\![t]\!]}\mathbb{G}_m$. Note that here $O[\![u]\!]$ is a finite $O[\![t]\!]$-algebra, where u satisfies an Eisenstein–Teichmüller type polynomial

$$u^n + a_1(t)u^{n-1} + \cdots + a_n(t) = 0 \tag{7.33}$$

in t based on some fixed choices of uniformizers π_K for K and π for F, see [FHLR22, beginning of Section 2.2]. Now, we claim for any $\sigma \in \mathrm{Gal}_F$, the element $\sigma z_u = u - \sigma\pi_K$ is a unit in $O_C[\![u]\!][z_t^{-1}]$ with $z_t = t - \pi$. Notice that its norm in $O_C[\![t]\!]$ equals

$$P(t) = \sigma\pi_K^n + a_1(t)\sigma\pi_K^{n-1} + \cdots + a_n(t) \tag{7.34}$$

which is the product of z_t with a unit of $O_C[\![t]\!]$. Indeed, we calculate the value $P(\pi) = 0$, also of the first derivative $P'(\pi) \in O_C^\times$ and apply the Taylor series inside $C[\![t]\!]$. Finally, we notice that σz_u reduces to the unit $\tau\pi_K - \sigma\pi_K$ for all $\tau \neq \sigma$ of $\mathrm{Gal}_F/\mathrm{Gal}_K$ in $C[\![u]\!][z_t^{-1}]$, and to a prime element in the factor indexed by σ due to norm considerations. The desired claim has been shown. ☐

7.4. **The Scholze–Weinstein conjecture.**

In this subsection, we finally prove the Scholze–Weinstein conjecture, see Theorem 7.21 and Theorem 7.23 below.

We start by adressing the representability problem as in [Lou20, Conjecture IV.4.18], which is one half of [SW20, Conjecture 21.4.1]. Recall that $F/\mathbb{Q}_p$ is a complete non-archimedean field with perfect residue field k, G is an arbitrary (connected) reductive F-group, μ is a dominant coweight of G_C and $\mathcal{G}$ an arbitrary parahoric O-model of G.

Theorem 7.21. *Let μ be minuscule. Then, there is a unique (up to unique isomorphism) flat, projective and weakly normal O_E-model $\mathcal{M}^{\mathrm{sch}}_{\mathcal{G},\mu}$ of the E-scheme $\mathcal{F}_{G,\mu}$ endowed with a $\mathcal{G}_{O_E}$-action for which*

$$\mathcal{M}^{\mathrm{sch},\diamondsuit}_{\mathcal{G},\mu} \cong \mathcal{M}_{\mathcal{G},\mu}, \tag{7.35}$$

prolonging $\mathcal{F}^{\diamondsuit}_{G,\mu} \cong \mathrm{Gr}_{G,\mu}$ equivariantly under $\mathcal{G}^{\diamondsuit}_{O_E}$.

Proof. First of all, let us work under Assumption 7.16. We know that the geometric fibers of $\mathcal{N}^{\mathrm{sch},\diamondsuit}_{\underline{\mathcal{I}},O_C,\mu}$ and $\mathcal{M}_{\mathcal{I},O_C,\mu}$ are uniquely equivariantly isomorphic by Lemma 7.18, Proposition 7.19. By uniqueness, this commutes with the Galois action, so it descends to the fibers over $\operatorname{Spd} O_E$.

Furthermore, thanks also to Proposition 7.20, Theorem 7.12, and Remark 7.14, we know that the specialization maps

$$\mathrm{sp}\colon \mathcal{F}_{G,\mu}(C) \to \mathcal{A}_{\mathcal{I},\mu}(\bar{k}), \tag{7.36}$$

arising respectively from the π-adic kimberlite $\mathcal{M}_{\mathcal{I},O_C,\mu}$ and $\mathcal{N}^{\mathrm{sch},\diamondsuit}_{\underline{\mathcal{I}},O_C,\mu}$ must coincide. By continuity for the constructible topology, we obtain an equivariant isomorphism of specialization triples:

$$\big(\mathcal{N}^{\mathrm{sch},\diamondsuit}_{\underline{\mathcal{I}},E,\mu}, \mathcal{N}^{\mathrm{sch},\diamondsuit}_{\underline{\mathcal{I}},\mu,k_E}, \mathrm{sp}_{\check{\mathcal{N}}^{\mathrm{sch}}_{\underline{\mathcal{I}},\mu}}\big) \cong \big(\mathcal{M}_{\mathcal{I},E,\mu}, \mathcal{M}_{\mathcal{I},\mu,k_E}, \mathrm{sp}_{\check{\mathcal{M}}_{\mathcal{I},\mu}}\big) \tag{7.37}$$

associated with v-sheaves over $\operatorname{Spd} O_E$. Observing that both v-sheaves satisfy the hypothesis of Theorem 2.37, we may directly appeal to it in order to get a necessarily equivariant isomorphism

$$\mathcal{N}^{\mathrm{sch},\diamondsuit}_{\underline{\mathcal{I}},\mu} \cong \mathcal{M}_{\mathcal{I},\mu}. \tag{7.38}$$

Now maintain the part of Assumption 7.16 that refers to G, but suppose $\mathcal{G}$ is now an arbitrary parahoric model such that $\mathcal{I} \to \mathcal{G}$. We get a v-cover

$$\mathcal{M}_{\mathcal{I},\mu} \to \mathcal{M}_{\mathcal{G},\mu} \tag{7.39}$$

and, paralelly, a scheme-theoretic projective cover

$$\mathcal{N}^{\mathrm{sch}}_{\underline{\mathcal{I}},\mu} \to \mathcal{N}^{\mathrm{sch}}_{\underline{\mathcal{G}},\mu} \tag{7.40}$$

by virtue of [FHLR22, Section 5.3]. Therefore, it is enough to verify that the v-sheaf-theoretic equivalence relations coincide along the left side identification.

By construction, this reduces to $\mathcal{I}_k^{\mathrm{perf}}$-equivariantly compare the surjection

$$\mathcal{A}_{\mathcal{I},\mu} \cong \mathcal{A}_{\mathcal{G},\mu} \tag{7.41}$$

to the one obtained in the equicharacteristic situation. This is entirely similar to what was done in Lemma 3.15 and Proposition 7.19, so we omit it.

Finally, suppose that G is arbitrary. Thanks to Proposition 4.16, $\mathcal{M}_{\mathcal{G},\mu}$ is isomorphic to $\mathcal{M}_{\mathcal{G}_{\mathrm{ad}},\mu_{\mathrm{ad}}}$ after base change to $\operatorname{Spd} O_E$, and decomposes into products, hence we may assume G is simple and adjoint. We can find a locally closed immersion

$$\mathcal{G} \to \tilde{\mathcal{G}}, \tag{7.42}$$

where $\tilde{\mathcal{G}}$ is a parahoric model of a Weil-restricted split form of G, which was treated in the previous paragraph. Since we have an inclusion $\mathcal{M}_{\mathcal{G},\mu} \subset \mathcal{M}_{\tilde{\mathcal{G}},\tilde{\mu}}$, it now suffices to take the absolute weak normalization of the flat closure of $\mathcal{F}_{G,\mu}$ inside the scheme-theoretic local model attached to $(\tilde{\mathcal{G}}, \tilde{\mu})$. □

Remark 7.22. Let us explain how representability can be proved for classical groups without resorting to the characterization of the specialization map found in Theorem 7.12. Indeed, for those groups we can directly understand the v-sheaves $\mathcal{M}^{\mathrm{tor}}_{\mathcal{G},\mu_i}$ by embedding them in a similar torsor attached to Weil-restricted PGL_n. Those had been studied already by Pappas–Rapoport, see [PR05, Proposition 5.2], and a careful analysis of the map in [SW20, Proposition 21.6.9] reveals that all proposed definitions coincide. The result follows by v-descent.

We have found certain finite type O_E-schemes $\mathcal{M}^{\mathrm{sch}}_{\mathcal{G},\mu}$ representing $\mathcal{M}_{\mathcal{G},\mu}$, but we still do not know a lot about the geometry of its special fiber, see the discussion after Conjecture 1.1. We recall the canonical deperfection $\mathcal{A}^{\mathrm{can}}_{\mathcal{G},\mu}$ of the μ-admissible locus introduced in Definition 3.11 and Definition 3.14.

Theorem 7.23. *Under Assumption 1.9 and Assumption 1.13, the special fiber of the O_E-scheme $\mathcal{M}^{\mathrm{sch}}_{\mathcal{G},\mu}$ of Theorem 7.21 is uniquely $\mathcal{G}_{k_E}$-equivariantly isomorphic to the canonical deperfection of the μ-admissible locus:*

$$\mathcal{M}^{\mathrm{sch}}_{\mathcal{G},\mu}|_{\mathrm{Spec}\, k_E} \cong \mathcal{A}^{\mathrm{can}}_{\mathcal{G},\mu} \tag{7.43}$$

In particular, $\mathcal{M}^{\mathrm{sch}}_{\mathcal{G},\mu}$ is normal, Cohen–Macaulay and has a reduced, weakly normal, Frobenius split special fiber.

Proof. During the proof of Theorem 7.21, we already saw that the algebraic local models $\mathcal{M}^{\mathrm{sch}}_{\mathcal{G},\mu}$ are actually the $\mathcal{N}^{\mathrm{sch}}_{\underline{\mathcal{G}},\mu}$ constructed in [FHLR22, Definition 5.11] by a variation on the techniques of Pappas–Zhu [PZ13], Levin [Lev16] and also the third author [Lou23a, Lou20]. For this, we may pass to a finite unramified extension of F, so G is quasi-split and residually split, so that $\mathcal{N}^{\mathrm{sch}}_{\underline{\mathcal{G}},\mu}$ is defined (under Assumption 1.9). Then, it embeds in a local model associated with a Weil-restricted split group, confer [FHLR22, Section 5.3.2] (this is where Assumption 1.13 is used). We conclude under the given hypothesis that the $\mathcal{M}^{\mathrm{sch}}_{\mathcal{G},\mu}$ are indeed normal, Cohen–Macaulay and have a Frobenius split special fiber by [FHLR22, Theorem 5.14]. Indeed, the special fiber of $\mathcal{N}^{\mathrm{sch}}_{\underline{\mathcal{G}},\mu}$ is reduced equal to an admissible locus $\mathcal{A}^{\mathrm{can}}_{\mathcal{G}',\mu'}$ in the equicharacteristic setting, which equivariantly identifies with $\mathcal{A}^{\mathrm{can}}_{\mathcal{G},\mu}$ by Lemma 3.15. □

Remark 7.24. More generally, [GL24, Corollary 1.4] proves Theorem 7.23 without Assumption 1.13 but still assuming Assumption 1.9. Invoking [Lou23a, Lou20] we get Theorem 7.23 except if $p = 2$ and G_{ad} has an odd unitary $\breve{F}$-factor defined by a ramified, quadratic root-of-unit extension. The remaining case is handled in [CL25], except for the assertion on Cohen–Macaulayness, compare with Remark 3.17.

To conclude, let us only use Theorem 7.21 –and not resort to the construction of local models in [FHLR22]– in order to study the geometry of the special fiber of $\mathcal{M}^{\mathrm{sch}}_{\mathcal{G},\mu}$.

First of all, we know that the perfection of $\mathcal{M}^{\mathrm{sch}}_{\mathcal{G},\mu,k_E}$ equals $\mathcal{A}_{\mathcal{G},\mu}$ by Theorem 6.16 and fully faithfulness of $\diamondsuit$ on perfect schemes, see [SW20, Proposition 18.3.1]. By the weak normality property and Lemma 7.6, we conclude that $\mathcal{M}^{\mathrm{sch}}_{\mathcal{G},\mu}$ admits a smooth open subscheme $\mathcal{M}^{\mathrm{sch},\circ}_{\mathcal{G},\mu}$ descending

$$\mathcal{M}^{\mathrm{sch},\circ}_{\mathcal{G},O_{\breve{E}},\mu} = \bigcup_{\lambda} \mathcal{G}_{O_{\breve{E}}}/\mathcal{P}^{-}_{\lambda}, \tag{7.44}$$

compare with the argument in [Ric16, Corollary 2.14]. It follows that we have a natural morphism

$$\mathcal{A}^{\mathrm{can}}_{\mathcal{G},\mu} \to \mathcal{M}^{\mathrm{sch}}_{\mathcal{G},\mu,k_E}. \tag{7.45}$$

The following conjecture is then the full p-adic coherence conjecture:

Conjecture 7.25. *The map (7.45) is always an isomorphism.*

This conjecture is settled in [CL25] in all cases, see Remarks 3.17 and 7.24. Indeed, the condition in Lemma 7.26 below has been verified in [GL24] after the first version of this paper was written. So, we know by [GL24, Corollary 1.4] that $\mathcal{M}^{\mathrm{sch}}_{\mathcal{G},\mu,k_E}$ is always reduced. To show that (7.45) is an isomorphism, we are hence reduced to comparing Hilbert polynomials via the dimension formula in Theorem 3.16. This applies in all cases, since Assumption 1.9 was lifted in [CL25].

Lemma 7.26. *Suppose $(\widehat{\mathcal{M}_{\mathcal{G},O_C,\mu}}_{/\bar{x}})_\eta$ is connected for every $\bar{k}$-valued point $\bar{x}$ of $\mathcal{A}_{\mathcal{G},\mu}$. Then $\mathcal{M}^{\mathrm{sch}}_{\mathcal{G},\mu,k_E}$ is geometrically reduced. Under Assumption 1.9, Conjecture 7.25 holds.*

Proof. By Proposition 2.39, we know that $\mathcal{M}^{\mathrm{sch}}_{\mathcal{G},\mu}$ is normal. In particular, $\mathcal{M}^{\mathrm{sch}}_{\mathcal{G},\mu,k_E}$ is S1, but it must also be R0, as it contains a smooth dense open $\mathcal{A}^{\mathrm{can},\circ}_{\mathcal{G},\mu}$. So Serre's criterion for reducedness furnishes the claim. As for identifying the special fiber with $\mathcal{A}^{\mathrm{can}}_{\mathcal{G},\mu}$ as per Conjecture 7.25, we appeal to Theorem 3.16, which computes the dimension of the vector spaces of global sections of ample line bundles. $\square$

Finally, let us also mention the following conjecture, arising from [FHLR22], on the singularities of $\mathcal{M}^{\mathrm{sch}}_{\mathcal{G},\mu}$:

Conjecture 7.27. *The local model $\mathcal{M}^{\mathrm{sch}}_{\mathcal{G},\mu}$ has pseudo-rational singularities.*

8. The test function conjecture

Throughout this section, we let $F/\mathbb{Q}_p$ be a finite field extension with ring of integers O and finite residue field k of cardinality q. We fix an algebraic closure $\bar{\mathbb{Q}}_p$, an embedding $F \hookrightarrow \bar{\mathbb{Q}}_p$ and denote by $\Gamma = \mathrm{Gal}(\bar{\mathbb{Q}}_p/F)$ the absolute Galois group of F with inertia subgroup I. Let G be a reductive F-group with parahoric O-model $\mathcal{G}$.

Furthermore, fix a square root $\sqrt{q}$, an auxiliary prime $\ell \nmid q$ and put $\Lambda = \mathbb{Q}_\ell(\sqrt{q})$. We let ${}^LG = \widehat{G}_\Lambda \rtimes \Gamma$ be the Langlands dual group viewed as a pro-algebraic Λ-group scheme. Each algebraic representation V of ${}^LG$ furnishes, by choosing a quasi-inverse to the geometric Satake equivalence, a semi-simple perverse Λ-sheaf $\mathrm{Sat}(V)$ of "weight zero" on the B^+_{dR}-affine Grassmannian $\mathrm{Gr}_G \to \mathrm{Spd}\, F$. Here $\sqrt{q}$ is needed to define a square root of the ℓ-adic cyclotomic character used when Tate twisting irreducible perverse sheaves supported on components of Gr_G of odd parity to be of "weight zero". More precisely, for a dominant coweight μ defined over F, we have

$$\mathrm{Sat}(V_\mu) = i_{\mu,*} j_{\mu,!*} \Lambda_{\mathrm{Gr}^\circ_\mu}\big(\tfrac{\langle 2\rho,\mu\rangle}{2}\big), \tag{8.1}$$

where $\mathrm{Gr}^\circ_{G,\mu} \xrightarrow{j_\mu} \mathrm{Gr}_{G,\mu} \xrightarrow{i_\mu} \mathrm{Gr}_G$ and V_μ is the irreducible representation of ${}^LG$ of highest weight μ. Every simple object is of this form, up to taking a finite Galois orbit of μ's and tensoring with simple Λ-local systems on $\mathrm{Spd}\, F$ of weight zero (corresponding to irreducible representations of Γ factoring through a finite quotient).

As in Section 6.5, we consider the functor of nearby cycles

$$\Psi_{\mathcal{G}} := i^* Rj_*(\text{-})|_{\mathrm{Spd}\, \mathbb{C}_p} \colon \mathrm{D}(\mathrm{Hk}_G, \Lambda) \to \mathrm{D}(\mathrm{Hk}_{\mathcal{G},\bar{k}}, \Lambda), \tag{8.2}$$

where $\mathrm{Hk}_{G,\mathbb{C}_p} \xrightarrow{j} \mathrm{Hk}_{\mathcal{G},O_{\mathbb{C}_p}} \xleftarrow{i} \mathrm{Hk}_{\mathcal{G},\bar{k}}$ are the inclusions of the geometric fibers.

Lemma 8.1. *For every finite dimensional algebraic ${}^LG$-representation V, the sheaf of nearby cycles $\Psi_{\mathcal{G}}(\mathrm{Sat}(V))$ naturally defines an object in the category*

$$\mathrm{D}_{\mathrm{cons}}\big([\underline{\Gamma}\backslash \mathrm{Hk}^{\mathrm{sch}}_{\mathcal{G},\bar{k}}], \Lambda\big)^{\mathrm{bd}} \tag{8.3}$$

of constructible Λ-sheaves with bounded support on the v-stack $[\underline{\Gamma}\backslash \mathrm{Hk}^{\mathrm{sch}}_{\mathcal{G},\bar{k}}]$. Here, $\underline{\Gamma}$ denotes the associated group v-sheaf and the action on the schematic Hecke stack $\mathrm{Hk}^{\mathrm{sch}}_{\mathcal{G},\bar{k}}$ is induced by the quotient map $\Gamma \to \mathrm{Gal}(\bar{k}/k)$. In particular, the cohomology sheaves $\mathrm{R}^n\Psi_{\mathcal{G}}(\mathrm{Sat}(V))$, $n \in \mathbb{Z}$ define $L^+_{\bar{k}}\mathcal{G}$-equivariant, constructible Λ-sheaves with bounded support on $\mathcal{F}\ell_{\mathcal{G},\bar{k}}$ equipped with an equivariant continuous Γ-action as defined in [SGA73, Exposé XIII] *compatibly with the $L^+_{\bar{k}}\mathcal{G}$-action.*

Proof. The group Γ is identified with the group of continuous automorphisms of $\mathbb{C}_p$ over F. Since the geometric fiber inclusions i and j are $\underline{\Gamma}$-equivariant, we obtain maps of v-stacks

$$\mathrm{Hk}_G = [\underline{\Gamma}\backslash \mathrm{Hk}_{G,\mathbb{C}_p}] \xrightarrow{[\underline{\Gamma}\backslash j]} [\underline{\Gamma}\backslash \mathrm{Hk}_{\mathcal{G},O_{\mathbb{C}_p}}] \xleftarrow{[\underline{\Gamma}\backslash i]} [\underline{\Gamma}\backslash \mathrm{Hk}_{\mathcal{G},\bar{k}}], \tag{8.4}$$

and define the Galois equivariant nearby cycles functor $[\underline{\Gamma}\backslash\Psi_{\mathcal{G}}] := [\underline{\Gamma}\backslash i]^* R[\underline{\Gamma}\backslash j]_*(\text{-})$ in analogy to (8.2). Consider the quotient map $v\colon \mathrm{Hk}_{\mathcal{G},\bar{k}} \to [\underline{\Gamma}\backslash\mathrm{Hk}_{\mathcal{G},\bar{k}}]$. We claim that the map $v^*[\underline{\Gamma}\backslash\Psi_{\mathcal{G}}](\mathrm{Sat}(V)) \to \Psi_{\mathcal{G}}(\mathrm{Sat}(V))$ induced by base change is an isomorphism.

Note that we can not apply the base change theorem directly because j is not quasi-compact and Γ is only profinite. Instead, we apply constant term functors, which commute with arbitrary base change: By finite étale descent, we may and do assume that G is quasi-split and residually split, so that every $\breve{F}$-Borel descends to F. Using the conservativity of constant term functors, see Proposition 6.4, we see again as in Proposition 6.12 that the equivariant integral extension $R[\underline{\Gamma}\backslash j]_*\,\mathrm{Sat}(V)$ is ULA over $[\underline{\Gamma}\backslash\,\mathrm{Spd}\,O_{\mathbb{C}_p}]$. In particular, so is its pullback to $\mathrm{Spd}\,O_{\mathbb{C}_p}$, which implies by Proposition 6.12 that it equals $Rj_*(\mathrm{Sat}(V)|_{\mathrm{Spd}\,\mathbb{C}_p})$. Restricting to geometric special fibers implies the claim.

It formally follows from Proposition A.5 and the construction of derived categories of Λ-sheaves that the comparison functor (A.3) induces an equivalence

$$\mathrm{D}([\underline{\Gamma}\backslash\mathrm{Hk}^{\mathrm{sch}}_{\mathcal{G},\bar{k}}],\Lambda)^{\mathrm{bd}} \cong \mathrm{D}([\underline{\Gamma}\backslash\mathrm{Hk}_{\mathcal{G},\bar{k}}],\Lambda)^{\mathrm{bd}} \tag{8.5}$$

under which constructible sheaves correspond to ULA sheaves, see Proposition 6.7. Note that both properties are preserved and detected under the functor v^*, respectively its schematic counterpart. Therefore, $[\underline{\Gamma}\backslash\Psi_{\mathcal{G}}](\mathrm{Sat}(V))$ naturally defines an object in the category (8.3) and its underlying sheaf is $\Psi_{\mathcal{G}}(\mathrm{Sat}(V))$.

For the final statement on the comparison with [SGA73, Exposé XIII], we reduce to the case where Λ is a finite ring by the construction of categories of ℓ-adic sheaves, see also (6.2). Then, for any qcqs k-scheme X, the category of abelian Λ-sheaves on $(X_{\bar{k}})_{\text{ét}}$ equipped with a continuous Γ-action as in [SGA73, Exposé XIII, Définition 1.1.2] embeds fully faithfully into the category of abelian Λ-sheaves on $[\underline{\Gamma}\backslash X_{\bar{k}}]$ inducing an equivalence on full subcategories of constructible sheaves. Applying this to closed subschemes $X \subset \mathcal{F}\ell_{\mathcal{G}}$ implies the lemma. $\square$

For every $\Phi \in \Gamma$, we define a function $\mathrm{Hk}^{\mathrm{sch}}_{\mathcal{G}}(k) \to \Lambda$ by the formula

$$\tau^{\Phi}_{\mathcal{G},V}(x) := (-1)^{d_V} \sum_{n\in\mathbb{Z}} (-1)^n \,\mathrm{trace}\left(\Phi \mid\ \mathrm{R}^n\Psi_{\mathcal{G}}\mathrm{Sat}(V)_{\bar{x}}\right) \tag{8.6}$$

whenever V is irreducible and extend the definition to general V by linearity. Here, $d_V = \langle 2\rho, \mu\rangle$ with μ being the highest weight of V. So the sign $(-1)^{d_V}$ in (8.6) only depends on the parity of the connected component of Gr_G that supports $\mathrm{Sat}(V)$.

Lemma 8.2. *For every finite dimensional algebraic ${}^LG$-representation V and every $\Phi \in \Gamma$, the function $\tau^{\Phi}_{\mathcal{G},V}$ naturally lies in the center of the parahoric Hecke algebra $\mathcal{H}(G(F),\mathcal{G}(O))_\Lambda$.*

Proof. Lang's lemma together with an approximation argument [RS20, Lemma A.3] implies that $\mathrm{H}^1_{\text{ét}}(k, L^+\mathcal{G})$ vanishes, so $\mathrm{Hk}^{\mathrm{sch}}_{\mathcal{G}}(k) = \mathcal{G}(O)\backslash G(F)/\mathcal{G}(O)$. As the function $\tau^{\Phi}_{\mathcal{G},V}$ is supported on finitely many double cosets, it lies in $\mathcal{H}(G(F),\mathcal{G}(O))_\Lambda$. Centrality follows from Proposition 6.17 and the usual sheaf function dictionary. $\square$

On the other hand, the theory of Bernstein centers defines another function: Namely, for every choice of lift $\Phi \in \Gamma$ of geometric Frobenius, we let $z^{\Phi}_{\mathcal{G},V}$ be the unique function in the center of $\mathcal{H}(G(F),\mathcal{G}(O))_\Lambda$ that acts on every smooth irreducible $\mathcal{G}(O)$-spherical representation π over Λ by the scalar

$$\mathrm{trace}\Big(s^{\Phi}(\pi) \;\Big|\; V\Big), \tag{8.7}$$

where $s^{\Phi}(\pi) \in [\widehat{G}^I \rtimes \Phi]_{\mathrm{ss}}/\widehat{G}^I$ is the Satake parameter for π with respect to Φ constructed in [Hai15].

Theorem 8.3. *For every finite dimensional algebraic ${}^LG$-representation V and every choice of lift Φ of geometric Frobenius, there is an equality*

$$\tau^{\Phi}_{\mathcal{G},V} = z^{\Phi}_{\mathcal{G},V} \tag{8.8}$$

of functions in the parahoric Hecke algebra.

Proof. As both sides of (8.8) are additive in V, we may freely assume that V is irreducible, and even further that $V|_{\widehat{G}\rtimes I}$ is irreducible: otherwise both sides in (8.8) are zero (hence, equal) by elementary considerations, see [HR21, Lemma 7.7].

Fix a maximal F-split torus $A \subset G$ whose Néron model embeds in $\mathcal{G}$ and a regular cocharacter $\lambda\colon \mathbb{G}_m \to A$. Then λ induces a minimal F-Levi M, respectively F-parabolic P in G. Denote by $\mathcal{M} \subset \mathcal{P}$ their flat closures in $\mathcal{G}$. Then the constant terms morphism [Hai14, Section 11.11] induces an injective morphism on the centers of the parahoric Hecke algebras

$$\mathrm{ct}_{\mathcal{P}}\colon \mathcal{Z}(G(F), \mathcal{G}(O))_\Lambda \hookrightarrow \mathcal{Z}(M(F), \mathcal{M}(O))_\Lambda. \tag{8.9}$$

As in [HR21, Lemma 7.8, Equation (7.15)], one checks the formulas

$$\mathrm{ct}_{\mathcal{P}}\left(\tau^{\Phi}_{\mathcal{G},V}\right) = \tau^{\Phi}_{\mathcal{M},V|_{{}^LM}}, \quad \mathrm{ct}_{\mathcal{P}}\left(z^{\Phi}_{\mathcal{G},V}\right) = z^{\Phi}_{\mathcal{M},V|_{{}^LM}}, \tag{8.10}$$

where ${}^LM = \widehat{M} \rtimes \Gamma$ is viewed as a closed subgroup of ${}^LG$. The second formula in (8.10) is straight forward. The first formula in (8.10) is based on the isomorphism

$$\mathrm{CT}_{\mathcal{P}}[\deg_{\mathcal{P}}] \circ \Psi_{\mathcal{G}} \cong \Psi_{\mathcal{M}} \circ \mathrm{CT}_P[\deg_P]\colon \mathrm{Sat}(\mathrm{Hk}_G, \Lambda) \to \mathrm{D}_{\mathrm{cons}}([\underline{\Gamma}\backslash \mathcal{F}\ell_{\mathcal{M},\bar{k}}], \Lambda)^{\mathrm{bd}}, \tag{8.11}$$

see Proposition 6.13, using that $\mathrm{CT}_P[\deg_P]$ corresponds to the restriction of representations $V \mapsto V|_{{}^LM}$ under the geometric Satake equivalence [FS21, Section VI]. (We note that the sign $(-1)^{d_V}$ in (8.6) appears when comparing $\mathrm{CT}_{\mathcal{P}}[\deg_{\mathcal{P}}]$ and $\mathrm{ct}_{\mathcal{P}}$ under the sheaf function dictionary, see also [HR21, Lemma 7.2].)

Hence, we reduce to the case where $G = M$ is a minimal F-Levi, so anisotropic modulo center, and $V|_{\widehat{G}\rtimes I}$ is irreducible. Let $\mathcal{M}_{\mathcal{G},V}$ be the v-sheaf theoretic closure of the support of $\mathrm{Sat}(V)$ in $\mathrm{Gr}_{\mathcal{G}}$, a finite union of $\mathcal{M}_{\mathcal{G},\mu}$ for μ ranging over the highest weights of V. The proof of [HR21, Lemma 7.13] is based on Iwahori-Weyl group combinatorics, hence applies to show that $\mathcal{M}_{\mathcal{G},V}$ has only a single $\mathrm{Spd}\, k$-valued point x_V. As Φ lifts the geometric Frobenius, we can apply the Grothendieck-Lefschetz trace formula to $\Psi_{\mathcal{G}}\mathrm{Sat}(V)$ viewed as an object in $\mathrm{D}_{\mathrm{cons}}([\underline{\Gamma}\backslash \mathcal{F}\ell_{\mathcal{G},\bar{k}}], \Lambda)^{\mathrm{bd}}$ to compute

$$\mathrm{trace}\left(\Phi \mid \Psi_{\mathcal{G}}\mathrm{Sat}(V)_{\overline{x_V}}\right) = \mathrm{trace}\left(\Phi \mid \mathrm{H}^*(\mathcal{F}\ell_{\mathcal{G},\bar{k}}, \Psi_{\mathcal{G}}\mathrm{Sat}(V))\right) \tag{8.12}$$

Since $Rj_*(\mathrm{Sat}(V)|_{\mathrm{Spd}\ \mathbb{C}_p})$ is ULA by Proposition 6.12, the latter cohomology group is Γ-equivariantly isomorphic to

$$\mathrm{H}^*(\mathrm{Gr}_{G,\mathbb{C}_p}, \mathrm{Sat}(V)) = \mathrm{H}^*(\mathrm{Gr}_{G^*,\mathbb{C}_p}, \mathrm{Sat}(V)), \tag{8.13}$$

where G^* is the unique quasi-split inner form of G. We note that there is a canonical identification ${}^LG = {}^LG^*$ so that on Satake categories $\mathrm{Sat}(\mathrm{Hk}_G, \Lambda) \cong \mathrm{Sat}(\mathrm{Hk}_{G^*}, \Lambda)$ by [FS21, Section VI]. Let $\mathcal{G}^*$ denote the parahoric corresponding to $\mathcal{G}$ (necessarily, an Iwahori) and $\mathcal{M}_{\mathcal{G}^*,V}$ the associated v-sheaf local model. On the other hand, we know [Hai14, Proposition 11.12.6] that $z^{\Phi}_{\mathcal{G},V}$ is supported at x_V with value

$$z^{\Phi}_{\mathcal{G},V}(x_V) = \sum_{x \in \mathcal{M}_{\mathcal{G}^*,V}(\mathrm{Spd}\ k)} z^{\Phi}_{\mathcal{G}^*,V}(x). \tag{8.14}$$

Now assuming the test function conjecture for the pair $(\mathcal{G}^*, V)$, that is, assuming $z^{\Phi}_{\mathcal{G}^*,V} = \tau^{\Phi}_{\mathcal{G}^*,V}$, we can apply the Grothendieck-Lefschetz trace formula again to see that (8.14) equals the trace of Φ on (8.13), up to the sign $(-1)^{d_V}$. So $z^{\Phi}_{\mathcal{G}^*,V} = \tau^{\Phi}_{\mathcal{G}^*,V}$ implies $z^{\Phi}_{\mathcal{G},V} = \tau^{\Phi}_{\mathcal{G},V}$.

Hence, we reduce to the case where $G = G^*$ is quasi-split. Now, the minimal Levi $M = T$ is a maximal torus, so (8.10) reduces us to the case where $G = T$ is a torus and $\mathcal{G} = \mathcal{T}$ its connected locally finite type Néron model. Without loss of generality, we assume that $V|_{\widehat{T}\rtimes I}$ is irreducible. Evidently, T is anisotropic modulo center so that both functions $\tau^{\Phi}_{\mathcal{T},V}, z^{\Phi}_{\mathcal{T},V}$ are supported at x_V. Using (8.12), the ULA property of $Rj_*(\mathrm{Sat}(V)|_{\mathrm{Spd}\,\mathbb{C}_p})$ and $\mathrm{H}^0(\mathrm{Gr}_{\mathcal{T},\mathbb{C}_p}, \mathrm{Sat}(V)) = V$, we see

$$\tau^{\Phi}_{\mathcal{T},V}(x_V) = (-1)^{d_V}\,\mathrm{trace}\left(\Phi \mid V\right) \tag{8.15}$$

which equals $z^{\Phi}_{\mathcal{T},V}(x_V)$ because $d_V = 0$. This finishes the proof. □

Lemma 8.4. *Theorem 8.3 implies Theorem 1.3.*

Proof. Let μ be a conjugacy class of geometric cocharacters in G. Denote by $E \subset \bar{\mathbb{Q}}_p$ its reflex field with maximal unramified subextension E_0/F. Their rings of integers are denoted by $O_E \supset O_{E_0}$ with residue fields $k_E = k_{E_0}$ and absolute Galois groups $\Gamma_E \subset \Gamma_{E_0}$. For every $\Phi \in \Gamma_E$ and $x \in \mathrm{Gr}_{\mathcal{G}}(k_E)$, there is an equality

$$\mathrm{trace}\left(\Phi \mid \Psi_{\mathcal{G},O_E}\mathrm{Sat}(V_\mu)_{\bar{x}}\right) = \mathrm{trace}\left(\Phi \mid \Psi_{\mathcal{G},O_{E_0}}\mathrm{Sat}(I^{E_0}_E(V_\mu))_{\bar{x}}\right), \tag{8.16}$$

where $I^{E_0}_E(V_\mu)$ is the induction to $\widehat{G} \rtimes \Gamma_{E_0}$ of the $\widehat{G} \rtimes \Gamma_E$-representation V_μ and $\mathrm{Sat}(V_\mu)$, $\mathrm{Sat}(I^{E_0}_E(V_\mu))$ the corresponding Satake sheaves on $\mathrm{Gr}_G|_{\mathrm{Spec}\,E}$, respectively $\mathrm{Gr}_G|_{\mathrm{Spec}\,E_0}$. Indeed, (8.16) follows from the commutation of nearby cycles with proper pushforward applied to the finite morphism $\mathrm{Gr}_{\mathcal{G}}|_{\mathrm{Spd}\,O_E} \to \mathrm{Gr}_{\mathcal{G}}|_{\mathrm{Spd}\,O_{E_0}}$, noting that it induces the induction of representations on Satake categories.

Now, we apply (8.16) to the pair $\mathcal{G}_0 := \mathcal{G}_{O_{E_0}}, V_{\mu,0} := I^{E_0}_E(V_\mu)$ and any choice of lift $\Phi \in \Gamma_E \subset \Gamma_{E_0}$ of geometric Frobenius to obtain

$$\tau^{\Phi}_{\mathcal{G}_0,V_{\mu,0}} = z^{\Phi}_{\mathcal{G}_0,V_{\mu,0}}. \tag{8.17}$$

The left hand side of (8.17) is equal to the function from Theorem 1.3 by (8.16) and so is the right hand side of (8.17) by a similar equality [Hai18, Lemma 8.1] for inductions of representations along the totally ramified extension E/E_0. That the function (8.17) takes, after multiplying by $(\sqrt{q_E})^{\langle 2\rho,\mu\rangle}$, values in $\mathbb{Z}$ independently of the choice of $\ell \neq p$, $\sqrt{q_E}$ and $E \hookrightarrow \bar{\mathbb{Q}}_p$ follows from [HR21, Theorem 7.15] where the statement is verified for the semi-simplified version of the right hand side of (8.17) without any assumptions on $(\mathcal{G},\mu)$. The same arguments apply here. □

A. Étale sheaves over v-stacks on perfect schemes

In this section, we extend some parts of [Sch17, Section 27] to v-stacks on perfect schemes, see also [Wu21, Appendix A]. Let k be a perfect field of characteristic $p > 0$. Let X be a perfect scheme over k, and let $X^\diamondsuit$ be the associated v-sheaf from Section 3.1 (under the tilting equivalence), that is, if $\mathrm{Spa}(R,R^+)$ is an affinoid perfectoid space over $\mathrm{Spa}(k,k)$ and X affine, then

$$X^\diamondsuit(\mathrm{Spa}(R,R^+)) = X(\mathrm{Spec}(R)), \tag{A.1}$$

while for X general $X^\diamondsuit$ is the analytic sheafification of the presheaf $\mathrm{Spa}(R,R^+) \mapsto X(\mathrm{Spec}(R))$. We note that the functor $(\text{-})^\diamondsuit$ does depend on k.

Fix a torsion ring Λ with $p \in \Lambda^\times$. We let $\mathrm{D}(X,\Lambda) := \widehat{\mathrm{D}}(X_{\text{ét}},\Lambda)$ be the left-completed étale derived ∞-category of X, see [Sch17, Section 27]. By [Sch17, Section 27] there is a morphism

$$c_X \colon (X^\diamondsuit)_v \to X_{\text{ét}} \tag{A.2}$$

of sites (even to the proétale site of X), and the induced functor of ∞-categories

$$c_X^* \colon \mathrm{D}(X,\Lambda) \to \mathrm{D}(X^\diamondsuit,\Lambda) \tag{A.3}$$

is fully faithful, [Sch17, Proposition 27.2.]. In general the functor c_X^* is not essentially surjective, for example, on topological spaces $|X^\diamondsuit| \to |X|$ is surjective, but very often not injective.

The functor c_X^* enjoys many compatibilities. If $f\colon Y \to X$ is a map of schemes, then $c_X^* \circ f^* \cong (f^\diamondsuit)^* \circ c_X^*$ and $c_X^*(A) \otimes_\Lambda^{\mathbb{L}} c_X^*(B) \cong c_X^*(A \otimes_\Lambda^{\mathbb{L}} B)$, see [Sch17, Proposition 27.1.]. If $f\colon Y \to X$ is separated perfectly of finite type, then $c_X^* \circ Rf_! \cong Rf_!^\diamondsuit \circ c_X^*$, see [Sch17, Proposition 27.4.]. As we now justify c_X^* also preserves ULA-objects.

Proposition A.1. *Let S be a qcqs perfect scheme in characteristic p, and let $f\colon X \to S$ be a separated perfect scheme perfectly of finite presentation over S. Let $A \in \mathrm{D}(X, \Lambda)$ such that A is ULA with respect to f. Then $c_X^* A$ is ULA with respect to $f^\diamondsuit$, and $c_X^* \mathbb{D}_{X/S}(A) \cong \mathbb{D}_{X^\diamond/S^\diamond}(c_X^* A)$ for the Verdier duals.*

Proof. As in [HS21, Section 3], we let $\mathcal{C}_S$ denote the 2-category whose objects are schemes over S as in the hypothesis, and where morphisms from X to Y are given by objects in $\mathrm{D}(X \times_S Y, \Lambda)$. Given two maps $A \in \mathrm{Hom}_{\mathcal{C}_S}(X_1, X_2)$ and $B \in \mathrm{Hom}_{\mathcal{C}_S}(X_2, X_3)$, we define their composition $A * B \in \mathrm{Hom}_{\mathcal{C}_S}(X_1, X_3)$ by the formula

$$A * B := R\pi_{1,3!}(\pi_{1,2}^* A \otimes_\Lambda^{\mathbb{L}} \pi_{2,3}^* B). \tag{A.4}$$

By [HS21, Proposition 3.4, Definition 3.2], the object $A \in \mathrm{Hom}_{\mathcal{C}_S}(X, S)$ is ULA with respect to f if and only if A is a left adjoint in $\mathcal{C}_S$. Analogously, let $\mathcal{C}_{S^\diamond}$ denote the category considered in [FS21, Section IV.2.3.3.]. By [FS21, Theorem IV.2.23.], the object $c_X^* A \in \mathrm{Hom}_{\mathcal{C}_S}(X^\diamondsuit, S^\diamondsuit)$ is ULA with respect to $f^\diamondsuit$ if it is a left adjoint in $\mathcal{C}_{S^\diamond}$. Now, we observe that the functors c_X^* can be promoted to a functor of 2-categories $c^*\colon \mathcal{C}_S \to \mathcal{C}_{S^\diamond}$ by the rule $c^* X = X^\diamondsuit$ and $c^*(A) = c_{X \times_S Y}^*(A)$ for $A \in \mathrm{Hom}_{\mathcal{C}_S}(X, Y)$. Here, we use that c^* commutes with the required operations by [Sch17, Propositions 27.1, 27.4]. But functors between 2-categories preserve the adjunctions between 1-morphisms which finishes the proof as the right adjoints are given by Verdier duals. $\square$

We move on to study stacks. Let Ani be the category of anima (also called spaces, ∞-groups or Kan complexes). By left Kan extension along the Yoneda embedding

$$\mathrm{SchPerf}_k \to \mathrm{Fun}(\mathrm{SchPerf}_k^{\mathrm{op}}, \mathrm{Ani}), \tag{A.5}$$

we can extend[7] the functors $\mathrm{D}(\text{-}, \Lambda), \mathrm{D}((\text{-})^\diamondsuit, \Lambda)$ using $*$-pullbacks and the natural transformation $c_{(\text{-})}^*$ to contravariant functors $\mathrm{D}(\text{-}, \Lambda), \mathrm{D}((\text{-})^\diamondsuit, \Lambda)^{\mathrm{Kan}}$ on $\mathrm{Fun}(\mathrm{SchPerf}_k^{\mathrm{op}}, \mathrm{Ani})$ with values in symmetric monoidal stable ∞-categories, sending colimits to limits. More concretely, if a functor (also known as, higher prestack)

$$\mathfrak{X} \cong \operatorname*{colim}_i X_i \in \mathrm{Fun}(\mathrm{SchPerf}_k^{\mathrm{op}}, \mathrm{Ani}) \tag{A.6}$$

is written as a colimit of representables, then

$$\mathrm{D}(\mathfrak{X}, \Lambda) \cong \lim_i \mathrm{D}(X_i, \Lambda) \tag{A.7}$$

and similarly for $\mathrm{D}(\mathfrak{X}^\diamondsuit, \Lambda)^{\mathrm{Kan}}$. By [Sch17, Proposition 27.2] and [BN19, Lemma B.6] (more precisely, its proof of 1.), the natural transformation

$$c_{\mathfrak{X}}^*\colon \mathrm{D}(\mathfrak{X}, \Lambda) \to \mathrm{D}(\mathfrak{X}^\diamondsuit, \Lambda)^{\mathrm{Kan}} \tag{A.8}$$

is fully faithful. Note that at this moment the right hand side is not the left completed derived étale category of some (higher) v-stack "$\mathfrak{X}^\diamondsuit$" on Perf_k, but depends on $\mathfrak{X}$ and is defined abstractly (therefore, we have added the superscript Kan).

[7]We take care of the set-theoretic issues by fixing a suitable cut-off cardinal, large enough to allow all the examples that we are interested in.

Assume now that $\mathfrak{X}$ is a stack (in 1-groupoids) with representable diagonal such that there exists a v-cover by a perfect scheme $X \to \mathfrak{X}$. Then

$$\mathfrak{X} \cong \mathrm{colim}_{\Delta^{\mathrm{op}}} X^{\bullet/\mathfrak{X}} \tag{A.9}$$

with the colimit of the Čech nerve of $X \to \mathfrak{X}$ taken in Ani-valued v-sheaves on $\mathrm{SchPerf}_k$. Using [HS21, Theorem 5.7], we get

$$\mathrm{D}(\mathfrak{X}, \Lambda) \cong \lim_{\Delta} \mathrm{D}(X^{\bullet/\mathfrak{X}}, \Lambda). \tag{A.10}$$

Indeed, by definition

$$\mathrm{D}(\mathfrak{X}, \Lambda) \cong \lim_{U \to \mathfrak{X}} \mathrm{D}(U, \Lambda) \tag{A.11}$$

where the limit is taken over all perfect schemes with a morphism to $\mathfrak{X}$, and thus

$$\begin{array}{cc} & \mathrm{D}(\mathfrak{X}, \Lambda) \\ = & \lim_{U \to \mathfrak{X}} \mathrm{D}(U, \Lambda) \\ \cong & \lim_{U \to \mathfrak{X}} \lim_{\Delta} \mathrm{D}(U \times_{\mathfrak{X}} X^{\bullet/\mathfrak{X}}, \Lambda) \\ \cong & \lim_{\Delta} \lim_{U \to \mathfrak{X}} \mathrm{D}(U \times_{\mathfrak{X}} X^{\bullet/\mathfrak{X}}, \Lambda) \\ \cong & \lim_{\Delta} \mathrm{D}(\mathfrak{X} \times_{\mathfrak{X}} X^{\bullet/\mathfrak{X}}, \Lambda) \\ \cong & \lim_{\Delta} \mathrm{D}(X^{\bullet/\mathfrak{X}}, \Lambda), \end{array} \tag{A.12}$$

where the first isomorphism comes from [HS21, Theorem 5.7] applied to the covering $X \times_{\mathfrak{X}} U \to U$, the second isomorphism just commutes two inverse limits, and the last two isomorphisms use that $\mathrm{D}(\text{-}, \Lambda)$ sends (by definition) colimits to limits and that $\mathrm{colim}_{U \to \mathfrak{X}} U \times_{\mathfrak{X}} X^{n/\mathfrak{X}} \cong X^{n/\mathfrak{X}}$.

Let $X^{\diamondsuit, \bullet/\mathfrak{X}}$ be the simplicial v-sheaf obtained by applying the functor $(\text{-})^{\diamondsuit}$ to $X^{\bullet/\mathfrak{X}}$. Now assume additionally that the projection maps in $X^{\diamondsuit, \bullet/\mathfrak{X}}$ are v-covers, and let $\mathfrak{X}^{\diamondsuit}$ be the colimit of $X^{\diamondsuit, \bullet/\mathfrak{X}}$ in v-stacks on Perf_k. Then $\mathfrak{X}^{\diamondsuit}$ is a small v-stack with well-defined $\mathrm{D}(\mathfrak{X}^{\diamondsuit}, \Lambda)$, and actually is the v-stackification of $\mathrm{Spa}(R, R^+) \mapsto \mathfrak{X}(\mathrm{Spec}(R))$. By [Sch17, Proposition 17.3], we can conclude that

$$\mathrm{D}(\mathfrak{X}^{\diamondsuit}, \Lambda) \cong \lim_{\Delta^{\mathrm{op}}} \mathrm{D}(X^{\diamondsuit, \bullet/\mathfrak{X}}, \Lambda). \tag{A.13}$$

Moreover, note that there exists a canonical morphism

$$\mathrm{D}(\mathfrak{X}^{\diamondsuit}, \Lambda)^{\mathrm{Kan}} \to \mathrm{D}(\mathfrak{X}^{\diamondsuit}, \Lambda), \tag{A.14}$$

which probably need not be an equivalence in general, but whose composite with $c^*_{\mathfrak{X}}$ is still fully faithful.

Lemma A.2. *If $f \colon Y \to X$ is a v-cover of perfect schemes which is a map of (perfectly) finite presentation, then $f^{\diamondsuit} \colon Y^{\diamondsuit} \to X^{\diamondsuit}$ is a v-cover.*

Proof. The functor $(\text{-})^{\diamondsuit}$ preserves open covers, so we may assume that $Y \to X$ are affine. In this case, a (perfectly) finite presented v-cover is a cofiltered limit of v-covers between (perfect) affine schemes of (perfect) finite presentation over $\mathrm{Spec}(k)$, see [BS17, Lemma 2.12]. As f is of (perfectly) finite presentation, we may assume that f is the base change of a v-cover between (perfect) affine schemes of (perfect) finite presentation over k. The functor $(\text{-})^{\diamondsuit}$ preserves fiber products, and base changes of v-cover of v-sheaves on Perf_k are again v-covers. Thus, we may reduce to the case that Y, X are of (perfect) finite presentation over k. Then the statement follows from [Gle24, Proposition 5.3]. ☐

In general, it need however not be true that for a v-cover $Y \to X$ of (perfect) schemes the map $Y^{\diamondsuit} \to X^{\diamondsuit}$ is a v-cover of small v-sheaves. We include the following example which shows that one needs to be careful when applying the above formalism to, say schematic Hecke stacks as in (A.23) as they arise as quotients of schematic loop groups which are of infinite type over the base field.

Example A.3. Let C be a perfect, non-archimedean field over $\mathbb{F}_p$, and fix a pseudo-uniformizer $\pi_C \in C$. Let

$$Z := \{\tfrac{1}{n} \mid n \in \mathbb{N}\} \cup \{0\} \subset [0,1] \tag{A.15}$$

with its subspace topology, which is profinite. We consider the space of continuous functions

$$X := \mathrm{Spec}(C^0(Z, C_{\mathrm{disc}})), \tag{A.16}$$

where C_{disc} is the field C equipped with the discrete topology. Note that $|X| \cong Z$. For $n \in \mathbb{N}$, let $g_n, h_n \colon Z \to C_{\mathrm{disc}}$ be the locally constant functions with

$$g_n(1/m) = 1/\pi_C^m, h_n(1/m) = 1 \tag{A.17}$$

for $m \leq n$ and $g_n(z) = h_n(z) = 0$ otherwise. Let

$$Y_n \subset \mathbb{A}^1_X = \mathrm{Spec}(C^0(Z, C_{\mathrm{disc}})[T]) \tag{A.18}$$

be the vanishing locus of $h_nT - g_n$. Then $Y_{n+1} \subset Y_n$, and we can set $Y := \lim_n Y_n$ which is a v-cover of X. Indeed, each Y_n is a v-cover, and inverse limits of v-covers between affine schemes are v-covers. More concretely, each map $\mathrm{Spec}(V) \to X$ with V a valuation ring must factor through a (closed) point of X, and then it suffices to see that $Y \to X$ is surjective (it is bijective over $X \setminus \{0\}$, and $\mathbb{A}^1_{C_{\mathrm{disc}}}$ over 0).

We claim that $Y^\diamondsuit \to X^\diamondsuit$ is not a v-cover. Set $R := C^0(Z, C)$ (now C given its valuation topology), and $R^+ = C^0(Z, \mathcal{O}_C)$. The canonical map $C^0(Z, C_{\mathrm{disc}}) \to R$ defines a map

$$S := \mathrm{Spa}(R, R^+) \to X^\diamondsuit, \tag{A.19}$$

which does not v-locally factor through $Y^\diamondsuit \subset (\mathbb{A}^1_X)^\diamondsuit$. Indeed, assume that $S' \to S$ is a v-cover with S' affinoid and $S' \to Y^\diamondsuit$ a lift of $S \to X^\diamondsuit$. Then the image of $S' \to Y^\diamondsuit \times_{X^\diamondsuit} S \subset \mathbb{A}^{1,\mathrm{ad}}_S$ must factor through some quasi-compact subset. But over each point $z_n := 1/n \in Z \cong |S|$ with $n \in \mathbb{N}$, we have that $Y^\diamondsuit \times_{X^\diamondsuit} \{z_n\}$ is the point $1/\pi_C^n \in \mathbb{A}^{1,\mathrm{ad}}_{z_n}$, and in $\mathbb{A}^{1,\mathrm{ad}}_S$ this set of points does not lie in a quasi-compact open.

We get the following consequence of Lemma A.2.

Lemma A.4. *Assume $\mathfrak{X}, \mathfrak{Y}$ are v-stacks on* $\mathrm{SchPerf}_k$ *with representable diagonal of perfectly finite presentation, and that $\mathfrak{X}, \mathfrak{Y}$ admit a (perfectly) finitely presented v-covers by a perfect schemes, which are of (perfect) finite presentation over* $\mathrm{Spec}(k)$*. Let $\mathfrak{X}^\diamondsuit$ be the v-stackification of the functor* $\mathrm{Spa}(R, R^+) \mapsto \mathfrak{X}(\mathrm{Spec}(R))$ *on* Perf_k*, and similarly for $\mathfrak{Y}$. Let $f \colon \mathfrak{X} \to \mathfrak{Y}$ be a morphism of v-stacks.*

(1) *Then the canonical morphism*

$$\mathrm{D}(\mathfrak{X}^\diamondsuit, \Lambda)^{\mathrm{Kan}} \to \mathrm{D}(\mathfrak{X}^\diamondsuit, \Lambda) \tag{A.20}$$

is an equivalence, and the composite (still denoted $c_{\mathfrak{X}}^*$)

$$\mathrm{D}(\mathfrak{X}, \Lambda) \xrightarrow{c_{\mathfrak{X}}^*} \mathrm{D}(\mathfrak{X}^\diamondsuit, \Lambda)^{\mathrm{Kan}} \cong \mathrm{D}(\mathfrak{X}^\diamondsuit, \Lambda) \tag{A.21}$$

is fully faithful.

(2) *The functors $c_{\mathfrak{X}}^* \circ f^*, (f^\diamondsuit)^* \circ c_{\mathfrak{Y}}^*$ are naturally isomorphic.*

(3) *If $\mathfrak{X} \cong [\mathrm{Spec}(k)/H]$ for some perfectly finitely presented group scheme H over k, then the functor*

$$\mathrm{D}(\mathfrak{X}, \Lambda) \to \mathrm{D}(\mathfrak{X}^\diamondsuit, \Lambda) \tag{A.22}$$

is an equivalence.

Proof. We prove the first point. From the assumptions on $\mathfrak{X}^\diamondsuit$ we can conclude that the morphisms $U \to \mathfrak{X}$ of perfectly finite presentation such that U is of perfectly finite presentation over k are cofinal among all maps $V \to \mathfrak{X}$ with V a perfect scheme. In particular, in the definition of $\mathrm{D}(\mathfrak{X}^\diamondsuit, \Lambda)^{\mathrm{Kan}}$ one can replace the limit over all $V's$ with a morphism to $\mathfrak{X}$ by the limit over all $U's$ with morphism to $\mathfrak{X}$. Using Lemma A.2 and the same argument as for $\mathrm{D}(\mathfrak{X}, \Lambda)$, we can then conclude that $\mathrm{D}(\mathfrak{X}^\diamondsuit, \Lambda)^{\mathrm{Kan}} \cong \mathrm{D}(\mathfrak{X}^\diamondsuit, \Lambda)$. Fully faithfulness follows from fully faithfulness of $c^*_{\mathfrak{X}}$. The second point follows by expressing the categories as limits over Δ by choosing v-covers $X \to \mathfrak{X}, Y \to \mathfrak{Y}$ of perfectly finite presentation with X, Y of perfectly finite presentation over k, such that $X \to \mathfrak{X} \xrightarrow{f} \mathfrak{Y}$ factors over $Y \to \mathfrak{Y}$. For the last point, we note that $\mathrm{D}(\mathrm{Spec}(k), \Lambda) \cong \mathrm{D}(\mathrm{Spa}(k,k)^\diamond, \Lambda)$ as both identify naturally with the derived category of Λ-modules, see [FS21, Theorem V.1.1]. Computing both sides via the Čech nerve of the covering $\mathrm{Spec}(k) \to [\mathrm{Spec}(k)/H]$, the statement follows from [BN19, Lemma B.6]. □

The results apply to the schematic Hecke stack as follows. Let O be a complete discrete valuation ring with residue field k, $\mathcal{G}$ a parahoric group scheme over O with generic fiber G and $\mathcal{F}\ell_{\mathcal{G}} = L_k G / L_k^+ \mathcal{G}$ the partial affine flag variety for $\mathcal{G}$ as in Section 3.2. Let

$$\mathrm{Hk}^{\mathrm{sch}}_{\mathcal{G},k} := [L_k^+\mathcal{G} \backslash L_k G / L_k^+ \mathcal{G}] \tag{A.23}$$

be the (schematic) Hecke stack where the quotient is taken in v-stacks on $\mathrm{SchPerf}_k$. Then $\mathrm{Hk}_{\mathcal{G},k} := (\mathrm{Hk}^{\mathrm{sch}}_{\mathcal{G},k})^\diamondsuit$ is the small v-stack on Perf_k considered in Section 6. We denote by

$$\mathrm{D}(\mathrm{Hk}^{\mathrm{sch}}_{\mathcal{G},k}, \Lambda)^{\mathrm{bd}},\ \mathrm{D}(\mathrm{Hk}_{\mathcal{G},k}, \Lambda)^{\mathrm{bd}} \tag{A.24}$$

the categories of objects with bounded support.

Proposition A.5. *The categories* $\mathrm{D}(\mathrm{Hk}^{\mathrm{sch}}_{\mathcal{G},k}, \Lambda)^{\mathrm{bd}},\ \mathrm{D}(\mathrm{Hk}_{\mathcal{G},k}, \Lambda)^{\mathrm{bd}}$ *are equivalent.*

Proof. Consider a closed substack

$$[L_k^+\mathcal{G} \backslash X] \subset \mathrm{Hk}^{\mathrm{sch}}_{\mathcal{G},k} \tag{A.25}$$

with $X \subset \mathcal{F}\ell_{\mathcal{G}}$ a closed $L_k^+\mathcal{G}$-stable subscheme, that is, a union of Schubert varieties. By the argument of [FS21, Proposition VI.4.1], the vanishing of the cohomology of the affine line over k implies that

$$\mathrm{D}([L_k^+\mathcal{G} \backslash X], \Lambda) \cong \mathrm{D}([H \backslash X], \Lambda) \tag{A.26}$$

for any perfectly finitely presented quotient H of $L_k^+\mathcal{G}$ by some congruence subgroup whose action on X is trivial. By Lemma A.4, we have, abusing notation, a natural fully faithful functor

$$c^*_{[H\backslash X]} \colon \mathrm{D}([H \backslash X], \Lambda) \to \mathrm{D}([H \backslash X]^\diamondsuit, \Lambda) \cong \mathrm{D}([H^\diamondsuit \backslash X^\diamondsuit], \Lambda), \tag{A.27}$$

and we claim that this functor is an equivalence. We prove this by induction on the number of Schubert strata contained in X. Let $i \colon Y \subset X$ be a closed H-stable (perfect) subscheme with non-empty open complement $j \colon U \to X$, for which $[H \backslash U]$ is a disjoint union of classifying stacks for closed subgroups of H. By Lemma A.4, we have

$$\mathrm{D}([H \backslash U], \Lambda) \cong \mathrm{D}([H^\diamondsuit \backslash U^\diamondsuit], \Lambda), \tag{A.28}$$

and by induction induction hypothesis

$$\mathrm{D}([H \backslash Y], \Lambda) \cong \mathrm{D}([H^\diamondsuit \backslash Y^\diamondsuit], \Lambda). \tag{A.29}$$

Let us note that as in [Sch17, Section 27] the functors $c^*_{[H\backslash X]}, c^*_{[H\backslash U]}, c^*_{[H\backslash Y]}$ admit right adjoints $Rc_{[H\backslash X],*}$, $Rc_{[H\backslash U],*}$, $Rc_{[H\backslash Y],*}$, and it suffices to see that $Rc_{[H\backslash X],*}$ is conservative. It is formal that

$$[H \backslash j]^* \circ Rc_{[H\backslash X],*} \cong Rc_{[H\backslash U],*} \circ [H\backslash j]^{\diamondsuit,*}, \tag{A.30}$$

where $[H\backslash j]\colon [H\backslash U] \to [H\backslash X]$ denotes the morphism induced by j. More precisely, there exists a natural morphism from the left hand side to the right hand side, and it suffices to see that the morphims induced on the left adjoints is an isomorphism. If $T \to [H\backslash X]$ is a v-cover with $T \to \mathrm{Spec}(k)$ of morphism of schemes of finite type, then it suffices (by Lemma A.2) to prove the statement on the isomorphism of left adjoints after pullback to $T^\diamondsuit$. Here, the functor c_T^* on the étale derived categories is induced by a morphism of topoi, and then (A.30) follows by general base change to open subtopoi.

Let $A \in \mathrm{D}([H\backslash X]^\diamondsuit, \Lambda)$ such that $Rc_{[H\backslash X],*}(A) = 0$. Then we deduce $[H\backslash j]^{\diamondsuit,*}(A) = 0$ because $Rc_{[H\backslash U],*}$ is an equivalence. In particular, $A \cong [H\backslash i]_*^\diamondsuit [H\backslash i]^{\diamondsuit,*}(A)$. Now note that

$$[H\backslash i]_* \circ Rc_{[H\backslash Y],*} \cong Rc_{[H\backslash X],*} \circ [H\backslash i]^\diamondsuit \tag{A.31}$$

as follows by adjunction from $[H\backslash i]^* \circ c^*_{[H\backslash X]} \cong c^*_{[H\backslash Y]} \circ [H\backslash i]^{\diamondsuit,*}$. We can conclude that

$$[H\backslash i]_* Rc_{[H\backslash Y],*}([H\backslash i]^{\diamondsuit,*}A) = 0, \tag{A.32}$$

which implies $[H\backslash i]^{\diamondsuit,*}(A) = 0$ because $[H\backslash i]_*$ is conservative and $Rc_{[H\backslash Y],*}$ an equivalence. This implies that $A = 0$ as desired.

The equivalence $c^*_{[H\backslash X]}$ is natural with respect to inclusions $X \to X'$ of H-stable subsets, and morphism $H' \to H$ of quotients of $L_k^+\mathcal{G}$ (which act trivially on X). More precisely, from $c^*_{[H\backslash X]}$ we get an equivalence

$$\mathrm{D}([L_k^+\mathcal{G}\backslash X], \Lambda) \cong \mathrm{D}([(L_k^+\mathcal{G})^\diamondsuit \backslash X^\diamondsuit], \Lambda) \tag{A.33}$$

using [FS21, VI.4.1], and then we can pass to the colimits of both sides along closed $L_k^+\mathcal{G}$-stable subschemes of $\mathcal{F}\ell_{\mathcal{G}}$. Then the left hand side is $\mathrm{D}(\mathrm{Hk}^{\mathrm{sch}}_{\mathcal{G},k}, \Lambda)^{\mathrm{bd}}$ while the right hand side is $\mathrm{D}(\mathrm{Hk}_{\mathcal{G},k}, \Lambda)^{\mathrm{bd}}$. $\square$

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TUBULAR NEIGHBORHOODS OF LOCAL MODELS

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Abstract. We show that the v-sheaf local models of [SW20] are unibranch. In particular, this proves that the scheme-theoretic local models defined in [AGLR22] are always normal with reduced special fiber, thereby establishing the remaining cases of the geometric part of the Scholze–Weinstein conjecture when $p \leq 3$. Our methods are general, topological and simplify those of [Zhu14] for tamely ramified groups in positive characteristic. As a technical input, we generalize a comparison theorem of nearby cycles of [Hub96] to the v-sheaf setup.

Contents

1. Introduction

Local models were introduced in the nineties to study the singularities of Shimura varieties, namely in the works of Chai–Norman [CN90], de Jong [dJ93] and Deligne–Pappas [DP94], and have found various applications. The theory was systematized in the book of Rapoport–Zink [RZ96], via linear algebraic moduli problems. Later, it underwent a significant transformation when Görtz [Gör01, Gör03] embedded their special fibers in certain infinite-dimensional flag varieties. This was subsequently exploited by Faltings, Pappas, Rapoport and Zhu [Fal03, PR08, Zhu14, PZ13] to great effect. More recently, Scholze–Weinstein [SW20] proposed a fully functorial avenue to study local models in mixed characteristic via perfectoid geometry. This program was pursued in [AGLR22]. We use the following notation in the paper.

Definition 1.1. Let F be a local field, O its ring of integers, and k its residue field. Let G be a reductive connected F-group, $\mathcal{G}$ a parahoric O-model of G, μ a geometric conjugacy class of cocharacters, and E its reflex field. We denote by $\mathcal{M}_{\mathcal{G},\mu}$ the v-sheaf given as the closure of $\mathrm{Gr}_{G,\mu} \subset \mathrm{Gr}_{G,E}$ inside the Beilinson–Drinfeld Grassmannian $\mathrm{Gr}_{\mathcal{G},O_E}$ of [SW20, FS21]. If F is of positive characteristic or μ is minuscule, we denote by $\mathcal{M}^{\mathrm{sch}}_{\mathcal{G},\mu}$ the canonical weakly normal[1] proper O_E-scheme representing $\mathcal{M}_{\mathcal{G},\mu}$.

If F is p-adic and μ is minuscule, then there is a unique weakly normal scheme $\mathcal{M}^{\mathrm{sch}}_{\mathcal{G},\mu}$ representing $\mathcal{M}_{\mathcal{G},\mu}$ by [AGLR22, Theorem 1.1]. If F has positive characteristic, then $\mathcal{M}^{\mathrm{sch}}_{\mathcal{G},\mu}$ is defined

[1] Recall that a scheme is weakly normal if every finite, birational, universally homeomorphic morphism with reduced source is an isomorphism.

as the weakly normal scheme representing $\mathcal{M}_{\mathcal{G},\mu}$ whose generic fiber is a Schubert variety in positive characteristic.

Historically, an important goal has been to show that the special fiber of $\mathcal{M}^{\mathrm{sch}}_{\mathcal{G},\mu}$ is reduced, see [PRS13] for various ad hoc definitions of $\mathcal{M}^{\mathrm{sch}}_{\mathcal{G},\mu}$. Görtz [Gör01, Gör03] proved this for split classical groups of PEL type via so-called straightening laws, but the proof does not extend to the general case. An important development is due to Pappas–Rapoport [PR08], who formulated the coherence conjecture to address reducedness via coherent cohomology of ample line bundles. Finally, Zhu [Zhu14] proved the conjecture for tame groups, by translating the problem to equicharacteristic, and constructing a global Frobenius splitting of the local model compatibly with the special fiber. Recall that a scheme X in characteristic p is Frobenius split if the $\mathcal{O}_X$-module homomorphism $\mathcal{O}_X \to \varphi_*\mathcal{O}_X$ given by Frobenius splits. Note that Frobenius split schemes are necessarily reduced. We refer to [BK05, BS13] for proper introductions to the subject.

Most modern results in the literature concerning the reduced structure of the local models rely on [Zhu14] through reductions and comparisons. Interestingly, the heart of Zhu's proof lies in characteristic p. This becomes problematic in the perfectoid perspective of [SW20, AGLR22], because it is not clear how to work with Frobenius splitting techniques in this context. Fortunately, we have the following crucial fact that allow us to bypass them:

Lemma 1.2. *If $\mathcal{M}^{\mathrm{sch}}_{\mathcal{G},\mu}$ is unibranch, then its special fiber is reduced.*

Proof. This is [AGLR22, Lemma 7.26], but we repeat the crux of the argument for future reference and the reader's convenience. We know already that the perfection of the special fiber of $\mathcal{M}^{\mathrm{sch}}_{\mathcal{G},\mu}$ equals the so-called μ-admissible locus $\mathcal{A}_{\mathcal{G},\mu}$, see [AGLR22, Theorem 6.16]. In particular, the union of maximal orbits defines a smooth open of $\mathcal{M}^{\mathrm{sch}}_{\mathcal{G},\mu}$ with dense geometric fibers, see [Ric16, Corollary 2.14] and [AGLR22, Equation (7.44)], so the special fiber satisfies Serre's condition R_0. As $\mathcal{M}^{\mathrm{sch}}_{\mathcal{G},\mu}$ is weakly normal per definition, the unibranch assumption in the statement implies normality already. In particular, the special fiber is S_1 by the Serre criterion for normality plus flatness. The Serre criterion for reducedness yields our claim. □

The unibranch property is already amenable to a formulation in terms of perfectoids, because it is topological in nature. Indeed, it suffices to know that the tubular neighborhoods of [Gle24, Definition 4.38] at all the closed points of the special fiber are connected. In turn, being connected is a cohomological invariant and it is natural to expect that a deeper study of the nearby cycles initiated in [AGLR22] would yield the result. Over a p-adic field and for non-minuscule coweights μ, the v-sheaf $\mathcal{M}_{\mathcal{G},\mu}$ does not come from a scheme. Nevertheless, $\mathcal{M}_{\mathcal{G},\mu}$ is a kimberlite, the v-sheaf analogue of a formal scheme, and being unibranch (or topologically normal) still makes sense in this context, see [Gle24, Definition 4.52]. Our main result is:

Theorem 1.3. *The kimberlite $\mathcal{M}_{\mathcal{G},\mu}$ is unibranch for all pairs $(\mathcal{G},\mu)$ from Definition 1.1.*

As an immediate corollary, we get the geometric part of the Scholze–Weinstein conjecture in full generality, removing certain exceptions found in [AGLR22, Theorem 7.23] for $p \leq 3$.

Corollary 1.4. *If $F = k(\!(t)\!)$ or μ is minuscule, the underlying scheme $\mathcal{M}^{\mathrm{sch}}_{\mathcal{G},\mu}$ is normal with reduced special fiber. If $p > 2$ or Φ_G is reduced, $\mathcal{M}^{\mathrm{sch}}_{\mathcal{G},\mu}$ is Cohen–Macaulay with Frobenius split special fiber.*

Proof. The first sentence follows from Theorem 1.3 and Lemma 1.2. In the last sentence, we must exclude the case $p = 2$ and Φ_G non-reduced as in [AGLR22, Assumption 1.9]. Our goal following [AGLR22, Conjecture 7.25] is to identify the special fiber of $\mathcal{M}^{\mathrm{sch}}_{\mathcal{G},\mu}$ with the canonical deperfection $\mathcal{A}^{\mathrm{sch}}_{\mathcal{G},\mu}$ of the μ-admissible locus, which follows from [FHLR22, Theorem 4.1, Corollary 5.9] in equicharacteristic and [AGLR22, Theorem 3.16] for minuscule μ. □

Remark 1.5. Another direct corollary of Theorem 1.3 is that moduli of p-adic shtukas in the sense of [Gle21, Definition 2.27] or the more restrictive [PR24, Definition 3.2.1] are also unibranch.

Let us explain the strategy behind Theorem 1.3. The main idea is that the stalks of the first non-trivial cohomology sheaf of the nearby cycles $R\Psi(\mathrm{IC}_\mu)$ detects the number of connected components of the tubular neighborhoods. In this way, a disconnection of a tubular neighborhood produces a disconnection étale locally. This however involves a comparison between the analytic nearby cycles defined in [Sch17] and the formal nearby cycles defined in [Gle24], see Theorem 4.7. In general, such a result only holds if the analytic nearby cycles are already algebraic, see Definition 4.6. Fortunately, algebraicity of $R\Psi(\mathrm{IC}_\mu)$ was proved in [AGLR22]. We expect this theorem to find broader applications in the study of integral models of local Shimura varieties and moduli of p-adic shtukas.

If $\mathcal{G}$ is Iwahori (enough for our purposes by Lemma 5.2 and [Gle24, Lemma 5.26]), one uses the theory of Wakimoto filtrations in mixed characteristic developed in [ALWY23] to bound the dimension of stalks of $R\Psi(\mathrm{IC}_\mu)$ along orbits in codimension at most 1 and to prove directly that $\mathcal{M}_{\mathcal{G},\mu}$ is unibranch away from a subset of codimension 2. To prove unibranchness in deeper strata, one uses the perversity of $R\Psi(\mathrm{IC}_\mu)$ due to [ALWY23] and a combinatorial argument. To deal with codimension 2 subsets in the special fiber, we observe that $\mathcal{A}_{\mathcal{G},\mu}$ is perfectly S_2 (i.e. the perfection of a scheme that satisfies Serre's condition S_2).[2] The S_2 property can be neatly expressed combinatorially due to [HH94], so it reduces to positive characteristic. Morally, the S_2 property implies that the normalization map has to be an isomorphism. Unfortunately, that map is not available for kimberlites, so we argue more carefully. We exploit the S_2 property and unibranchness on codimension 1 strata to prove that an étale-formal local disconnection of the generic fiber forces a large open subset to specialize to a small stratum contradicting the perversity of $R\Psi(\mathrm{IC}_\mu)$.

In positive characteristic, we reprove normality for tame groups using $\mathbb{G}_{m,k}$-actions, following techniques of Le–Levin–Le Hung–Morra [LHLM22], who study the unibranch property for the more singular crystalline local models. The strategy for proving that the local model is unibranch in this case was therefore known to the authors of [LHLM22], but they did not seem to know of Lemma 1.2.

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[2] Under the conditions of Corollary 1.4, we know that $\mathcal{A}^{\mathrm{sch}}_{\mathcal{G},\mu}$ is Cohen–Macaulay, compare with [HR22]. If F is p-adic and μ is not minuscule, it is reasonable to still expect $\mathcal{A}^{\mathrm{sch}}_{\mathcal{G},\mu}$ to be Cohen–Macaulay.

2. New proof of Zhu's theorem

In this section, we establish a particular case of Theorem 1.3, originally due to [Zhu14], via global methods specific to positive characteristic. Let k be algebraically closed of characteristic p and G a connected reductive $\mathbb{G}_{m,k}$-group that splits over the finite étale cover $\mathbb{G}_{m,k} \to \mathbb{G}_{m,k}$ given by raising to the e-th power, with $(e,p)=1$. Let $\mathcal{G}$ be the $\mathbb{A}^1_k$-model of G built out of a parahoric $k[\![t]\!]$-model of $G_{k(\!(t)\!)}$ and G via Beauville–Laszlo descent. We regard the Beilinson–Drinfeld Grassmannian $\mathrm{Gr}_{\mathcal{G}}$ as being an ind-(perfect scheme) defined over the perfection $\mathbb{A}^{1,\mathrm{pf}}_k$ of the degree e cover $\mathbb{A}^1_k \to \mathbb{A}^1_k$ that splits G away from the origin, the local model $\mathcal{M}_{\mathcal{G},\mu}$ given as the closure of $\mathrm{Gr}_{G,\mu}$ inside $\mathrm{Gr}_{\mathcal{G}}$, and finally $\mathcal{M}^{\mathrm{sch}}_{\mathcal{G},\mu}$ as the canonical weakly normal deperfection of $\mathcal{M}_{\mathcal{G},\mu}$ finitely presented over $\mathbb{A}^1_k$. Apologies to the reader are in order for deviating from the notation in Definition 1.1, which referred to the associated v-sheaves over a complete local ring, but it is not hard to reconcile both perspectives.

We aim to reprove [Zhu14, Theorem 3]. Thanks to Lemma 1.2, this can proceed along the lines of [LHLM22, Section 3].

Theorem 2.1 ([Zhu14])**.** *The flat projective scheme $\mathcal{M}^{\mathrm{sch}}_{\mathcal{G},\mu}$ is normal with Frobenius split, reduced fiber over 0.*

A special feature of tame groups in equicharacteristic is that $\mathrm{Gr}_{\mathcal{G}}$ carries the so-called rotation $\mathbb{G}^{\mathrm{pf}}_{m,k}$-action which lifts the e-th power of the natural one on the base $\mathbb{A}^{1,\mathrm{pf}}_k$, see [Zhu14, Section 5]. Also, we can regard a maximal F-split torus $S \subset G$ as defined over k up to unique isomorphism, and we get a natural action of S^{pf} on $\mathrm{Gr}_{\mathcal{G}}$ linear over $\mathbb{A}^{1,\mathrm{pf}}_k$. For any coweight χ of $S \times \mathbb{G}_{m,k}$, the induced $\mathbb{G}^{\mathrm{pf}}_{m,k}$-action on $\mathrm{Gr}_{\mathcal{G}}$ is Zariski locally linearizable in the sense of [Ric19]. This is seen by reduction to GL_n, where we reason via lattices as in [HR21, Lemma 3.3]. Now, the attractor $\mathrm{Gr}^+_{\mathcal{G}}$ exists by [Ric19, Theorem 1.8] and [HR21, Theorem 2.1], and is representable by a disjoint union of locally closed sub-ind-schemes. By compactifying $\mathrm{Gr}_{\mathcal{G}}$ to $\mathbb{P}^{1,\mathrm{pf}}_k$ (simply extend $\mathcal{G}$ further to a parahoric $\mathbb{P}^1_k$-group scheme, see [Lou23, Définition 4.2.8]), we see by [Ric19, Lemma 1.11] that the attractor $\mathrm{Gr}^+_{\mathcal{G}}$ maps surjectively to $\mathrm{Gr}_{\mathcal{G}}$ if χ is contracting on $\mathbb{A}^{1,\mathrm{pf}}_k$. We denote by $\mathcal{F}\ell_{\mathcal{G}}$ the fiber of $\mathrm{Gr}_{\mathcal{G}}$ over 0.

Lemma 2.2. *For every S-fixed point $w \in \mathcal{F}\ell_{\mathcal{G}}$, there exists $\chi_w \colon \mathbb{G}_{m,k} \to S \times \mathbb{G}_{m,k}$ such that $w \in \mathcal{F}\ell^0_{\mathcal{G}}$ is isolated and the connected component of $\mathcal{F}\ell^+_{\mathcal{G}}$ containing w is open in $\mathcal{F}\ell_{\mathcal{G}}$.*

Proof. Recall that $S \times \mathbb{G}_{m,k}$ supports a Kac–Moody root system, see [Lou23, Définition 4.2.1, Lemme 4.2.2]. Let $\chi \colon \mathbb{G}_{m,k} \to S \times \mathbb{G}_{m,k}$ be a coweight lying in the w-conjugate of the anti-dominant facet of type corresponding to $\mathcal{G}$. Then, the connected component of $\mathcal{F}\ell^+_{\mathcal{G}}$ containing w equals the open left translate $w \cdot L^{--}\mathcal{G} \cdot e \subset \mathcal{F}\ell_{\mathcal{G}}$ of the big cell, see [Lou23, Corollaire 4.2.11]. □

Notice that χ_w is anti-dominant for the Kac–Moody root system, meaning it acts on the variable t_1 defining the flag variety of $\mathcal{F}\ell_{\mathcal{G}}$ by negative powers, whereas we expect χ_w to contract the affine line $\mathbb{A}^{1,\mathrm{pf}}_k$ that serves as base of the local model. The change of sign occuring here is explained by the fact that the definition of $\mathrm{Gr}_{\mathcal{G}}$ involves an auxiliary formal variable t_2, so that $r^{-1}t_1 - t_2$ and $t_1 - rt_2$ define the same Cartier divisor.

The next task is to globalize the open set of the previous lemma. In the case of the Iwahori model of GL_n, the desired open neighborhood is constructed explicitly in [LHLM22, Lemma 3.2.7]. Instead, we provide an abstract argument.

Lemma 2.3. *Let χ_w be as in Lemma 2.2. Then, $w \in \mathrm{Gr}^0_{\mathcal{G}}$ is isolated and the connected component of $\mathrm{Gr}^+_{\mathcal{G}}$ containing w is open in $\mathrm{Gr}_{\mathcal{G}}$.*

Proof. Choose a presentation of a connected component of $\mathrm{Gr}_{\mathcal{G}}$ containing w by an increasing union of $\mathbb{G}^{\mathrm{pf}}_{m,k}$-stable perfect varieties X. A simple finiteness argument with reduced words, see also [HLR24, Theorem 2.5], shows that every sufficiently large element of the Iwahori–Weyl group is bigger than w. Thus, we may and do assume that every irreducible component of the fiber X_0 over 0 already contains the point w. Let U be the connected component of X^+ containing the generic point of X. Since $U \subset X$ is locally closed and X is irreducible, it must be an open subset. Also, its fiber U_0 over 0 is non-empty, as it contains the fixed points U^0. As every generic point of X_0 specializes to w, we see that U_0 intersects $w \cdot L^{--}\mathcal{G} \cdot e$ non-trivially. In particular, U is contracting to w. □

We also need the following helpful criterion to detect the unibranch property.

Lemma 2.4 ([LHLM22])**.** *Let X be a perfect k-variety with monoid $\mathbb{A}_k^{1,\mathrm{pf}}$-action, such that $X^0 = \{x\}$. Then X is unibranch at x.*

Proof. We explain the idea of [LHLM22, Lemma 3.4.8] for the reader's convenience. Note that the $\mathbb{A}_k^{1,\mathrm{pf}}$-action extends to the normalization Y. The closed subspace Y^0 is the fiber over x. So now the Białynicki-Birula map $Y \to Y^0$ shows that the right side is connected, as Y is irreducible. □

Proof of Theorem 2.1. Thanks to Lemma 2.3, we can produce for any given $S^{\mathrm{pf}} \times \mathbb{G}_{m,k}^{\mathrm{pf}}$-stable point $w \in \mathcal{M}_{\mathcal{G},\mu}(k)$, a χ_w-stable open neighborhood $\mathcal{N}_w$ of w in $\mathcal{M}_{\mathcal{G},\mu}$ such that $\mathcal{N}_w = \mathcal{N}_w^+$ and $\mathcal{N}_w^0 = w$. Now, $\mathcal{N}_w$ is irreducible, so Lemma 2.4 shows that $\mathcal{M}_{\mathcal{G},\mu}$ is unibranch at w. By $L^+\mathcal{G}$-equivariance, this holds at any closed point of the special fiber of $\mathcal{M}_{\mathcal{G},\mu}$, hence it must be unibranch. By Lemma 1.2, we conclude that the special fiber of $\mathcal{M}_{\mathcal{G},\mu}^{\mathrm{sch}}$ is reduced. We remind the reader that, even though we only introduce nearby cycles in Section 4, they are already implicit here, since Lemma 1.2 relies on the calculation of the special fiber from [AGLR22, Theorem 6.16].

Now, we will not require the global rotation action, and need to work with the scheme-theoretic $\mathcal{M}_{\mathcal{G},\mu}^{\mathrm{sch}}$, so we base change our entire situation to the complete local ring $k[\![t]\!]$ of $\mathbb{A}_k^1$. In order to see that the special fiber is Frobenius split, we must consider the flat closure $\mathcal{M}_{\mathcal{G},\mu}^{\mathrm{fl}}$ inside the ind-scheme $\mathrm{Gr}_{\mathcal{G}}^{\mathrm{sch}}$, i.e. before taking perfections. In general, there is a universal homeomorphism $\mathcal{M}_{\mathcal{G},\mu}^{\mathrm{sch}} \to \mathcal{M}_{\mathcal{G},\mu}^{\mathrm{fl}}$, but it is typically only an isomorphism if $p \nmid \pi_1(G_{\mathrm{der}})$. This last condition is necessary and sufficient to avoid certain pathological non-normal Schubert varieties discovered in [HLR24, Theorem 2.5] that force $\mathcal{M}_{\mathcal{G},\mu}^{\mathrm{fl}}$ to not be normal. However, notice that $\mathcal{M}_{\mathcal{G},\mu}^{\mathrm{sch}}$ does not vary under central extensions by [AGLR22, Proposition 4.16], so we may and do assume until the end of the proof that G_{der} is simply connected, by passing to a z-extension.

If $G_{\mathrm{der}} = G_{\mathrm{sc}}$, we claim that the natural morphism $\mathcal{M}_{\mathcal{G},\mu}^{\mathrm{sch}} \to \mathcal{M}_{\mathcal{G},\mu}^{\mathrm{fl}}$ is an isomorphism. Since it is a finite universal homeomorphism which is an isomorphism on generic fibers by [Fal03, Theorem 8], we may pass to special fibers and check that it becomes a closed immersion by Nakayama's lemma. Indeed, if we have a finite injection $R \to S$ of local rings and we know that $R/I \to S/IS$ is surjective for some ideal I contained in the maximal ideal of R, then $R \to S$ is an isomorphism by Nakayama applied to S/R. But we know by [PR08, Theorem 8.4, Proposition 9.7] that Schubert varieties in the scheme-theoretic flag variety are compatibly Frobenius split (thanks to the equality $G_{\mathrm{der}} = G_{\mathrm{sc}}$), hence their unions are Frobenius split and thus weakly normal. In particular, the deperfected admissible locus $X := \mathcal{A}_{\mathcal{G},\mu}^{\mathrm{sch}}$ identifies with the reduction of the special fiber Z of $\mathcal{M}_{\mathcal{G},\mu}^{\mathrm{fl}}$. On the other hand, X is the weak normalization of the special fiber Y of $\mathcal{M}_{\mathcal{G},\mu}^{\mathrm{sch}}$ which is reduced. We get a sequence of maps

$$X \to Y \to Z \tag{2.1}$$

between separated schemes whose composition is a closed immersion. Hence, the map $X \to Y$ is a closed immersion and a universal homeomorphism of reduced schemes, thus an isomorphism. We conclude that the map on special fibers $Y \to Z$ must be a closed immersion. □

Remark 2.5. There are two ways in which the results in [Zhu14] seem to differ from Theorem 2.1:

(1) The original [Zhu14, Theorem 3] refers to the local model $\mathcal{M}^{\mathrm{fl}}_{\mathcal{G},\mu}$ defined via flat closure under the assumption $p \nmid \pi_1(G_{\mathrm{der}})$. During our proof of Theorem 2.1, we explained this difference and how it relates to [HLR24, Theorem 2.5].

(2) The finer [Zhu14, Theorem 6.10] asserts that the local model is compatibly Frobenius split with its special fiber. We stress that our methods do not yield this stronger claim, as we work over the perfection until we apply the normality result of [PR08] in the special fiber, which is considerably weaker.

3. Combinatorics of admissible sets

In contrast with the introduction, we no longer assume that F is a local field, but rather a complete discretely valued field with algebraically closed residue field k of positive characteristic. Let G be a connected reductive F-group and $\mathcal{I}$ a Iwahori O-model of G.

Fix a quadruple (G, S, T, B) where S is a maximal F-split torus whose apartment contains the alcove fixed by $\mathcal{I}$, T is the maximal torus given as $Z_G(S)$, and B is some F-Borel containing S. We denote $N = N_G(S)$ and let $\tilde{W} = N(F)/\mathcal{T}(O)$ be the Iwahori–Weyl group of G. It is an extension of $W_0 = N(F)/T(F)$ by $T(F)/\mathcal{T}(O)$. We consider the Kottwitz map $T(F)/\mathcal{T}(O) \to X_*(T)_I$ to the group of inertia coinvariants of T-coweights, inducing a bijection with inverse $\bar{\nu} \mapsto t_{\bar{\nu}}$. Note that, according to the sign conventions of [BT72, BT84], $t_{\bar{\nu}}$ acts on the apartment of S by $-\bar{\nu}$.

Recall that the Witt flag variety $\mathcal{F}\ell_{\mathcal{I}}$ of [Zhu17] is an ind-(perfect scheme) by [BS17] stratified in $L^+\mathcal{I}$-orbits indexed by $\tilde{W}$. The reduction of the local model $\mathcal{M}_{\mathcal{I},\mu}$ embeds in $\mathcal{F}\ell_{\mathcal{I}}$ and it was shown in [AGLR22, Theorem 6.16] that it coincides with the μ-admissible locus $\mathcal{A}_{\mathcal{I},\mu}$. This perfect $L^+\mathcal{I}$-stable subscheme is defined via the μ-admissible set $\mathrm{Adm}(\mu)$ of Kottwitz–Rapoport [KR00]: the lower poset generated by $t_{\bar{\nu}}$ with $\bar{\nu}$ running over the $\hat{T}^I$-weights $\Omega(\mu)$ of the highest weight $\hat{G}$-representation V_μ. Its maximal elements form a W_0-orbit [Hai18, Theorem C] and we denote by $\Lambda(\mu) \subset X_*(T)_I$ its image under the Kottwitz map.

Lemma 3.1 ([Hai05])**.** *Suppose $x \in \mathrm{Adm}(\mu)$ has colength 1. Then, there are at most two distinct $\bar{\nu}_i \in \Lambda(\mu)$ such that $x \leq t_{\bar{\nu}_i}$ for $i = 1, 2$.*

Proof. We proceed as in the last paragraph of [Hai05, Proposition 8.7]. There is an affine reflection s_i, such that $x = s_i t_{\bar{\nu}_i}$ due to the colength hypothesis. Mapping x to the group W_0 of euclidean transformations, we see that the fixed hyperplanes of the s_i must all be parallel, so the $\bar{\nu}_i$ lie in a $\mathbb{R}$-line. But they all have the same length by maximality, and a $\mathbb{R}$-line cannot cross a $\mathbb{R}$-sphere more than twice. □

The next remark was explained to us by Haines. Although it will not be needed, we leave it for the interested reader.

Remark 3.2 (Haines)**.** [Hai05, Proposition 8.7] can be adapted to describe the set $\mathrm{Irr}(x) \subset \Lambda(\mu)$ whose translations lie above x with colength 1. Write $x = t_{\bar{\nu}} s_\beta$ for some $\bar{\nu} \in \Omega(\mu)$ and some positive root β of the échelonnage root system Σ, see [BT72, 1.4.1].

(a) If $x < s_\beta x$, then $\mathrm{Irr}(x) = \{\bar{\nu}, s_\beta(\bar{\nu})\}$.

(b) If $s_\beta x < x$, then $\mathrm{Irr}(x) = \{\bar{\nu} + \beta^\vee, s_\beta(\bar{\nu} + \beta^\vee)\}$. If μ is minuscule with respect to Σ, this does not occur.

Another important tool for us will be the S_2 property of Serre for μ-admissible loci. We say that a perfect k-scheme X perfectly of finite presentation is perfectly S_2 if it is the perfection of an S_2 finite type scheme.

Lemma 3.3. *If X is equidimensional, there is up to isomorphism a unique perfectly finite birational morphism $X^{S_2} \to X$ such that X^{S_2} is perfectly S_2 and identifies with the right side away from codimension 2. In particular, X is perfectly S_2 if and only if some (equiv. every) weakly normal finite type deperfection X_0 is S_2.*

Proof. Take a finite type reduced deperfection X_0 of X, an S_2 open subset $U_0 \subset X_0$ with complement of codimension 2 and consider the S_2-ification $X_0^{S_2} \to X_0$ given as the normalization of $U_0 \to X_0$, see [Čes21, Lemma 2.11, Corollary 2.14]. Passing to the perfection, we get the desired morphism with the stated property. Indeed, given a finite universal homeomorphism $X_1 \to X_0$ of reduced schemes such that the preimage $U_1 \to U_0$ is also S_2, the local sections of $\mathcal{O}_{U_1}$ are iterated p-th roots of those of $\mathcal{O}_{U_0}$, so the same holds for the integral closures. For the last claim, observe that X being perfectly S_2 implies $X_0^{S_2} \to X_0$ is a finite universal homeomorphism and birational, hence an isomorphism by weak normality of X_0. $\square$

In the next result, we are going to consider the following subset

$$\mathrm{Codim}_{\leq 1}(x) = \{y \mid y \text{ has colength} \leq 1 \text{ in } \mathrm{Adm}(\mu) \text{ and } y \geq x\} \tag{3.1}$$

of the μ-admissible set $\mathrm{Adm}(\mu)$. Recall that this is endowed with the Bruhat order, which in turn defines the so-called Bruhat graph by specifying edges between the vertices.

Proposition 3.4. *The following are equivalent:*

(1) *The μ-admissible locus $\mathcal{A}_{\mathcal{I},\mu}$ is perfectly S_2.*
(2) *For any $x \in \mathrm{Adm}(\mu)$, the (undirected) Bruhat graph of* $\mathrm{Codim}_{\leq 1}(x)$ *is connected.*

Proof. Let $\mathcal{A}^{\mathrm{sch}}_{\mathcal{I},\mu}$ be the canonical deperfection in the sense of [AGLR22, Definition 3.14], which is weakly normal by construction. We know by Lemma 3.3 that this scheme is S_2 if and only if $\mathcal{A}_{\mathcal{I},\mu}$ is perfectly S_2. Consider the Hochster–Huneke graph $\mathrm{HH}(x)$ of the local ring of $\mathcal{A}^{\mathrm{sch}}_{\mathcal{I},\mu}$ at some closed point in the x-stratum, see [HH94, Definition 3.4]. Recall that the vertices of $\mathrm{HH}(x)$ are enumerated by the irreducible components of the local ring, and the edges connecting two of those by prime divisors contained in their intersection. Hence, $\mathrm{HH}(x)$ and $\mathrm{Codim}_{\leq 1}(x)$ have the same number of connected components. Now, as the irreducible components of $\mathcal{A}^{\mathrm{sch}}_{\mathcal{I},\mu}$ are unibranch by [AGLR22, Proposition 3.7], the Hochster–Huneke graph does not change under completion in our situation, so we apply [HH94, Theorem 3.6], which states that the fibers of closed points in the x-stratum of the S_2-ification of $\mathcal{A}^{\mathrm{sch}}_{\mathcal{I},\mu}$ are singletons exactly when $\mathrm{HH}(x)$ is connected. Here, we also use that S_2-ifications commute with completion, see [HH94, Proposition 3.8].

At this point, we have concluded that if $\mathcal{A}_{\mathcal{I},\mu}$ is perfectly S_2, then the graphs $\mathrm{Codim}_{\leq 1}(x)$ are connected for all x. Conversely, if the graphs are all connected, then the S_2-ification of $\mathcal{A}_{\mathcal{I},\mu}$ is bijective at k-valued points. Since our situation does not change when replacing k by a larger algebraically closed field, the S_2-ification is a universally bijective perfectly finite map, and hence an isomorphism. $\square$

It would be interesting to find a purely combinatorial proof of the S_2 property of admissible sets. Instead, we apply the previous criterion twice to reduce to positive characteristic geometry.

Corollary 3.5. *The μ-admissible locus $\mathcal{A}_{\mathcal{I},\mu}$ is perfectly S_2.*

Proof. We may and do assume that G is adjoint. By Proposition 3.4, it suffices to prove an analogous combinatorial property for $\mathrm{Adm}(\mu)$. Observe that $\tilde{W}$ embeds into the group of affine transformations of the échelonnage root system Σ, its quotient W_0 identifies with vector transformations of Σ, and $\mathrm{Adm}(\mu)$ is the lower poset generated by the W_0-orbit of some translation element, thanks to [Hai18, Theorem C]. But this can be realized as $\mathrm{Adm}(\mu')$ for a split $k(\!(t)\!)$-group G' by the classification of affine Coxeter groups, see [BT72, 1.3.17, 1.4.6]. Applying again Proposition 3.4, we need to show that, under the previous restrictions, the local rings of $\mathcal{A}_{\mathcal{I}',\mu'}$ are connected away from codimension 2. But this is a consequence of Theorem 2.1 and Grothendieck's connectedness theorem, see [GR05, Exposé XIII, Théorème 2.1]. $\square$

4. Comparison of nearby cycles

In this section, we compare the analytic nearby cycles of [Sch17] to the formal ones of [Gle24, Remark 4.29] under an algebraicity assumption. In the classical setting of formal schemes, our results are known by the work of Huber [Hub96], so the scheme-theoretically inclined reader can safely skip this section.

Fix a rank 1 valuation ring O and a prekimberlite X over $\operatorname{Spd} O$ in the sense of [Gle24, Definition 4.15]. Let $j : Y = X^{\mathrm{an}} \to X$, $i : Z^\diamond \to X$ be the natural inclusions, where $Z = X^{\mathrm{red}}$. Given an ℓ-torsion coefficient ring Λ with $\ell \neq p$, the first author defines in [Gle24] a naive nearby cycles functor

$$R\Psi' : D_{\text{ét}}(Y, \Lambda) \to D_{\text{ét}}(Z, \Lambda). \tag{4.1}$$

This arises as the (left-completion of the) derived pushforward of a morphism of sites $\Psi' \colon Y_{\text{ét}} \to Z_{\text{ét}}$ induced by canonical liftings of étale neighborhoods [Gle24, Theorem 4.27]. On the other hand, we have the nearby cycles functor of Scholze's theory $R\Psi : D_{\text{ét}}(Y, \Lambda) \to D_{\text{ét}}(Z^\diamond, \Lambda)$ given by $R\Psi = i^* Rj_*$.

In order to compare them, we can consider the fully faithful functor (Proposition 4.2):

$$D_{\text{ét}}(Z, \Lambda) \xrightarrow{c_Z^*} D_{\text{ét}}(Z^{\diamondsuit}, \Lambda) \xrightarrow{t_Z^*} D_{\text{ét}}(Z^\diamond, \Lambda) \tag{4.2}$$

where c_Z^* is the functor from [Sch17, §27] and $t_Z : Z^\diamond \to Z^{\diamondsuit}$ is the natural inclusion. We refer the reader to [AGLR22, Definition 2.10] for the construction of the functors $Z \mapsto Z^\diamond$ and $Z \mapsto Z^{\diamondsuit}$. By construction, $R\Psi'$ lands in the full subcategory of $D_{\text{ét}}(Z^\diamond, \Lambda)$ just described, while in general $R\Psi$ might not. In this section, we prove that under some conditions if $R\Psi(A) \in D_{\text{ét}}(Z, \Lambda)$ then $R\Psi(A) = t_Z^* c_Z^* R\Psi'(A)$. We do this in a series of lemmas.

Lemma 4.1. *Let U be a perfect scheme separated over $\mathbb{F}_p$. When U is affine denote by $t_{U^\dagger} \colon U^\dagger \to U^{\diamondsuit}$ the inclusion of the closure of $U^\diamond$ in $U^{\diamondsuit}$. For every $B \in D_{\text{ét}}(U, \Lambda)$ we have canonical identifications $R\Gamma(U^\diamond, t_U^* c_U^* B) = R\Gamma(U^{\diamondsuit}, c_U^* B) = R\Gamma(U, B)$. Moreover, if U is affine $R\Gamma(U^\dagger, t_{U^\dagger}^* c_U^* B) = R\Gamma(U^\diamond, t_U^* c_U^* B)$.*

Proof. The second equality is [Sch17, Proposition 27.2]. The first equality is done following the same reduction steps in the proof loc. cit. reducing to the case where $U = \operatorname{Spec}(\mathbb{A}^I)$ for a set I. This case is settled during the proof of [Sch17, Proposition 27.2] by appealing to the invariance under change of base field [Sch17, Theorem 19.5] and comparing cohomology on affine space $\mathbb{A}^I_C$ and the unit ball $\mathbb{B}^I_C$.

If $i : U^\diamond \to U^\dagger$ is the natural inclusion, then a Postnikov tower argument, quasicompact basechange, [Sch17, Proposition 17.6], and that $c_U^* B$ is overconvergent imply that $Ri_* t_{U^\diamond}^* c_U^* B = t_{U^\dagger}^* c_U^* B$. This proves the second claim. $\square$

Proposition 4.2. *Let U be a perfect scheme separated over $\mathbb{F}_p$, the functor $t_U^* c_U^*$ is fully faithful.*

Proof. We may assume $A \in D^+_{\text{ét}}(U, \Lambda)$, since full faithfulness extends formally to left-completions. By descent, we may also assume U is defined over $\operatorname{Spec}(\overline{\mathbb{F}}_p)$. For $A \in D_{\text{ét}}(U, \Lambda)$ we verify $A \to Rc_{U,*} Rt_{U,*} t_U^* c_U^* A$ is an isomorphism by checking this on sections. Let $B = c_U^* A$, we also denote by B the evident pullback to different loci. By Lemma 4.1 one reduces to proving $R\Gamma(Q, B) = R\Gamma(V^\diamond, B)$ for sufficiently small étale neighborhoods $V \to U$, for $Q = U^\diamond \times_{U^\diamondsuit} V^{\diamondsuit}$. Let $W \subseteq U$ be an open subset and let $V_W = V \times_U W$, observe that $W^\diamond \times_{U^\diamond} Q = W^\diamond \times_{W^\diamondsuit} V_W^{\diamondsuit}$. Applying descent to an open cover $\coprod W \to U$ and shrinking V we may assume that U and V

are affine. Let $\overline{V}$ be the closure of $V^\diamond$ in Q, arguing as in Lemma 4.1 we have $R\Gamma(V^\diamond, B) = R\Gamma(\overline{V}, B)$ so it suffices to see that $R\Gamma(Q, j_! B) = 0$ for the open immersion $j : Q \setminus \overline{V} \to Q$. By Noetherian approximation $V \to U$ arises as the base change of an étale map $S \to T$ with S and T the spectrum of finite type $\overline{\mathbb{F}}_p$-algebras. All spaces above come from basechange by the map $\pi : U^\diamond \to T^{\diamondsuit}$ which is qcqs. Indeed, Q corresponds to $S^{\diamondsuit}$, $V^\diamond$ corresponds to $S^\diamond$ and $\overline{V}$ corresponds to $S^\dagger$. To see the last identification recall that $U^\dagger \times_{T^\dagger} S^\dagger = V^\dagger$ as it is evident from the moduli description [Gle24, Proposition 2.22], and that $V^\diamond$ is always dense in $V^\dagger$ [Gle24, Proposition 2.24]. By [Sch17, Proposition 17.6], it suffices to prove $R\Gamma(S^{\diamondsuit}, Rs_! s^* R\pi_* B) = 0$ where $s : S^{\diamondsuit} \setminus S^\dagger \to S^{\diamondsuit}$ is the natural inclusion. We claim $R\Gamma(S^{\diamondsuit}, Rs_! K) = 0$ for any K. Finding a closed immersion $S \to \mathbb{A}^N_{\overline{\mathbb{F}}_p}$, it suffices to prove this if $S = \mathbb{A}^N_{\overline{\mathbb{F}}_p}$. This follows from [FS21, Theorem IV.5.3], since $S^{\diamondsuit} \setminus S^\dagger$ is a spatial diamond partially proper over $\mathrm{Spd}(\overline{\mathbb{F}}_p)$. Indeed, it is the pointed formal completion of the divisor at infinity in $\mathbb{P}^N_{\overline{\mathbb{F}}_p}$. □

Lemma 4.3. *Let $A \in D_{\text{ét}}(Y, \Lambda)$. Then $R\Gamma(X, Rj_! A) = 0$, or equivalently $R\Gamma(Y, A) = R\Gamma(Z^\diamond, R\Psi(A))$.*

Proof. Since $j_!$ commutes with canonical truncations a Postnikov limit argument allows us to assume $A \in D^+_{\text{ét}}(Y, \Lambda)$. Since X is a specializing v-sheaf there is a hypercover of $X_\bullet \to X$ where each X_i is of the form $\coprod_{j \in I_i} \mathrm{Spd}(R_j^+)$ for I_j a set and the $\mathrm{Spa}(R_j, R_j^+)$ are strictly totally disconnected perfectoid spaces. By v-hyperdescent [Sch17, Proposition 17.3] and proper basechange we may assume $X = \mathrm{Spd}(R^+)$. At this point we may cite [FS21, Remark V.4.3]. Let us explain a detail. A choice of pseudouniformizer defines a qcqs map $\mathrm{Spd}(R^+) \to \mathrm{Spd}(W(k)[\![t]\!])$ and [Sch17, Proposition 17.6] reduces the computation to the case $X = \mathrm{Spd}(W(k)[\![t]\!])$ which follows from [FS21, Theorem IV.5.3]. □

We also have a map of sites $f \colon X_v \to Z_{\text{ét}}$ induced again by canonical liftings of étale neighborhoods. Restricted to $Z_v^\diamond$, $f \circ i_v$ sends an étale map $U \to Z$ to $U^\diamond \to Z^\diamond$. In the case of Y_v, $f \circ j_v$ factors as the composition of $\nu_Y \colon Y_v \to Y_{\text{ét}}$ with $\Psi' \colon Y_{\text{ét}} \to Z_{\text{ét}}$. We get a functor $Rf_{v,*} : D(X_v, \Lambda) \to D_{\text{ét}}(Z, \Lambda)$ that factors through $D_{\text{ét}}(X, \Lambda)$, since its right adjoint factors through the inclusion $D_{\text{ét}}(X, \Lambda) \subseteq D(X_v, \Lambda)$. We denote by Rf_* the induced map on this latter category.

Lemma 4.4. *Let $A \in D_{\text{ét}}(Y, \Lambda)$. Then $R(f \circ i)_* R\Psi(A) = R\Psi' A$.*

Proof. We have an exact triangle $Rj_! A \to Rj_* A \to i_* R\Psi(A)$ in $D_{\text{ét}}(X, \Lambda)$, to which we apply Rf_*. Now, by [Sch17, Proposition 14.10, Proposition 14.11] $Rf_* Rj_* = R(f_v \circ j_v)_* \nu_Y^* = R\Psi' \nu_{Y,*} \nu_Y^* = R\Psi'$. It suffices to prove $Rf_* j_! A = 0$. It suffices to prove $R\Gamma(U, Rf_* j_! A) = 0$ for all $U \in Z_{\text{ét}}$. We compute directly $R\Gamma(U, Rf_* j_! A) = R\Gamma(\widehat{X}_U, j_! A) = 0$. The last equality follows from Lemma 4.3, and the fact that passing to étale formal neighborhoods preserves being a prekimberlite. □

Lemma 4.5. *We have canonical identifications $R(f \circ i)_* t_Z^* c_Z^* A = Rc_{Z,*} c_Z^* A = A$.*

Proof. The second one is [Sch17, Proposition 27.2]. Let $h \colon U \to Z$ any étale neighborhood and let $t_U : U^\diamond \to U^{\diamondsuit}$ denote the natural map. We compute as follows:

$$\begin{aligned} R\Gamma(U, h^* Rc_{Z,*} c_Z^* A) &= R\Gamma(U^{\diamondsuit}, h^{\diamondsuit,*} c_Z^* A) \\ &= R\Gamma(U^{\diamondsuit}, c_U^* h^* A) \\ &= R\Gamma(U^\diamond, t_U^* h^{\diamondsuit,*} c_Z^* A) \\ &= R\Gamma(U^\diamond, h^{\diamond,*} t_Z^* c_Z^* A) \\ &= R\Gamma(U, h^* R(f \circ i)_* t^* c_Z^* A), \end{aligned} \tag{4.3}$$

where we have applied [Sch17, Proposition 27.1] twice to commute c^* and h^*, and also Lemma 4.1 in the middle equality. $\square$

From now on assume Y is a spatial diamond that has finite cohomological dimension as in [Sch17, Proposition 20.10] so that $D(Y_{\text{ét}}, \Lambda) = D_{\text{ét}}(Y, \Lambda)$. We also assume that Z is perfectly of finite type over the residue field of O so that $D(Z_{\text{ét}}, \Lambda) = D_{\text{ét}}(Z, \Lambda)$. With these hypothesis $R\Psi' : D_{\text{ét}}(Y, \Lambda) \to D_{\text{ét}}(Z, \Lambda)$ is defined site theoretically (without having to left-complete) and taking stalks is well-behaved.

Definition 4.6. Identify $D_{\text{ét}}(Z, \Lambda)$ with its essential image in $D_{\text{ét}}(Z^\diamond, \Lambda)$ under $t_Z^* c_Z^*$. We say $K \in D_{\text{ét}}(Z^\diamond, \Lambda)$ is algebraic if $K \in D_{\text{ét}}(Z, \Lambda)$.

Recall that for a closed subset $S \subset U$ and an étale map $U \to Z$, we denote by $\widehat{X}_{/S}$ the formal neighborhood at S in the sense of [Gle24, Definition 4.18] and by $X^{\circledcirc}_{/S} := \widehat{X}_{/S} \times_X X^{\text{an}}$ the tubular neighborhood in the sense of [Gle24, Definition 4.38]. (The references given assume $U = Z$, but it extends to étale neighborhoods by [Gle24, Theorem 4.27].)

Theorem 4.7. *Let the notation be as above and $A \in D_{\text{ét}}(Y)$. The following hold:*

1. *If $\overline{x} \in Z$ is a closed point, then $(R\Psi A)_{\overline{x}} = R\Gamma(X^{\circledcirc}_{/\overline{x}}, A)$.*
2. *$(R\Psi' A)_{\overline{x}} = \varinjlim_{\overline{x} \in U} R\Gamma(X^{\circledcirc}_{/U}, A)$, where U runs over all étale neighborhoods of Z at $\overline{x}$.*
3. *If $R\Psi A$ is algebraic then $R\Psi A = R\Psi' A$ and $R\Gamma(X^{\circledcirc}_{/\overline{x}}, A) = \varinjlim_{\overline{x} \in U} R\Gamma(X^{\circledcirc}_{/U}, A)$.*

Proof. The third claim follows directly from Lemma 4.5 and Lemma 4.4, and from the second and first claim. Let $\iota_{\overline{x}}$ denote the inclusion of $\overline{x}$. Notice that $\iota_{\overline{x}}$ factors through the open immersion $\widehat{X}_{/\overline{x}} \to X$. By smooth base change and Lemma 4.3 applied to $\widehat{X}_{/\overline{x}} \to X$ we get $(R\Psi A)_{\overline{x}} = R\Gamma(X^{\circledcirc}_{/\overline{x}}, A)$, proving the first claim. Finally, by [Sch17, Proposition 27.1.(ii)] $\iota_{\overline{x}}^{\diamondsuit,*} c_Z^*(R\Psi' A) = c_{\overline{x}}^*(R\Psi' A_{\overline{x}})$ and the latter is by definition computed by $\varinjlim_{\overline{x} \in U} R\Gamma(X^{\circledcirc}_{/U}, A)$, since it is site theoretic. $\square$

5. Proof of unibranchness

In this section, we prove Theorem 1.3. During this section, we set $\Lambda = \mathbb{F}_\ell$ and consider $\mathcal{M}_{\mathcal{G},\mu}$ already after base change to $\operatorname{Spd} O_C$, where C is a complete algebraic closure of F with ring of integers O_C. The reader who is only interested in the scheme-theoretic local models can imagine this takes place in the realm of formal and rigid-analytic geometry.

The object $\mathcal{Z}_\mu = R\Psi(\mathrm{IC}_\mu)$ allows us to read off the set of connected components of tubes.

Lemma 5.1. *There is an equality $\#\pi_0(\mathcal{M}^{\circledcirc}_{\mathcal{G},\mu/x}) = \dim_{\mathbb{F}_\ell} \mathcal{H}_x^{-\langle 2\rho,\mu\rangle} \mathcal{Z}_\mu$.*

Proof. Let $j_\mu \colon \mathrm{Gr}^\circ_{G,\mu} \to \mathrm{Gr}_G$ denote the orbit inclusion in the generic fiber. It follows from perverse left t-exactness of $Rj_{\mu*}$ and Schubert varieties being unibranch that

$$\mathcal{H}^{-\langle 2\rho,\mu\rangle}(\mathrm{IC}_\mu) = \mathcal{H}^0(Rj_{\mu*}\mathbb{F}_\ell) = \mathbb{F}_\ell \tag{5.1}$$

is a constant sheaf on the generic fiber. (Strictly speaking, the B^+_{dR}-affine Grassmannian is not an ind-scheme, but facts such as these can be reduced via [FS21, Corollary VI.6.7] to the Witt affine Grassmannian of the split form of G, which is an ind-perfect scheme.) This implies that we may replace the right side of the claimed equality by $\dim_{\mathbb{F}_\ell} i_x^* R^0 j_* \mathbb{F}_\ell$, with $j \colon \mathrm{Gr}_{G,\mu} \to \mathcal{M}_{\mathcal{G},\mu}$ the generic fiber inclusion and $i_x \colon \operatorname{Spd} k \to \mathcal{M}_{\mathcal{G},\mu}$ the inclusion of the point x. But the latter equals $\#\pi_0(\mathcal{M}^{\circledcirc}_{\mathcal{G},\mu/x})$ by Theorem 4.7. $\square$

Thanks to [Gle24, Lemma 5.26], we can reduce the proof of Theorem 1.3 to the case when $\mathcal{G}$ is Iwahori, provided we verify the following.

Lemma 5.2. *If $\mathcal{I} \to \mathcal{G}$ is a Iwahori dilation, then the geometric fibers of $\pi\colon \mathcal{A}_{\mathcal{I},\mu} \to \mathcal{A}_{\mathcal{G},\mu}$ are connected.*

Proof. By $\mathcal{I}$-equivariance, we are reduced to considering the fiber over the image $w_{\mathcal{G}}$ of a $L^+\mathcal{T}$-fixed point $w \in \mathcal{A}_{\mathcal{I},\mu}$. We may and do assume that w is minimal in its right $W_{\mathcal{G}}$-coset. Using Demazure resolutions, one sees that the intersection of $\pi^{-1}(w_{\mathcal{G}})$ with any Schubert variety is connected. As all of those subschemes must contain w by minimality, the fiber itself must be connected. □

From now on, we work with a Iwahori model $\mathcal{I}$. To calculate in codimension 1, we are going to apply the Wakimoto filtration of the Gaitsgory central functor studied in [Gai01, AB09] in equicharacteristic and in [AGLR22, ALWY23] in mixed characteristic.

Theorem 5.3 ([ALWY23])**.** *The functor $R\Psi$ is perverse t-exact. Moreover, the perverse sheaf $R\Psi(\mathrm{Sat}(V))$ admits a filtration with subquotients isomorphic to $\mathcal{J}_{\bar{\nu}} \otimes V(w_0\bar{\nu})$.*

The Wakimoto sheaves $\mathcal{J}_{\bar{\nu}}$ depend crucially on the choice of a Borel subgroup $B \subset G$ and we exploit this degree of freedom in Proposition 5.5 by conveniently choosing B. For the images $\bar{\nu}$ of B-dominant coweights, $\mathcal{J}_{\bar{\nu}}$ are defined as the costandard object $\nabla_{\bar{\nu}} = Rj_{\bar{\nu},*}\mathbb{F}_\ell[\ell(t_{\bar{\nu}})]$, which admits the standard object $\Delta_{\bar{\nu}} = j_{\bar{\nu},!}\mathbb{F}_\ell[\ell(t_{\bar{\nu}})]$ as a left and right inverse for convolution. Here $j_{\bar{\nu}} : \mathcal{F}\ell^\circ_{\mathcal{I},t_{\bar{\nu}}} \to \mathcal{F}\ell_{\mathcal{I}}$ is the natural orbit inclusion, which is affine, so the derived functors are perverse t-exact by [BBDG18, Corollaire 5.1.3]. The definition extends by convolution and linearity to the other $\bar{\nu}$. Just as in [AB09, Theorem 5], it turns out that $\mathcal{J}_{\bar{\nu}}$ is a perverse sheaf concentrated in $\mathcal{F}\ell_{\mathcal{I},t_{\bar{\nu}}}$ and restricts to $\mathbb{F}_\ell[\ell(t_{\bar{\nu}})]$ on $\mathcal{F}\ell^\circ_{\mathcal{I},t_{\bar{\nu}}}$. The proof of Theorem 5.3 in [ALWY23] relies not only on the $\mathcal{J}_{\bar{\nu}}$, but also on $R\Psi(\mathrm{Sat}(V))$ being central, see [AGLR22, Proposition 6.17], and the computation of its constant terms, see [AGLR22, Equation (6.32)].

Lemma 5.4. *For any $x \in \mathcal{F}\ell_{\mathcal{I}}(k)$, the $\mathbb{F}_\ell$-vector space $\mathcal{H}_x^{-\ell(t_{\bar{\nu}})}\mathcal{J}_{\bar{\nu}}$ is either zero or one-dimensional.*

Proof. Consider the adjunction map $\mathcal{J}_{\bar{\nu}} \to \nabla_{\bar{\nu}}$. It follows that the kernel and cokernel have support contained in $\mathcal{F}\ell_{\mathcal{I},t_{\bar{\nu}}} \setminus \mathcal{F}\ell^\circ_{\mathcal{I},t_{\bar{\nu}}}$. In particular, they are concentrated on degrees strictly larger than $-\ell(t_{\bar{\nu}})$ by perversity, so that

$$\mathcal{H}_x^{-\ell(t_{\bar{\nu}})}\ker(\mathcal{J}_{\bar{\nu}} \to \nabla_{\bar{\nu}}) = \mathcal{H}_x^{-\ell(t_{\bar{\nu}})}\operatorname{coker}(\mathcal{J}_{\bar{\nu}} \to \nabla_{\bar{\nu}}) = 0. \tag{5.2}$$

Taking the long exact sequence of cohomology, this yields an inclusion $\mathcal{H}_x^{-\ell(t_{\bar{\nu}})}\mathcal{J}_{\bar{\nu}} \subset \mathcal{H}_x^{-\ell(t_{\bar{\nu}})}\nabla_{\bar{\nu}}$. The latter sheaf is constant equal to $\mathbb{F}_\ell$, because the Schubert perfect variety $\mathcal{F}\ell_{\mathcal{I},t_{\bar{\nu}}}$ is normal. □

Proposition 5.5. *Given $x \in \mathcal{A}_{\mathcal{I},\mu}(k)$ whose $L^+\mathcal{I}$-orbit has codimension at most 1, we have an equality $\mathcal{H}_x^{-\langle 2\rho,\mu\rangle}\mathcal{Z}_\mu = \mathbb{F}_\ell$.*

Proof. This is evident in codimension 0. By abuse of notation, we denote by x the corresponding element in $\mathrm{Adm}(\mu)$ and let $x < t_{\bar{\nu}_i}$ with $\bar{\nu}_i \in \Lambda(\mu)$ with $i = 1, 2$ be the only (possibly equal) maximal elements above x by Lemma 3.1. Assume without loss of generality that $\bar{\nu}_1 \in X_*(T)_I^-$ by replacing our initial choice of Borel if necessary. Consider the Wakimoto filtration of the perverse sheaf $\mathcal{Z}_\mu$ given in Theorem 5.3. Only the $\mathcal{J}_{\bar{\nu}_i} \otimes V(w_0\bar{\nu}_i)$ for $i = 1, 2$ contribute to the stalk at x. Since we chose $\mathcal{J}_{\bar{\nu}_1}$ to be standard, it is concentrated exclusively on a maximal orbit and hence does not contribute to the stalk at x. Consequently, $\mathcal{H}_x^{-\langle 2\rho,\mu\rangle}\mathcal{Z}_\mu = V(w_0\bar{\nu}_2) \otimes \mathcal{H}_x^{-\langle 2\rho,\mu\rangle}(\mathcal{J}_{\bar{\nu}_2})$. As $V_\mu(w_0\bar{\nu}_2) = \mathbb{F}_\ell$ for extremal weights, Lemma 5.4 bounds the dimension above by 1. On the other hand, Lemma 5.1 and density of the generic fiber bound the dimension below by 1. (This also implies $\bar{\nu}_1 \neq \bar{\nu}_2$.) □

We have now all the necessary tools at our disposal to finish the proof of the main theorem.

Proof of Theorem 1.3. Assume there is $x \in \mathcal{A}_{\mathcal{I},\mu}(k)$ such that $\mathcal{M}^{\circledcirc}_{\mathcal{I},\mu/x}$ is disconnected. By Theorem 4.7 there is a connected étale neighborhood $x \in U \to \mathcal{A}_{\mathcal{I},\mu}$ such that

$$\widehat{\mathcal{M}}^{\mathrm{an}}_{\mathcal{I},\mu/U} = V_1 \sqcup V_2 \tag{5.3}$$

is a union of two non-empty clopen subsets. Let $\iota\colon W \subset U$ be the open subset obtained by pulling back the union of $L^+\mathcal{I}$-orbits of $\mathcal{A}_{\mathcal{I},\mu}$ of codimension at most 1. The scheme W is connected by Corollary 3.5, as the S_2 property is stable under étale maps, otherwise $\mathcal{O}_U = \iota_*\mathcal{O}_W$ would contain non-trivial idempotents. Apply Lemma 5.1, Proposition 5.5 and [Gle24, Lemma 4.55] to conclude that $\widehat{\mathcal{M}}^{\mathrm{an}}_{\mathcal{I},\mu/W}$ is connected as well, so $\mathrm{sp}^{-1}(W)$ is entirely contained in V_1, say.

On the other hand, the disjoint union induces a direct sum decomposition $\mathcal{Z}_\mu|_U = A_1 \oplus A_2$ into non-zero perverse sheaves, where the A_i are the nearby cycles of IC_μ after pulling it back to V_i. We know by construction that $\mathcal{H}^{-\langle 2\rho,\mu\rangle} A_2$ does not vanish, and also that the support of A_2 has codimension at least 2 in U by the previous paragraph. These two facts contradict A_2 being perverse, so our initial assumption was wrong. □

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SINGULARITIES OF LOCAL MODELS

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Abstract. We construct local models of Shimura varieties and investigate their singularities, with special emphasis on wildly ramified cases. More precisely, with the exception of odd unitary groups in residue characteristic 2 we construct local models, show reducedness of their special fiber, Cohen–Macaulayness and in equicharacteristic also (pseudo-)rationality. In mixed characteristic we conjecture their pseudo-rationality.

This is based on the construction of parahoric group schemes over two dimensional bases for wildly ramified groups and an analysis of singularities of the attached Schubert varieties in positive characteristic using perfect geometry.

Contents

1. Introduction

1.1. **Background.** Let O be a complete discretely valued ring with fraction field K and with residue field k of characteristic $p > 0$, which for simplicity we assume is algebraically closed. Let G be a (connected) reductive group over K.

The local models we consider in this paper are certain flat projective O-schemes which model the singularities of integral O-models of Shimura varieties (in the case of mixed characteristic) and of G-shtukas (in the equicharacteristic case) with parahoric level structure.

Local models attached to PEL type Shimura varieties with parahoric level structure at a given prime number were developed in the book of Rapoport and Zink [RZ96], and were defined there in a linear algebra style using moduli spaces of self-dual lattice chains in certain skew-Hermitian vector spaces. The local models were proved to be étale locally isomorphic to the corresponding integral models of the Shimura varieties defined using analogous chains of polarized abelian schemes with additional structure. This has two important consequences:

(1) The singularities in the special fiber of the Shimura variety coincide with those of its local model, which can be studied more directly;
(2) The sheaf of nearby cycles on its special fiber can be determined from the corresponding object on the local model.

The approach in (1) goes back to de Jong [dJ93], who used it to determine the singularities appearing in Siegel modular 3-folds with Iwahori level structure at p (Shimura varieties attached

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to GSp(4)); it was also exploited by many other authors, see [CN90, DP94, Pap95, Fal01, Gör01, PR05, PRS13, PZ22] for example. The method in (2) is a key ingredient in the study of local Hasse–Weil zeta functions of Shimura varieties with parahoric level structure; see the survey articles of Rapoport [Rap90] and the second named author [Hai05, Hai14].

For more general Shimura varieties (as well as for moduli spaces of shtukas) a more purely group-theoretic construction of local models – also satisfying (1) and (2) above – is desirable, in parallel to Deligne's group-theoretic axiomatic construction of Shimura varieties [Del71]. Such constructions also have the benefit of tying the theory of Shimura varieties more closely to Schubert varieties, loop groups, and other objects appearing in the geometric Langlands program. This also gives hints about how to make the construction itself, with the help of Beilinson–Drinfeld affine Grassmannians.

The sought-after local models, which we denote by $\widetilde{M}_{\mathcal{G},\mu}$, arise as the seminormalizations of certain orbit closures $M_{\mathcal{G},\mu}$ inside a Beilinson–Drinfeld Grassmannian, and are associated to a parahoric group scheme $\mathcal{G}$ over O extending G, a geometric conjugacy class μ of cocharacters of G and certain auxiliary additional data in the mixed characteristic case, see Section 5.2. The schemes are constructed by Zhu in [Zhu14] and by Pappas–Zhu in [PZ13] for all G splitting over a tamely ramified extension of K. Their construction in the mixed characteristic setting is extended by Levin in [Lev16] to all groups G which are restrictions of scalars of tamely ramified groups, so covering all G (up to central isogeny) in the cases where $p \geq 5$. In the equicharacteristic setting, the construction for arbitrary groups is given in [Ric16]. (In all these cases, flatness of the local models so defined is automatic, in contrast with the lattice-theoretic proposals in [RZ96], which in certain cases failed to be flat, as first pointed out by Pappas. On the other hand, unlike [RZ96], these group-theoretic local models are not given by explicit moduli problems.)

One of the main results of [Zhu14] and [PZ13] is that when $p \nmid |\pi_1(G_{\mathrm{der}})|$ the orbit closures $M_{\mathcal{G},\mu}$ are normal (hence coincide with $\widetilde{M}_{\mathcal{G},\mu}$) with reduced special fiber, all of whose components are normal, Cohen–Macaulay and compatibly Frobenius split. They also conjecture that under the same conditions the local models are always Cohen–Macaulay [PZ13, Remark 9.5 (b)]. This is proved by He in [He13] in the case that G is unramified and μ is minuscule and by the second and fourth named author [HR22, Theorem 2.3] for $p > 2$ in all cases where local models had been constructed. In the case when $p \mid |\pi_1(G_{\mathrm{der}})|$, it is known by [HLR24], that the orbit closures $M_{\mathcal{G},\mu}$ are not normal in general, so instead one passes to their seminormalizations $\widetilde{M}_{\mathcal{G},\mu}$ which then have the aforementioned properties.

The paper at hand extends the above results to all G and all p with the exception of one family of examples: ramified odd unitary groups G in the case $p = 2$, see also Remark 2.2. More precisely, excluding this family we construct local models $\widetilde{M}_{\mathcal{G},\mu}$ also for wildly ramified groups G which are not necessarily restrictions of scalars of tamely ramified groups, and we prove that these models are normal, Cohen–Macaulay and have reduced special fibers all of whose components are also normal, Cohen–Macaulay and compatibly Frobenius split. The reader is referred to Lemma 5.23 for the relation with the construction of local models via z-extensions from [HPR20, Section 2.6]. Let us now explain our main results in more detail.

1.2. **Main results.** Fix $O \subset K$ with residue field k and G as above. Denote by Φ_G the relative root system of G. If G ranges through all absolutely simple groups, then Φ_G is non-reduced if and only if G is an odd unitary group, see Section 2. Our first main result is Theorem 5.4 in the main text and concerns local models in equicharacteristic:

Theorem 1.1. *Assume that $K \simeq k(\!(t)\!)$ has characteristic $p > 0$. Also assume that $p > 2$ or Φ_G is reduced. Then the local model $\widetilde{M}_{\mathcal{G},\mu}$ is Cohen–Macaulay, has rational singularities, and reduced special fiber equal to the admissible locus $\widetilde{A}_{\mathcal{G},\mu}$.*

For the definition of the admissible locus $\widetilde{A}_{\mathcal{G},\mu}$, the reader is referred to Definition 5.3. We also note that $\widetilde{M}_{\mathcal{G},\mu} = M_{\mathcal{G},\mu}$ when $p \nmid |\pi_1(G_{\mathrm{der}})|$, see Remark 5.5 and Remark 5.15. In Corollary 5.8, we also calculate the Picard group of $\widetilde{M}_{\mathcal{G},\mu}$.

Our second main result is Theorem 5.14 in the main text and concerns mixed characteristic local models:

Theorem 1.2. *Assume that K has characteristic 0. Also assume that $p > 2$ or Φ_G is reduced. Then the local model $\widetilde{M}_{\underline{\mathcal{G}},\mu}$ is Cohen–Macaulay and has a reduced special fiber equal to the μ'-admissible locus $\widetilde{A}_{\mathcal{G}',\mu'}$. If $\widetilde{A}_{\mathcal{G}',\mu'}$ is irreducible (for example, $\mathcal{G}$ special parahoric), then $\widetilde{M}_{\underline{\mathcal{G}},\mu}$ has pseudo-rational singularities.*

Here $\mathcal{G}'$ and μ' are equicharacteristic analogues of $\mathcal{G}$ and μ associated to them via a choice of $O[\![t]\!]$-group lift $\underline{\mathcal{G}}$, see Section 2. As above, $\widetilde{M}_{\underline{\mathcal{G}},\mu} = M_{\underline{\mathcal{G}},\mu}$ when $p \nmid |\pi_1(G_{\mathrm{der}})|$, and see Corollary 5.19 for its Picard group. Theorem 1.2 is slightly weaker than Theorem 1.1 in that we do not prove that the singularities of $\widetilde{M}_{\underline{\mathcal{G}},\mu}$ are always pseudo-rational. However, we conjecture that this is always the case, see Conjecture 5.20.

1.3. **Methods.** We now explain our methods and the structure of this paper. The main input needed to construct the local model (in mixed characteristic) is, as in [PZ13], the construction of a lifting of the parahoric group scheme $\mathcal{G}$ over O to a group scheme $\underline{\mathcal{G}}$ over $O[\![t]\!]$. The special fiber of the local model is then a closed subscheme of a partial affine flag variety over k and to analyze this we also need to construct lifts of parahoric group schemes over $k[\![t]\!]$ to $W(k)[\![t]\!]$. These steps were carried out for tame groups in [PR08, PZ13].

So we need to extend these constructions to wild groups. The group lifts are constructed in Section 2 using ideas from [Lou23]: we define suitable integral models of maximal tori and root groups separately which induce birational models and then apply the result that such a model extends to a unique group scheme. The reason that we have to exclude the case of odd unitary groups stems from this very first step since we are unable to construct the lifts of root groups in the case of multipliable roots when G is ramified and $p = 2$, see Remark 2.2.

In Section 3 we start with a review of F-singularities and (pseudo-)rational singularities. These techniques are central to the study of singularities of local models in later sections. Conjecture 3.6 states a conjectural mixed characteristic analogue of a result of Schwede and Singh [HMS14, Appendix A], which would imply pseudo-rationality of mixed characteristic local models, see also the discussion below.

The first step in analyzing the singularities of local models is the study of the singularities of Schubert varieties S_w in affine flag varieties. We carry this out in Section 4, first proving in Theorem 4.1 that the seminormalizations $\widetilde{S}_w$ are always normal, Cohen–Macaulay, compatibly Frobenius split and have rational (in fact, even F-rational) singularities. We use the by now standard method of applying the Mehta–Ramanathan criterion for Frobenius splitting, but we need some extra arguments for $p = 2, 3$. In Theorem 4.25, we then show that if $p > 2$ or Φ_G is reduced, then all Schubert varieties S_w are normal if and only if p does not divide the order of $\pi_1(G_{\mathrm{der}})$.

In Section 5, we construct our local models and prove our main results. In the equicharacteristic case, the local model is canonical. In mixed characteristic, it depends on the choice of the group lift $\underline{\mathcal{G}}$ constructed in Proposition 2.6. For minuscule μ, which is the case relevant to Shimura varieties, it is expected that these are independent of all choices, see [SW20, Conjecture 21.4.1], [HPR20, Conjectures 2.12, 2.15] and also [AGLR22]. To identify the special fiber and prove that it is reduced, we follow the method of [Zhu14] and [PZ13] based on the coherence conjecture. This is fairly straightforward, given the results of Section 4. It then remains to prove that the special fiber is Cohen–Macaulay. To do this, we use a variant of the argument used in [HR22, Section 6], which has the advantage of also being applicable in characteristic 2 since it does not depend on Zhu's global Frobenius splitting [Zhu14, Theorem 6.5]. The proof uses some results in commutative algebra by Schwede and Singh [HMS14, Appendix A] to deduce that in equicharacteristic p the local models are Cohen–Macaulay and have F-rational (hence pseudo-rational) singularities. In the case of mixed characteristic, we get the reducedness and Cohen–Macaulayness of the special fiber of the local model by comparing with the equicharacteristic case. However, it does not seem possible to immediately transfer pseudo-rationality from equal characteristic to mixed characteristic. Motivated by this we discuss the above mentioned conjectural mixed characteristic analogue (Conjecture 3.6) of one of the results of Schwede and Singh which, given our other results, would suffice to deduce the pseudo-rationality of local models in mixed characteristic.

1.4. **Relationship with the perfectoid theory.** Let us comment on the relationship between this work and the other recent works [AGLR22, GL24] by some of the authors. The first paper [AGLR22] studied at length a perfectoid analogue of the local model constructed in Scholze–Weinstein's book [SW20]. An important conjecture in [SW20] postulated that these perfectoid local models, despite only being v-sheaves, should be representable by a flat, normal, and projective scheme over O_E with reduced special fiber. This was proved in [AGLR22, Section 7] under Hypothesis 2.1 and Hypothesis 5.24, using the constructions of this paper as an input and comparing them to the v-sheaves of perfectoids via a specialization principle. However, we stress that the results in [AGLR22] concerning the singularities of local models like reducedness of their special fiber and Cohen–Macaulayness rely on the present paper. As for [GL24], it gives a new proof that local models are normal with reduced special fiber, including the missing cases of Hypothesis 2.1 and Hypothesis 5.24. The statements in [GL24] related to Frobenius splittings of the special fiber or Cohen–Macaulayness rely again on the present paper.

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2. Group lifts to two-dimensional bases

In our presentation we follow [Lou23, Sections 2–3] to construct group lifts via gluing from birational group laws. The method works for Witt lifts (equicharacteristic) and Breuil–Kisin lifts (mixed characteristic) in the same way which we, however, treat in the separate Sections 2.1 and 2.2 for readability. We start by fixing some notation.

Let O denote a complete discretely valued ring with fraction field K and perfect residue field k of characteristic $p > 0$. Let $\breve{O}/O$ be the completion of the maximal unramified extension with fraction field $\breve{K}/K$. Let G be a reductive K-group that is quasi-split (automatic if $K = \breve{K}$ by

Steinberg's theorem) and either simply connected or adjoint. Denote by $\breve{G} := G \otimes_K \breve{K}$ the base change.

Assume G is also almost K-simple. Then $G = \mathrm{Res}_{L/K}(G_0)$, for some finite separable field extension L/K, of an absolutely almost simple L-group G_0 [BT65, Section 6.21 (ii)], which is necessarily quasi-split and simply connected or adjoint, respectively. Choose a separable field extension M/L of minimal degree such that G_0 splits over its Galois hull. As the only non-trivial automorphism groups of connected Dynkin diagrams are $\mathbb{Z}/2$ and S_3, the extension M/L is of degree ≤ 3.

In this section, we also work under the following:

Hypothesis 2.1. If $p = 2$, then the relative root system $\Phi_{\breve{G}}$ is reduced.

An examination of the tables in [Tit79] shows that $\Phi_{\breve{G}}$ is non-reduced if and only if the associated absolutely almost simple group $\breve{G}_0 = G_0 \otimes_K \breve{K}$ is isomorphic to an odd unitary group. So Hypothesis 2.1 excludes this case if $p = 2$.

Fix a maximal K-split torus $S \subset G$ with centralizer equal to a maximal torus T and a Borel subgroup B containing it. Let $H/\mathbb{Z}$ be the split form of G equipped with a pinning. Choose a Chevalley–Steinberg system for H, see [Lou23, Section 2.1]. Let K^s/K be a Galois extension splitting G, and fix an isomorphism

$$G \otimes_K K^s \xrightarrow{\simeq} H \otimes_{\mathbb{Z}} K^s \tag{2.1}$$

preserving the chosen maximal tori and Borel subgroups such that the $\mathrm{Gal}(K^s/K)$-action transported to the target acts by pinned automorphisms, so $G = \mathrm{Res}_{K^s/K}(H \otimes_{\mathbb{Z}} K^s)^{\mathrm{Gal}(K^s/K)}$ by Galois descent.

The Chevalley–Steinberg system for H induces a Chevalley quasi-system for the quasi-split group G in the sense of [Lou23, Définition 2.2.6, Proposition 2.2.7]. Essentially, this is the choice of the pair $S \subset B$ in G along with a family of isomorphisms

$$x_a \colon U_a \xrightarrow{\simeq} \begin{cases} \mathrm{Res}_{L_a/K}\mathbb{G}_a \\ \mathrm{Res}_{L_{2a}/K}\mathbb{H}_{L_a/L_{2a}} \end{cases} \tag{2.2}$$

for all $a \in \Phi_G^{\mathrm{nd}}$ with $\Phi_G^{\mathrm{nd}} \subset \Phi_G$ the subset of non-divisible roots and U_a the corresponding root subgroup. Here, if Φ_G is reduced, then $L_a = M$ if $a \in \Phi_G^{<}$ is short and $L_a = L$ if $a \in \Phi_G^{>}$ is long. If Φ_G is non-reduced, then $L_a = M \supset L = L_{2a}$ if $2a \in \Phi_G$ and $\mathbb{H}_{L_a/L_{2a}}$ is the L_{2a}-group described in [BT84, 4.1.9]. Here, the quadratic extension L_a/L_{2a} is allowed to be ramified if $p > 2$ but must be unramified if $p = 2$ by Hypothesis 2.1. This induces a Chevalley valuation of $\mathscr{A}(G, S, K)$, see [BT84, 4.2.2], which we then regard as the origin of that affine space, which then becomes identified with $\mathscr{V}(S) = X_*(S) \otimes \mathbb{R}$.

Remark 2.2. Let us comment on the various hypotheses on G.

(1) If we wished to include the case where $p = 2$ and Φ_G is non-reduced, the structure of U_a would be arithmetically more involved, particularly as the subset $M^0 \subset M$ of trace zero elements does not behave so well, see [BT84, Sections 4.1.10, 4.2.20]. For instance, the valuation of M^0 divides the quadratic separable extensions into those given by roots of primes and the rest of them, see [BT84, Lemmes 4.3.3, 4.3.4]. Root-of-prime extensions are treated in [Lou23] relying on the theory of pseudo-reductive groups. For the other quadratic extensions, we do not know, for example, how to construct the groups $\underline{U_a}$ that appear below.

(2) The case of quasi-split and simply connected (respectively, adjoint) groups G appears to be most important when studying the geometry of Schubert varieties and local models. Note that for such G the maximal torus T is induced [BT84, Proposition 4.4.16], which is

a technical convenience, see the proof of Lemma 2.5. If we wished to include more general central extensions of G with induced maximal torus, we could follow the construction in [Lou23, Section 2.4], see also Section 5.3.1 for a particular interesting case. Further, it should be possible, though difficult, to extend the construction of group lifts below to not necessarily quasi-split groups using étale descent [BT84, Section 5].

2.1. **Witt lifts.** In this subsection, we assume that K is a Laurent series field of characteristic $p > 0$. Choosing uniformizers u of L and t of K, we identify their rings of integers $O_L = k_L[\![u]\!]$ and $O = k[\![t]\!]$ as k-algebras. The uniformizers satisfy an Eisenstein equation:

$$u^e + a_{e-1}(t)u^{e-1} + \cdots + a_1(t)u + a_0(t) = 0 \tag{2.3}$$

where each of the

$$a_i(t) = \sum b_{ij}t^j \tag{2.4}$$

is a power series with $b_{ij} \in k_L$, $b_{i0} = 0$ and $b_{01} \neq 0$. Consider now the defining equation

$$u^e + [a_{e-1}(t)]u^{e-1} + \cdots + [a_1(t)]u + [a_0(t)] = 0 \tag{2.5}$$

where each of the

$$[a_i(t)] = \sum [b_{ij}]t^j \tag{2.6}$$

is a power series in $W(k_L)[\![t]\!]$ obtained by taking Teichmüller representatives of the coefficients. Then (2.5) defines the finite free $W(k)[\![t]\!]$-algebra $W(k_L)[\![u]\!]$, which reduces modulo p to the $k[\![t]\!]$-algebra $k_L[\![u]\!]$. Similarly, we lift O_M/O_L to $W(k_M)[\![v]\!]/W(k_L)[\![u]\!]$ via a choice of uniformizers.

Lemma 2.3. *The quasi-pinned K-group $(G, B, S, (x_a)_{a\in\Phi_G^{\mathrm{nd}}})$ lifts to $(\underline{G}, \underline{B}, \underline{S}, (\underline{x_a})_{a\in\Phi_G^{\mathrm{nd}}})$ defined over the maximal open subset $U \subset \operatorname{Spec} W(k)[\![t]\!]$ over which the extension $W(k_M)[\![v]\!]/W(k)[\![t]\!]$ is étale.*

Proof. Firstly, the split form $H/\mathbb{Z}$ of G with its Chevalley–Steinberg system induces a split form $H_0/\mathbb{Z}$ of G_0 with such a system. As quasi-pinnings are compatible with restriction of scalars along finite étale maps, we reduce to the case $G = G_0$ is absolutely almost simple and without loss of generality also non-split. Let $\widetilde{V}$ be the Galois hull of the finite étale map $V := f^{-1}(U) \to U$ where $f\colon \operatorname{Spec} W(k_M)[\![v]\!] \to \operatorname{Spec} W(k)[\![t]\!]$. As f is ramified at $\{t = 0\}$, we have $U \subset \operatorname{Spec} W(k)(\!(t)\!)$ and the reduction of $\widetilde{V} \to U$ modulo p defines a Galois ring extension $\widetilde{K}/K$ splitting G. Hence, $\operatorname{Gal}(\widetilde{V}/U) \to \operatorname{Gal}(\widetilde{K}/K)$ acts through (2.1) by pinning preserving automorphisms on H, replacing K^s by $\widetilde{K}$ if necessary. We define

$$\underline{G} = \operatorname{Res}_{\widetilde{V}/U}(H \otimes_{\mathbb{Z}} \widetilde{V})^{\operatorname{Gal}(\widetilde{V}/U)}, \tag{2.7}$$

equipped with the quasi-pinning induced from the chosen Chevalley–Steinberg system for H, which satisfies the requirements of the lemma. $\square$

Note that $W(k)(\!(t)\!)$ is a Euclidean domain which is not local. Even though the extension $W(k_M)(\!(v)\!)/W(k)(\!(t)\!)$ is ramified in general, we can extend $\underline{G}$ from U over $\operatorname{Spec} W(k)(\!(t)\!)$ via a birational extension process as follows. Note that we have the maximal torus $\underline{T}$ in $\underline{G}$ defined over U. We consider the family of group schemes consisting of the connected Néron $W(k)(\!(t)\!)$-model of $\underline{T}$ denoted by the same symbol, and the unipotent group schemes

$$\underline{U_a} = \begin{cases} \operatorname{Res}_{W(k_a)(\!(t_a)\!)/W(k)(\!(t)\!)}\mathbb{G}_a \\ \operatorname{Res}_{W(k_{2a})(\!(t_{2a})\!)/W(k)(\!(t)\!)}\mathbb{H}_{W(k_a)(\!(t_a)\!)/W(k_{2a})(\!(t_{2a})\!)} \end{cases} \tag{2.8}$$

for every non-divisible root $a \in \Phi_G$, extending the quasi-pinning defined in Lemma 2.3. Here, the symbols k_a denote the residue field of the root fields L_a, and the variables t_a are either one of the prescribed lifts u or v of the uniformizer of L_a, depending on whether it equals L or M.

Lemma 2.4. *The models $(\underline{T}, \underline{U_a})$ glue birationally to a smooth, affine $W(k)(\!(t)\!)$-group $\underline{G}$ with connected fibers extending* (2.7).

Proof. This follows from the method of [Lou23, Proposition 3.3.4]. Here we give an overview of the argument.

First, we must show that the axioms of [BT84, Définition 3.1.1] are satisfied: These involve showing that the conjugation action of $\underline{T}$ on the $\underline{U_a}$, the commutator morphisms between $\underline{U_a}$ and $\underline{U_b}$ for linearly independent roots, and a rationally defined morphism exchanging the order of $\pm a$ in a rank 1 big cell extend from U (defined in Lemma 2.3) to all of $\operatorname{Spec} W(k)(\!(t)\!)$. In the rank 1 case, we can construct $\underline{G}$ explicitly by extending the definition of $\underline{G}$ over U, isogenous to a restriction of scalars of either SL_2 or SU_3, to the more general ring extensions that we consider; this provides us with the first and third morphisms using the Néron property of $\underline{T}$. Hence, the main concern are commutator morphisms. Over the generic fiber, these morphisms are given explicitly in [BT84, Section A.6], up to sign and conjugation, and only involve natural operations such as sum, multiplication, trace and norm, so they are still well-defined over $W(k)(\!(t)\!)$. For example, if Φ_G is reduced, and a, b are short roots with long sum $c = a + b \in \Phi_G$, then the commutator $\gamma_{a,b}$ is given on points under the fixed pinnings by

$$(x, y) \to \operatorname{tr}_{R[t_a]/R}(xy), \tag{2.9}$$

where R is any $W(k_c)(\!(t_c)\!)$-algebra, and $x, y \in R[t_a] = R \otimes_{W(k_c)(\!(t_c)\!)} W(k_a)(\!(t_a)\!)$, up to ignoring sign and conjugation. It is now a consequence of [Lou23, Théorème 3.2.5] that there is a smooth affine $W(k)(\!(t)\!)$-group $\underline{G}$ with connected fibers glued from these closed subgroups. Here, for affineness we use the fact that $W(k)(\!(t)\!)$ is a Dedekind ring. $\square$

We already know that $\underline{G}$ is reductive over $k(\!(t)\!)$ and $K_0(\!(t)\!)$, where $K_0 = W(k)[p^{-1}]$. We can compare a portion of their Bruhat–Tits theory.

Lemma 2.5. *There are identifications*

$$\mathscr{A}(\underline{G}, \underline{S}, k(\!(t)\!)) \simeq \mathscr{A}(\underline{G}, \underline{S}, K_0(\!(t)\!)), \tag{2.10}$$

of apartments, equivariant along a natural identification of the Iwahori–Weyl groups.

Proof. Our method of proof is similar to [Lou23, Proposition 3.4.1]. We fix as origin of the apartments the Chevalley–Steinberg valuations determined by the quasi-pinning inherited from (2.8). Then, both identify with the real vector space $\mathscr{V}(\underline{S})$ generated by the coweights of the split torus $\underline{S}$ compatibly with the hyperplanes.

As G is assumed to be either simply connected or adjoint, the maximal torus T is induced and so is $\underline{T}$ over U. We denote by $\underline{\mathcal{T}}$ its connected Néron $W(k)[\![t]\!]$-model, see [Lou23, Définition 3.3.3] and [Lou20, Part IV, Proposition 3.8]. Let $\underline{N}$ be the normalizer of $\underline{S}$ in $\underline{G}$. In order to identify the Iwahori–Weyl groups, we prove that they are isomorphic to

$$\underline{N}(W(k)(\!(t)\!))/\underline{\mathcal{T}}(W(k)[\![t]\!]) \tag{2.11}$$

via the natural maps as follows. It suffices to show that the natural maps

$$\underline{T}(W(k)(\!(t)\!))/\underline{\mathcal{T}}(W(k)[\![t]\!]) \to \underline{T}(L(\!(t)\!))/\underline{\mathcal{T}}(L[\![t]\!]) \tag{2.12}$$

and

$$\underline{N}(W(k)(\!(t)\!))/\underline{T}(W(k)(\!(t)\!)) \to \underline{N}(L(\!(t)\!))/\underline{T}(L(\!(t)\!)), \tag{2.13}$$

are isomorphisms, where L equals either k or K_0. The first case (2.12) is verified by decomposing $\underline{T}$ as a product of restriction of scalars of multiplicative group schemes. The second case (2.13) is a consequence of constancy of the Weyl group of a split torus and vanishing of H^1 for $\underline{T}$. One sees readily that these comparison isomorphisms are compatible with those of the apartments and the corresponding group actions. $\square$

For any point x in the apartments, we have a certain optimal quasi-concave function $f_x \colon \Phi_G \to \mathbb{R}$ in the sense of [BT84, Section 4.5], defined with respect to the chosen origin, the Chevalley–Steinberg valuation. We use this to define the $W(k)[\![t]\!]$-models $\underline{\mathcal{U}_{a,x}}$ via

$$\underline{\mathcal{U}_{a,x}} = \begin{cases} \mathrm{Res}_{W(k_a)[\![t_a]\!]/W(k)[\![t]\!]}\big(t_a^{e_a f_x(a)}\mathbb{G}_a\big) \\ \mathrm{Res}_{W(k_{2a})[\![t_{2a}]\!]/W(k)[\![t]\!]}\big(t_a^{(e_a f_x(a), e_a f_x(2a))}\mathbb{H}_{W(k_a)[\![t_a]\!]/W(k_{2a})[\![t_{2a}]\!]}\big) \end{cases}, \tag{2.14}$$

where the e_a are the ramification degrees of the root field extension L_a/K, and by construction the $e_a f_x(a)$ are integers.

Proposition 2.6. *The models $\underline{\mathcal{T}}$ and $\underline{\mathcal{U}_{a,x}}$ for all $a \in \Phi_G^{\mathrm{nd}}$ birationally glue to a smooth, affine $W(k)[\![t]\!]$-group scheme $\underline{\mathcal{G}_x}$ with connected fibers. Its reductions to $k[\![t]\!]$ and $K_0[\![t]\!]$ are parahoric group schemes coming from facets which correspond under* (2.10).

Proof. To see that the models $\underline{\mathcal{T}}$ and $\underline{\mathcal{U}_{a,x}}$ for all $a \in \Phi_G^{\mathrm{nd}}$ satisfy the axioms of [BT84, Définition 3.1.1], we can proceed as in [Lou23, Proposition 3.4.5]: due to the equality $W(k)(\!(t)\!) \cap K_0[\![t]\!] = W(k)[\![t]\!]$, it suffices to apply [BT84, Théorème 3.8.1] to prove the existence of a birational group law. So it glues to a smooth and separated group scheme $\underline{\mathcal{G}_x}$ with connected fibers due to [Lou23, Théorème 3.2.5].

This group scheme is quasi-affine and admits a smooth affine hull, whose geometric fibers are connected outside the unique closed point of $\mathrm{Spec}(W(k)[\![t]\!])$, by [Lou23, Proposition 3.2.7]. In order to check affineness, we apply verbatim the proof in [Lou23, Théorème 3.4.10]: indeed, this relies on the identification of the Iwahori–Weyl groups given in Lemma 2.5. □

2.2. **Breuil–Kisin lifts.** In this subsection, we assume that K has characteristic zero. So L/K is a finite extension of complete discretely valued fields of characteristic zero with perfect residue fields k_L/k of characteristic $p > 0$. Define also L^{nr}/K as the maximal unramified subextension of L/K, so L/L^{nr} is totally ramified. Choosing uniformizers π_L of L and π_K of K, they satisfy an Eisenstein equation:

$$\pi_L^e + a_{e-1}(\pi_K)\pi_L^{e-1} + \cdots + a_0(\pi_K) = 0, \tag{2.15}$$

where each of the $a_i(\pi_K) \in \pi_K O_{L^{\mathrm{nr}}}$ is a power series in π_K with coefficients being Teichmüller representatives of elements in k_L and satisfying the usual constraints, compare with (2.3). Assume without loss of generality that there exists i with $(i,p) = 1$ and

$$a_i(\pi_K) \neq 0. \tag{2.16}$$

This can be achieved by replacing π_L by $\pi_L + \pi_K$, if needed. Consider, in analogy to (2.5), the equation

$$u^e + [a_{e-1}(t)]u^{e-1} \cdots + [a_0(t)] = 0, \tag{2.17}$$

where u and t are indeterminates, each of the $[a_i(t)] \in W(k_L)[\![t]\!]$ is obtained from $a_i(\pi_K)$ by taking the coefficients and by replacing π_K by t. Equation (2.17) defines the finite free $W(k)[\![t]\!]$-algebra $W(k_L)[\![u]\!]$. We repeat this procedure for M/L: choose a uniformizer π_M of M satisfying the analogue of (2.16) with respect to π_L, an indeterminate v and define the finite free $W(k_L)[\![u]\!]$-algebra $W(k_M)[\![v]\!]$. Tensoring with O over $W(k)$, we arrive at the finite free ring extensions

$$O[\![t]\!] \subset O_{L^{\mathrm{nr}}}[\![u]\!] \subset O_{M^{\mathrm{nr}}}[\![v]\!], \tag{2.18}$$

where L^{nr} and M^{nr} are the maximal unramified subextensions of L/K and M/K, respectively. The tower (2.18) reduces modulo $t - \pi_K$ to $O \subset O_L \subset O_M$; its reduction modulo π_K is $k[\![t]\!] \subset k_L[\![u]\!] \subset k_M[\![v]\!]$ with separable fraction field extensions by (2.16).

As for the group G endowed with its quasi-pinning $(B, S, (x_a)_{a \in \Phi_G^{\mathrm{nd}}})$, these data also lift to an open neighborhood $U \subset \mathrm{Spec}\, O[\![t]\!]$ of the points (π_K) and $(t - \pi_K)$ in analogy to Lemma 2.3, and we denote the resulting U-groups by $\underline{G}$, $\underline{B}$, $\underline{T}$ and $\underline{S}$ as before. To extend $\underline{G}$ from U over

$\operatorname{Spec} O(\!(t)\!)$, we proceed again via a gluing procedure using extensions of birational group laws. Consider the family of group schemes consisting of the connected Néron $O(\!(t)\!)$-model $\underline{T}$ (note that $O(\!(t)\!)$ is a Dedekind domain), and the unipotent group schemes

$$\underline{U_a} = \begin{cases} \operatorname{Res}_{O_{L_a^{\mathrm{nr}}}(\!(t_a)\!)/O(\!(t)\!)}\mathbb{G}_a \\ \operatorname{Res}_{O_{L_{2a}^{\mathrm{nr}}}(\!(t_{2a})\!)/O(\!(t)\!)}\mathbb{H}_{O_{L_a^{\mathrm{nr}}}(\!(t_a)\!)/O_{L_{2a}^{\mathrm{nr}}}(\!(t_{2a})\!)} \end{cases} \tag{2.19}$$

for every root $a \in \Phi_G^{\mathrm{nd}}$, extending the generic quasi-pinning. Here, the variables t_a are either t, u or v depending on the cases for the root fields L_a for $a \in \Phi_G$ explicated in (2.2). We arrive at the following result:

Lemma 2.7. *The models* $(\underline{T}, (\underline{U_a})_{a\in\Phi_G^{\mathrm{nd}}})$ *birationally glue to a smooth, affine* $O(\!(t)\!)$*-group* $\underline{G}$ *with connected fibers. Furthermore, its fibers over* K *and* $\kappa(\!(t)\!)$ *for* $\kappa = k, K$ *are reductive, and there are identifications of apartments*

$$\mathscr{A}(G, S, K) \simeq \mathscr{A}(\underline{G}, \underline{S}, \kappa(\!(t)\!)), \tag{2.20}$$

equivariantly for the respective Iwahori–Weyl groups.

Proof. The proofs of Lemma 2.4 and Lemma 2.5 translate literally. □

For any point x in the apartments (2.20), we have the quasi-concave function $f_x \colon \Phi_G \to \mathbb{R}$, compare with (2.14). We define the $O[\![t]\!]$-models $\underline{\mathcal{U}_{a,x}}$ by

$$\underline{\mathcal{U}_{a,x}} = \begin{cases} \operatorname{Res}_{O_{L_a^{\mathrm{nr}}}[\![t_a]\!]/O[\![t]\!]}\big(t_a^{e_a f_x(a)}\mathbb{G}_a\big) \\ \operatorname{Res}_{O_{L_{2a}^{\mathrm{nr}}}[\![t_{2a}]\!]/O[\![t]\!]}\big(t_a^{(e_a f_x(a), e_a f_x(2a))}\mathbb{H}_{O_{L_a^{\mathrm{nr}}}[\![t_a]\!]/O_{L_{2a}^{\mathrm{nr}}}[\![t_{2a}]\!]}\big) \end{cases}, \tag{2.21}$$

where the e_a are the ramification degrees of the extensions L_a/K and by construction the $e_a f_x(a)$ are integers. Let $\underline{\mathcal{T}}$ be the connected Néron $O[\![t]\!]$-model of the induced torus $\underline{T_U}$, compare with the proof of Lemma 2.5.

Proposition 2.8. *The models* $\underline{\mathcal{T}}$ *and* $\underline{\mathcal{U}_{a,x}}$ *for all* $a \in \Phi_G^{\mathrm{nd}}$ *birationally glue to a smooth, affine* $O[\![t]\!]$*-group scheme* $\underline{\mathcal{G}_x}$ *with connected fibers. Its reductions to* O *and* $\kappa[\![t]\!]$*, with* $\kappa = k, K$ *are parahoric group schemes coming from facets which correspond under* (2.20)*.*

Proof. The proof of Proposition 2.6 applies verbatim. □

3. A conjecture on pseudo-rationality

We recall some definitions and facts from the theory of singularities, especially in positive characteristic. Conjecture 3.6 below is a mixed characteristic analogue of a result of Schwede–Singh recalled in Lemma 3.2. Its proof would imply that mixed characteristic local models also have pseudo-rational singularities, see Conjecture 5.20.

3.1. **Review of F-singularities.** A Noetherian scheme X over $\mathbb{F}_p$ is said to be F-*finite* if the absolute Frobenius morphism $F \colon X \to X$ is a finite morphism (for example, finite type schemes over F-finite fields). It is said to be F-*split* if the canonical morphism $\mathcal{O}_X \to F_*\mathcal{O}_X$ has an $\mathcal{O}_X$-linear splitting. We say X is *stably* F-*split* if for some $e > 0$ the map $\mathcal{O}_X \to F_*^e\mathcal{O}_X$ splits, and the two notions are equivalent by [BS13, Lemma 5.0.3]. Moreover, a closed subscheme $Y \subset X$ is *compatibly (stably)* F-*split* if the corresponding splittings respect the closed immersion, and again the stable notion is an equivalent one by [BS13, Lemma 6.0.4]. A local $\mathbb{F}_p$-algebra $(R, \mathfrak{m})$ is said to be F-*injective* if the map on local cohomology $F_* \colon H^\bullet_{\mathfrak{m}}(R) \to H^\bullet_{\mathfrak{m}}(R)$ is injective (for example, local rings of F-split schemes).

A Noetherian reduced F-finite $\mathbb{F}_p$-algebra R is said to be F-*regular* if every prime ideal localization $R_{\mathfrak{p}}$ has all its ideals tightly closed, see [HH89, Section 1]. If every parameter ideal of such an $R_\mathfrak{p}$ is tightly closed, we say $R_\mathfrak{p}$ is *F-rational*; see [HH94, Definition 4.1], and also [FW89] or [Smi97]. We say a Noetherian reduced F-finite $\mathbb{F}_p$-scheme has *F-rational singularities* if all of its local rings are F-rational.

A projective scheme X over an F-finite field is said to be *globally F-regular* provided that for every ample invertible sheaf $\mathcal{L}$, the section ring $\bigoplus_{n\in\mathbb{Z}_{\geq 0}} H^0(X,\mathcal{L}^{\otimes n})$ is a *strongly F-regular* ring, in the sense of [HH89, Section 3] (see also [Cas22, Definition 5.2]). By [HH89, Theorem 3.1(d)], any strongly F-regular ring is F-regular (the converse is expected but appears to be an open question in general). A key property of strong F-regularity is that it passes to all prime localizations of the ring.

We shall use the following results, extracted from [HH89, Smi00, HMS14].

Lemma 3.1. *A globally F-regular projective variety* $\mathrm{Proj}(S)$ *over a perfect field is F-rational.*

Proof. By [Smi00, Theorem 3.10], S is strongly F-regular. By [HH89, Theorem 3.1], all localizations of S and all direct summands of such are strongly F-regular. This means the local rings of $\mathrm{Proj}(S)$ are strongly F-regular. Now by [HH89, Theorem 3.1(d)], they are also F-regular, which means that all ideals are tightly closed. In particular these local rings are F-rational. □

Lemma 3.2. *Let R be an F-finite Noetherian local ring and t a non-zero divisor. If $R/(t)$ is F-injective and $R[t^{-1}]$ is F-rational, then R is F-rational.*

Proof. This is Schwede–Singh [HMS14, Corollary A.4]. □

3.2. **(Pseudo-)rational singularities.** We follow [LT81, Section 2] (see also [Smi97, Definition 1.8]). An excellent (thus Noetherian) local ring $(R,\mathfrak{m})$ is said to be *pseudo-rational* if it is normal, Cohen–Macaulay, admits a dualizing complex, and if for each proper birational morphism $\pi\colon Y\to \mathrm{Spec}(R)$ with Y normal, the canonical map

$$f_*\omega_Y \to \omega_R \tag{3.1}$$

is an isomorphism (or equivalently, is surjective on global sections [LT81, Section 4], or equivalently by duality theory $H^d_\mathfrak{m}(R)\to \mathbb{H}^d_\mathfrak{m}(Rf_*\mathcal{O}_Y) = H^d_{f^{-1}(\mathfrak{m})}(\mathcal{O}_Y)$ is injective for $d=\dim(R)$). An excellent scheme has *pseudo-rational singularities* (or is *pseudo-rational*) if each of its local rings is pseudo-rational.

Remark 3.3. In order to establish pseudo-rationality, one may restrict to the class of *projective* birational morphisms $\pi\colon Y\to\mathrm{Spec}(R)$ with Y normal, by an application of Chow's lemma. Further, we note that the definition of pseudo-rationality in [MS21, Definition 2.6] is weaker in that R is not required to be excellent or normal.

Lemma 3.4. *Any excellent local $\mathbb{F}_p$-algebra R which is F-rational is also pseudo-rational.*

Proof. This is [Smi97, Theorem 3.1]. □

The next lemma is used in Theorem 5.14 to establish pseudo-rationality of special local models:

Lemma 3.5. *Let R be a local ring of mixed characteristic $(0,p)$ which is excellent, normal and admits a dualizing complex. Let $\pi\in\mathfrak{m}$ be a non-zero divisor such that $R/(\pi)$ is an $\mathbb{F}_p$-algebra. If $R/(\pi)$ is F-rational, then R is pseudo-rational.*

Proof. Since R is assumed to be normal, this is [MS21, Theorem 3.8]. □

The following conjecture is a mixed characteristic analogue of Lemma 3.2. Since it does not appear in the literature (but see the discussion at `https://mathoverflow.net/q/396462`), we write it down here:

Conjecture 3.6. *In the situation of Lemma 3.5, if $R/(\pi)$ is F-finite and F-injective, and $R[\pi^{-1}]$ is pseudo-rational, then R is pseudo-rational.*

We conclude this section by recalling a stronger notion of rationality over a perfect field k. Let X be a finite type k-scheme. We say X has *rational singularities* if it is Cohen-Macaulay and there exists a proper birational morphism $f\colon Y \to X$ of k-schemes with Y smooth over k (in which case we say X has a *resolution of singularities*) such that the natural map $\mathcal{O}_X \to Rf_*\mathcal{O}_Y$ is an isomorphism. It follows using Grothendieck–Serre duality that if X has rational singularities then $R^i f_*\omega_Y = 0$ for all $i > 0$. Moreover, it also follows that X has pseudo-rational singularities by [Kov17, Lemma 9.3], but we do not use this fact anywhere in the present paper. We note that this notion of rational singularities is independent of the choice of resolution by [CR11, Theorem 1].

4. Schubert varieties

Let k be an algebraically closed field of characteristic $p > 0$, $K = k(\!(t)\!)$ be the corresponding Laurent series field and $O = k[\![t]\!]$ the power series ring.

Let G be a reductive K-group. For each facet $\mathbf{f} \subset \mathscr{B}(G, K)$ of the Bruhat–Tits building, we denote by $\mathcal{G} = \mathcal{G}_{\mathbf{f}}$ the associated parahoric O-group scheme extending G, see [BT84, Définition 5.2.6 ff.].

The loop group LG, respectively positive loop group $L^+\mathcal{G}$, is the functor on the category of k-algebras R given by $LG(R) = G(R(\!(t)\!))$, respectively $L^+\mathcal{G}(R) = \mathcal{G}(R[\![t]\!])$. Then $L^+\mathcal{G} \subset LG$ is a subgroup functor, and the (*twisted partial*) *affine flag variety* is the étale quotient

$$\mathcal{F}\ell_{\mathcal{G}} = LG/L^+\mathcal{G}, \tag{4.1}$$

which is represented by an ind-projective k-ind-scheme by [PR08, Theorem 1.4].

In the following, we fix two facets $\mathbf{f}, \mathbf{f}' \subset \mathscr{B}(G, K)$ and denote by $\mathcal{G} = \mathcal{G}_{\mathbf{f}}$, $\mathcal{G}' = \mathcal{G}_{\mathbf{f}'}$ the associated parahorics. Given an element $w \in L^+\mathcal{G}'(k)\backslash LG(k)/L^+\mathcal{G}(k)$, the *Schubert variety* S_w is the reduced $L^+\mathcal{G}'$-orbit closure of $\tilde{w} \cdot e$ in $\mathcal{F}\ell_{\mathcal{G}}$, where $\tilde{w} \in LG(k)$ is any representative of w and e the base point of $\mathcal{F}\ell_{\mathcal{G}}$, see [PR08, Definition 8.3] and compare with [HR22, Section 3]. Then S_w is a projective k-variety admitting the $L^+\mathcal{G}'$-orbit C_w of $\dot{w} \cdot e$ as a dense open subset. This induces a presentation on reduced ind-schemes

$$(\mathcal{F}\ell_{\mathcal{G}})_{\mathrm{red}} = \operatorname{colim} S_w, \tag{4.2}$$

where w runs through the double cosets as above, and all transition maps $S_v \to S_w$ are closed immersions.

4.1. **F-singularities of seminormalized Schubert varieties.** Let $\widetilde{S}_w \to S_w$ be the seminormalization [Sta23, 0EUK], that is, the initial scheme mapping universally homeomorphically to S_w with the same residue fields. In this subsection we show the following result for general reductive K-groups:

Theorem 4.1. *The seminormalized Schubert varieties $\widetilde{S}_w$ are normal, Cohen–Macaulay, compatibly F-split and have rational singularities. Furthermore, the $\widetilde{S}_w$ are globally F-regular, hence have F-rational singularities.*

Here *compatibly* F-split carries the following meaning. By functoriality of seminormalizations [Sta23, 0EUS], there are maps $\widetilde{S}_v \to \widetilde{S}_w$ lifting the closed immersions $S_v \to S_w$ from (4.2), yielding the (a priori non-strict) ind-scheme

$$\widetilde{\mathcal{F}\ell}_{\mathcal{G}} = \operatorname{colim} \widetilde{S}_w. \tag{4.3}$$

In the course of the proof of Theorem 4.1, we show that $\widetilde{S}_v \to \widetilde{S}_w$ are closed immersions (see Lemma 4.5) and that $\widetilde{S}_w$ is F-split compatibly with all closed subvarieties $\widetilde{S}_v$.

Remark 4.2. The methods from [Fal03, Theorem 8], [PR08, Theorem 8.4] and [Cas22, Theorem 1.4] essentially imply Theorem 4.1 for all groups whose adjoint simple factors are Weil restrictions of scalars of tamely ramified groups. Theorem 4.1 is new whenever one of the absolutely simple factors is wildly ramified, therefore covering general reductive K-groups.

Remark 4.3. There exist surfaces which have rational, but not F-rational, singularities [HW96, Example 2.11]. Further, we note that by the proof of Lemma 3.1, we know something slightly stronger than F-rationality, namely, the local rings of $\widetilde{S}_w$ are F-regular.

4.1.1. *Preliminary reductions for the proof of Theorem 4.1.* Recall the notation $\mathcal{G} = \mathcal{G}_{\mathbf{f}}$, $\mathcal{G}' = \mathcal{G}_{\mathbf{f}'}$. Let S be a maximal K-split torus with $\mathbf{f}, \mathbf{f}' \subset \mathscr{A}(G, S, K)$, see [BT72, Theorem 7.4.18 (i)]. Fix an alcove $\mathbf{a}$ in the apartment containing $\mathbf{f}$ in its closure, and denote by $\mathcal{I} = \mathcal{G}_{\mathbf{a}}$ the associated Iwahori O-group scheme. The affine Weyl group W_{af} (respectively, its subgroup $W_{\mathcal{G}}$) is the Coxeter group generated by the simple reflections along the hyperplanes meeting the closure of $\mathbf{a}$ (respectively, passing through $\mathbf{f}$). There is a natural bijection $W_{\mathrm{af}}/W_{\mathcal{G}} \cong L^+\mathcal{I}(k)\backslash LG^0(k)/L^+\mathcal{G}(k)$ where LG^0 denotes the neutral component. In order to prove Theorem 4.1, we may and do assume without loss of generality that $\mathcal{G}' = \mathcal{I}$ and $w \in W_{\mathrm{af}}/W_{\mathcal{G}}$, as every Schubert variety is isomorphic to one of this particular form by [HR22, Section 3.1, Corollary 3.2].

In the following we identify the Bruhat order on the coset space $W_{\mathrm{af}}/W_{\mathcal{G}}$ compatibly with the Bruhat order on the subset of right $W_{\mathcal{G}}$-minimal representatives in W_{af}, see [Ric13, Lemma 1.6]. Suppose $w \in W_{\mathrm{af}}$ is right $W_{\mathcal{G}}$-minimal. Fix a reduced decomposition as a product of simple reflections $\dot{w} = s_1 \cdot \ldots \cdot s_d$ in W_{af}. Denote by $D_{\dot{w}}$ the Demazure variety for $\dot{w}$, denoted $D(\tilde{w})$ in [PR08, Proposition 8.8]. By [HR22, Section 3.3], there is a projective morphism

$$D_{\dot{w}} \to S_w, \tag{4.4}$$

which is an isomorphism over the open Schubert cell C_w, hence birational and surjective. For any $v \leq w$ in the Bruhat order, the reduced decomposition $\dot{w}$ induces a (not necessarily unique) reduced decomposition $\dot{v}$ of v, so there exists a closed immersion $D_{\dot{v}} \to D_{\dot{w}}$ covering $S_v \to S_w$. The following lemma makes the connection to the normalized Schubert varieties appearing in [HLR24, HR22]:

Lemma 4.4. *The seminormalized Schubert varieties $\widetilde{S}_w$ are normal.*

Proof. The normalization morphism $S_w^{\mathrm{nor}} \to S_w$ is a universal homeomorphism [HR22, Proposition 3.1 i)], which induces an isomorphism over C_w (because it is regular). By the universal property of seminormalizations, it remains to show that $S_w^{\mathrm{nor}} \to S_w$ induces an isomorphism on *all* residue fields. We observe that there are transition maps $S_v^{\mathrm{nor}} \to S_w^{\mathrm{nor}}$ lifting the closed immersions $S_v \to S_w$, see the proof of [PR08, Proposition 9.7 (b)] using the functoriality of the Demazure resolution (4.4). Now, given a point $x \in S_w$ lying in some cell C_v, it induces a tower of residue field extensions $\kappa(S_v^{\mathrm{nor}}, x) \supset \kappa(S_w^{\mathrm{nor}}, x) \supset \kappa(S_w, x) = \kappa(C_v, x)$. As $S_v^{\mathrm{nor}} \to S_v$ is an isomorphism over C_v, all inclusions are equalities which implies the lemma. □

Lemma 4.4 implies that (4.4) factors through $\widetilde{S}_w \to S_w$ inducing the birational projective morphism

$$f\colon D_{\dot{w}} \to \widetilde{S}_w, \tag{4.5}$$

with the property $f_*\mathcal{O}_{D_{\dot{w}}} = \mathcal{O}_{\widetilde{S}_w}$, compare [HR22, Section 3.3]. The proof of the next lemma follows the arguments from [LRPT06, Cas22] and reduces Theorem 4.1 to the corresponding result for Demazure varieties. The latter case is proved in Section 4.1.4, see (4.26) for details.

Lemma 4.5. *Assume that the Demazure variety $D_{\dot{w}}$ is compatibly stably F-split with the divisors $D_{\dot{v}}$ for all subwords $\dot{v}$ of $\dot{w}$ of colength 1. Then Theorem 4.1 holds true.*

Proof. Recall that compatibly stably F-split varieties are compatibly F-split by [BS13, Lemmas 5.0.3, 6.0.4]. Now, if $D_{\dot w}$ is compatibly F-split with the divisors $D_{\dot v}$ for $\dot v$ of colength 1, then $D_{\dot w}$ is compatibly F-split both with their union $\partial D_{\dot w}$ and $D_{\dot v}$ for all subwords $\dot v$ of $\dot w$ by [BK05, Proposition 1.2.1].

Compatibility with $\partial D_{\dot w}$ implies that $D_{\dot w}$ is globally F-regular (following the second part of the argument in [Cas22, Proposition 5.8] which applies verbatim). Recall that we have an identity $f_*\mathcal{O}_{D_{\dot w}} = \mathcal{O}_{\widetilde{S}_w}$ where f denotes the map (4.5), compatibly with the Frobenius, which allows us to descend any F-splitting along the proper cover f. More generally, we can apply [LRPT06, Lemma 1.2] to f and deduce $\widetilde{S}_w$ is globally F-regular. Lemma 3.1 implies that $\widetilde{S}_w$ has F-rational singularities. Then by Lemma 3.4, $\widetilde{S}_w$ is pseudo-rational, and in particular, is Cohen–Macaulay.

Next, consider the scheme-theoretic image $T_{v,w}$ of the map $\widetilde{S}_v \to \widetilde{S}_w$ and follow the argument in [PR08, Proposition 9.7 (b)]. This is a k-variety with seminormalization equal to $\widetilde{S}_v$. Since $D_{\dot v}$ is compatibly F-split inside $D_{\dot w}$, we deduce that $T_{v,w}$ is also compatibly F-split with $\widetilde{S}_w$ by pushforward along the map f. But F-split schemes are weakly normal by [BK05, Proposition 1.2.5], and in particular seminormal, so we have that $T_{v,w} \simeq \widetilde{S}_v$. In other words, the maps $\widetilde{S}_v \to \widetilde{S}_w$ are closed immersions for all $v \leq w$ and compatibly F-split.

Finally, we handle rationality of $\widetilde{S}_w$. We factor f as partial Demazure resolutions having fibers of dimension at most 1

$$f_i\colon D_{\dot u_i}\tilde{\times}\widetilde{S}_{v_i} \to D_{\dot u_{i+1}}\tilde{\times}\widetilde{S}_{v_{i+1}}, \tag{4.6}$$

where we write $\dot w = \dot u_i \cdot \dot v_i$ with $\dot u_i = s_1 \cdot \dots \cdot s_{d-i}$ and $\dot v_i = s_{d-i+1} \cdot \dots \cdot s_d$. By induction, it suffices to show vanishing of the higher direct images of the structure sheaf along f_i. Moreover, we may even ignore the factor $D_{\dot u_{i+1}} \subset D_{\dot u_i}$ and reduce to the study of $g\colon S_s\tilde{\times}\widetilde{S}_v \to \widetilde{S}_w$ with $w = sv$ being a reduced expression. We claim that for any (not necessarily closed) point $x \in \widetilde{S}_w$ the fiber $g^{-1}(x)$ is either isomorphic to $\mathrm{Spec}(\kappa(x))$ or to $\mathbb{P}^1_{\kappa(x)}$: Indeed, if $g^{-1}(x)$ is 0-dimensional, then the birational map g becomes a universal homeomorphism of normal varieties around x, thus a local isomorphism by Zariski's main theorem; if $g^{-1}(x)$ is 1-dimensional, then x belongs to $\widetilde{S}_u$ with $u < v$ and $su < u$, and we can directly see that the fibers of $h\colon S_s\tilde{\times}\widetilde{S}_u \to \widetilde{S}_u$ are projective lines. Therefore, we have $H^i(g^{-1}(x), \mathcal{O}_{S_s\tilde{\times}\widetilde{S}_v}) = 0$ for all $i > 0$, which upgrades in the presence of an F-splitting to $R^i g_*\mathcal{O}_{S_s\tilde{\times}\widetilde{S}_v} = 0$ for all $i > 0$ by [BK05, Lemma 1.2.11]. Since $\widetilde{S}_w$ is Cohen–Macaulay, Grothendieck–Serre duality yields also $R^i f_*\omega_{D_{\dot w}} = \omega_{\widetilde{S}_w}$. This means $\widetilde{S}_w$ has rational singularities, as desired. □

Remark 4.6. The map $G_{\mathrm{sc}} \to G$ from the simply connected group extends to the Iwahori O-models, and the induced map on Demazure varieties $D_{\mathrm{sc},\dot w} \to D_{\dot w}$ is an isomorphism, see [HR22, Proof of Lemma 3.8]. Further, $D_{\mathrm{sc},\dot w}$ factors as a product of Demazure varieties according to the almost simple factors of G_{sc}, and products of (stably) compatibly F-split varieties are (stably) compatibly F-split [BK05, Section 1.3.E (8)]. Therefore, in order to verify the assumption of Lemma 4.5, we may assume whenever convenient that $G = G_{\mathrm{sc}}$ is simply connected and (by the Weil restriction of scalars case in [HR22, Lemma 3.9]) *absolutely* almost simple and that $\mathcal{G} = \mathcal{I}$ is the Iwahori group scheme.

4.1.2. *Picard groups of perfected Schubert varieties.* In this subsection, we calculate the Picard groups of perfected Schubert varieties and the induced map on the Demazure resolution. This plays a role later in proving the existence of F-splittings for Demazure varieties, which requires the construction of a certain divisor that is more easily done on the Schubert varieties.

For any $v \in W_{\mathrm{af}}$ we consider the corresponding $(\mathcal{I}, \mathcal{G})$-Schubert variety S_v and its seminormalization $\widetilde{S}_v$. For the right $W_{\mathcal{G}}$-minimal element w above, we fix a choice of reduced expression

$\dot{w} = s_1 \cdot \ldots \cdot s_d$ and consider the Demazure resolution $f\colon D_{\dot{w}} \to \widetilde{S}_w$ as in (4.5). For each simple reflection $s \in W_{\mathrm{af}}$ and any choice of isomorphism of $D_{\dot{s}} \cong \mathbb{P}^1_k$, the degree of line bundles induces a well-defined isomorphism $\deg\colon \mathrm{Pic}(D_{\dot{s}}) \cong \mathbb{Z}$.

Lemma 4.7. *There is an isomorphism*

$$\mathrm{Pic}(D_{\dot{w}}) \xrightarrow{\cong} \mathbb{Z}^d, \quad \mathcal{L} \mapsto (\deg(\mathcal{L}|_{D_{\dot{s}_i}}))_{i=1,\ldots,d}. \tag{4.7}$$

Proof. The method of [HZ20, Proposition 3.4] applies as follows. Writing $\dot{w} = \dot{s} \cdot \dot{v}$ with $\dot{s} = s_1$, $\dot{v} = s_2 \cdot \ldots \cdot s_d$ induces an étale locally trivial fibration $D_{\dot{w}} \to D_{\dot{s}}$ with general fiber $D_{\dot{v}}$. The fibration is Zariski locally trivial by [PR08, Proposition 8.7 (b)]. Hence, [Mag75, Theorem 5] gives an exact sequence $0 \to \mathrm{Pic}(D_{\dot{s}}) \to \mathrm{Pic}(D_{\dot{w}}) \to \mathrm{Pic}(D_{\dot{v}}) \to 0$, which splits by using the section $D_{\dot{s}} \to D_{\dot{w}}$. The lemma follows by induction. □

The universal homeomorphism $\widetilde{S}_v \to S_v$ induces an isomorphism on perfections [BS17, Lemma 3.8], and we denote by S_v^{pf} its common value. For each simple reflection $s \in W_{\mathrm{af}} \backslash W_{\mathcal{G}}$, we have an isomorphism $S_s = \widetilde{S}_s = D_{\dot{s}} \cong \mathbb{P}^1_k$, and the degree map uniquely extends to an isomorphism $\deg\colon \mathrm{Pic}(S_s^{\mathrm{pf}}) \cong \mathbb{Z}[p^{-1}]$ (see [BS17, Lemma 3.5]); further $\mathrm{Pic}(\widetilde{S}_s) \cong \mathrm{Pic}(S_s^{\mathrm{pf}}) = 0$ if $s \in W_{\mathcal{G}}$, since $\widetilde{S}_s \cong S_s \cong \mathrm{Spec}(k)$ in that case.

Lemma 4.8. *There is an isomorphism*

$$\mathrm{Pic}(S_w^{\mathrm{pf}}) \xrightarrow{\cong} \bigoplus_s \mathbb{Z}[p^{-1}], \quad \mathcal{L} \mapsto (\deg(\mathcal{L}|_{S_s^{\mathrm{pf}}}))_s \tag{4.8}$$

where the sum runs over all $s \in \{s_1, \ldots, s_d\}$ with $s \notin W_{\mathcal{G}}$. Further, the pullback map $\mathrm{Pic}(\widetilde{S}_w) \to \mathrm{Pic}(S_w^{\mathrm{pf}})$ is injective, and its image is a $\mathbb{Z}$-lattice.

Proof. The argument in [HZ20, Proposition 3.9] applied to $f\colon D_{\dot{w}} \to \widetilde{S}_w$ translates verbatim, and we sketch it for the reader's convenience. The pullback map $\mathrm{Pic}(\widetilde{S}_w) \to \mathrm{Pic}(D_{\dot{w}})$ is injective using the projection formula and the relation $f_*\mathcal{O}_{D_{\dot{w}}} = \mathcal{O}_{\widetilde{S}_w}$ from (4.5). Under the isomorphism $\mathrm{Pic}(D_{\dot{w}}) \cong \mathbb{Z}^d$ from Lemma 4.7, in the i-th component, for all $i = 1, \ldots, d$, the map is given by $\mathcal{L} \mapsto \deg(\mathcal{L}|_{S_{s_i}})$ if $s_i \notin W_{\mathcal{G}}$ and $\mathcal{L} \mapsto 0$ else. (Note that if $s \leq w$ and $i_{\dot{s}}\colon D_{\dot{s}} \to D_{\dot{w}}$ covers $i_s\colon \widetilde{S}_s \hookrightarrow \widetilde{S}_w$, then the map $\mathrm{Pic}(\widetilde{S}_w) \to \mathrm{Pic}(D_{\dot{w}}) \xrightarrow{i_{\dot{s}}^*} \mathrm{Pic}(D_{\dot{s}})$ factors as $\mathrm{Pic}(\widetilde{S}_w) \xrightarrow{i_s^*} \mathrm{Pic}(\widetilde{S}_s) \to \mathrm{Pic}(D_{\dot{s}})$, hence is zero if $s \in W_{\mathcal{G}}$.) For any qcqs $\mathbb{F}_p$-scheme X one has $\mathrm{Pic}(X^{\mathrm{pf}}) = \mathrm{Pic}(X)[p^{-1}]$ by [BS17, Lemma 3.5]. So, passing to perfections implies injectivity of (4.8) and that $\mathrm{Pic}(\widetilde{S}_w)$ defines a $\mathbb{Z}$-lattice in $\mathrm{Pic}(S_w^{\mathrm{pf}})$.

To prove surjectivity of (4.8), let $(\lambda_s) \in \oplus_s \mathbb{Z}[p^{-1}] \subset \mathbb{Z}[p^{-1}]^d$. This induces a line bundle $\mathcal{D} = \mathcal{D}(\lambda_s)$ on $D_{\dot{w}}^{\mathrm{pf}}$. It suffices to show that $\mathcal{D}$ is trivial along the fibers of f^{pf}, for once we know this $\mathcal{D}$ descends to S_w^{pf} by v-descent for vector bundles on perfect varieties [BS17, Theorem 6.13]. First, we factor f^{pf} as partial Demazure resolutions having fibers of dimension at most 1 as in the proof of Lemma 4.5. By induction we may and do replace f^{pf} by a corresponding map $g^{\mathrm{pf}}\colon S_s^{\mathrm{pf}} \tilde{\times} S_v^{\mathrm{pf}} \to S_w^{\mathrm{pf}}$ with $w = sv$ being a reduced expression. Also, by the proof of Lemma 4.5 the map g^{pf} has non-trivial fibers exactly over the union of all S_u^{pf}, with $u \leq v$ such that $su < u$. So, we restrict our attention to the fibers of the map $h^{\mathrm{pf}}\colon S_s^{\mathrm{pf}} \tilde{\times} S_u^{\mathrm{pf}} \to S_u^{\mathrm{pf}}$. By induction on w, we know that $\mathrm{Pic}(S_u^{\mathrm{pf}})$ is a free $\mathbb{Z}[1/p]$-module generated on its $L^+\mathcal{I}$-stable projective lines. Following the argument in Lemma 4.7, the same assertion holds for $\mathrm{Pic}(S_s^{\mathrm{pf}} \tilde{\times} S_u^{\mathrm{pf}})$. In particular, we see that the restriction of $\mathcal{D}$ to $S_s^{\mathrm{pf}} \tilde{\times} S_u^{\mathrm{pf}}$ is the pullback of some line bundle $\mathcal{L}$ on S_u^{pf} along h^{pf}. Hence, it is trivial along the fibers. □

As perfections preserve closed immersions [BS13, Lemma 3.4 (viii)], there is a strict k-ind-scheme

$$\mathcal{F}\ell_{\mathcal{G}}^{\mathrm{pf}} = \mathrm{colim}\, S_w^{\mathrm{pf}} \tag{4.9}$$

lying over $\widetilde{\mathcal{F}\ell}_{\mathcal{G}} = \operatorname{colim} \widetilde{S}_w$ from (4.3). Their Picard groups are defined as the limit of the Picard groups of the respective Schubert varieties.

Corollary 4.9. *There is an isomorphism*

$$\operatorname{Pic}(\mathcal{F}\ell_{\mathcal{G}}^{0,\mathrm{pf}}) \xrightarrow{\cong} \bigoplus_s \mathbb{Z}[p^{-1}], \quad \mathcal{L} \mapsto (\deg(\mathcal{L}|_{S_s^{\mathrm{pf}}}))_s \tag{4.10}$$

where $\mathcal{F}\ell_{\mathcal{G}}^0$ denotes the neutral component and the sum runs over all simple reflections $s \in W_{\mathrm{af}} \backslash W_{\mathcal{G}}$. The pullback map $\operatorname{Pic}(\widetilde{\mathcal{F}\ell}_{\mathcal{G}}) \to \operatorname{Pic}(\mathcal{F}\ell_{\mathcal{G}}^{\mathrm{pf}})$ is injective.

Proof. This is immediate from Lemma 4.8: For $v \leq w$ in W_{af} with large enough length, the pullback map $\operatorname{Pic}(S_w^{\mathrm{pf}}) \to \operatorname{Pic}(S_v^{\mathrm{pf}})$ is an isomorphism, which is the identity map under (4.8). □

4.1.3. *The central charge.* We assume in this subsection (for simplicity) that G is almost simple and simply connected, compare with Remark 4.6. In particular, the affine flag variety $\mathcal{F}\ell_{\mathcal{G}}$ is connected.

The quotient $L^+\mathcal{G} \to \mathcal{G}_k$ induces maps $\mathcal{F}\ell_{\mathcal{G}} = LG/L^+\mathcal{G} \to [\operatorname{Spec} k/L^+\mathcal{G}] \to [\operatorname{Spec} k/\mathcal{G}_k]$ to the respective quotient stacks. Passing to Picard groups we obtain

$$X^*(\mathcal{G}_k) \cong X^*(L^+\mathcal{G}) \to \operatorname{Pic}(\mathcal{F}\ell_{\mathcal{G}}) \tag{4.11}$$

Here, the first isomorphism holds because the kernel of $L^+\mathcal{G} \to \mathcal{G}_k$ is pro-unipotent. The Picard groups of the quotient stacks are the respective character groups because giving a line bundle on such a stack is the same as giving a 1-dimensional representation of the group, that is, a character.

Lemma 4.10. *The group $\operatorname{Pic}([LG\backslash\mathcal{F}\ell_{\mathcal{G}}])$ of isomorphism classes of line bundles on $\mathcal{F}\ell_{\mathcal{G}}$ equipped with an LG-equivariant structure naturally identifies with $X^*(\mathcal{G}_k)$ via the induction map*

$$\mu \mapsto \mathcal{L}(\mu) := LG \times^{L^+\mathcal{G}} \mathcal{O}_\mu, \tag{4.12}$$

where $\mathcal{O}_\mu$ is the equivariant line bundle on $L^+\mathcal{G}$ attached to μ.

Proof. This is immediate from the isomorphisms in (4.11): the inverse to the induction map is given by pullback of LG-equivariant line bundles to the origin of $\mathcal{F}\ell_{\mathcal{G}}$, noticing that they carry an action of $L^+\mathcal{G}$, that is, $[LG\backslash\mathcal{F}\ell_{\mathcal{G}}] = [\operatorname{Spec} k/L^+\mathcal{G}]$ in terms of (étale) stacks. □

We now pass to perfections in order to make the map (4.11) explicit, compare Corollary 4.9. So choosing any presentation of LG by affine schemes, we denote by LG^{pf} the colimit of the perfections of the constituents. As k is perfect, we can equivalently use the relative Frobenius over k to form LG^{pf}, so it is naturally an ind-affine k-group ind-scheme.

After perfection, we deduce from Lemma 4.10 and Equation (4.11) the homomorphism

$$X^*(\mathcal{G}_k)[p^{-1}] \to \operatorname{Pic}(\mathcal{F}\ell_{\mathcal{G}}^{\mathrm{pf}}), \tag{4.13}$$

whose image identifies with the line bundles admitting an LG^{pf}-equivariant structure. In order to explicitly describe (4.13), we fix the standard basis $\epsilon_i = (0, \dots, 1, \dots, 0)$ of $\operatorname{Pic}(\mathcal{F}\ell_{\mathcal{I}}^{\mathrm{pf}}) \cong \oplus_s \mathbb{Z}[p^{-1}]$ (see Corollary 4.9 for $\mathcal{G} = \mathcal{I}$ being the Iwahori). It will be convenient for us to fix a certain enumeration of the simple reflections.

Lemma 4.11. *There exists a simple reflection s_0 such that the unique standard maximal parahoric $\mathcal{G}_0$ with $s_0 \notin W_{\mathcal{G}_0}$ satisfies the following: the reductive quotient of the special fiber $\mathcal{G}_{0,k}$ is simply connected and its root system equals the non-multipliable roots of Φ_G.*

Proof. For any positive simple affine root α_s in the affine root system Ψ_G in the sense of [KP23, Definition 4.3.4] associated with a simple reflection s, let $a_s \in \Phi_G$ be the gradient of α_s. For any enumeration $s_0, \ldots, s_n$ of the simple reflections, we write $a_i := a_{s_i}$ for $i = 0, \ldots, n$. We claim that there exists a choice of enumeration such that the a_i for $i > 0$ form a basis of Φ_G^{nm}, the sub-root system of non-multipliable roots. In order to see that this is possible, we consider the folowing cases: either Φ_G is reduced, and this amounts to the choice of a special vertex in the fundamental alcove not fixed by s_0; or Φ_G is not reduced, and we need to ensure the existence of special vertices which are not extra special in the sense of [KP23, Proposition 1.5.39], which can be verified in [KP23, Table 1.5.51]. From now on, we fix such an enumeration and claim that the standard maximal parahoric $\mathcal{G}_0$ attached to s_0 satisfies the conditions in the lemma.

In what follows, we canonically identify the character and cocharacter groups of the K-split torus S with those of the special fiber $\mathcal{S}_k$ of its connected Néron O-model $\mathcal{S}$. Note that $\mathcal{S}_k$ defines a maximal split torus of the reductive quotient of $\mathcal{G}_{0,k}$, because k is algebraically closed. By construction, the a_i for $i > 0$ define roots of the reductive quotient of $\mathcal{G}_{0,k}$. In particular, the coroots $a_i^\vee$ for $i > 0$ (which are non-divisible) form a basis of the dual root system $\Phi_G^\vee$ and hence of $X_*(S)$ by our assumption on G. $\square$

From now on, we fix an enumeration $s_0, s_1, \ldots, s_n$ of the simple reflections with s_0 being as in Lemma 4.11. With our numbering system above in terms of our choice of special vertex, this has the following explicit description: for $i > 0$, a_i is the non-multipliable relative root whose reflection is s_i, and a_0 is the negative of the highest multipliable relative root.

Lemma 4.12. *Let $a_i^\vee \in X_*(S)$ be the coroot associated to the root a_i as defined above. Under the isomorphism (4.10), the map (4.13) is given by*

$$\mu \mapsto \sum \langle a_i^\vee, \mu\rangle \epsilon_i, \tag{4.14}$$

where the sum runs over all $i = 0, \ldots, n$ with $s_i \notin W_{\mathcal{G}}$. Thus, (4.13) is injective and has cokernel free over $\mathbb{Z}[p^{-1}]$ of rank 1.

Proof. Let $\mathcal{P}_i$ be the minimal standard parahoric such that $L^+\mathcal{P}_i/L^+\mathcal{I} = S_{s_i}$. The reductive quotient of the special fiber of $\mathcal{P}_i$ has simply connected cover isomorphic to SL_2 with positive coroot $a_i^\vee$. Therefore, the pullback to $S_{s_i}^{\mathrm{pf}}$ of the equivariant line bundle $\mathcal{L}(\mu)$ attached to a weight $\mu \in X^*(S)[p^{-1}] \cong X^*(\mathcal{I}_k)[p^{-1}]$ is isomorphic to $\mathcal{O}(\langle a_i^\vee, \mu\rangle)$, hence (4.14) holds. It is well-known from the theory of algebraic groups that $X^*(\mathcal{G}_k)$ is a direct summand of $X^*(S)$, compare with [CGP15, Corollary A.2.7]. Hence, to deduce injectivity of (4.13) and freeness of its cokernel, we may and do assume that $\mathcal{G} = \mathcal{I}$ is the Iwahori. Due to the fact that $\mathcal{S}_k$ identifies with a maximal torus in the reductive quotient of $\mathcal{G}_{0,k}$, which is simply connected with roots Φ_G^{nm}, the coroots $a_i^\vee$ for $i > 0$ form a basis of $X_*(S)$. So its dual basis ω_i form a basis of $X^*(S)$, and thus (4.14) admits a section. Finally, to see that the cokernel has rank 1 for arbitrary $\mathcal{G}$, we proceed as follows. First, we notice that for any $i = 0, \ldots, n$, the set a_j for $j \neq i$ forms a basis of $X^*(S)_{\mathbb{Q}}$, because otherwise the affine reflections s_j for $j \neq i$ would have a positive-dimensional intersection in $\mathscr{A}(G, S)$. Suppose $W_{\mathcal{G}}$ contains exactly $m < n + 1$ many simple reflections and notice that the associated relative roots are still linearly independent in $X_*(S)_{\mathbb{Q}}$ by our previous observation. Let $\mathcal{S}_k^{\mathrm{der}}$ denote the maximal torus of the derived subgroup of $\mathcal{G}_k^{\mathrm{red}}$ and notice that $X^*(\mathcal{S}_k^{\mathrm{der}})$ has rank m. We deduce that the cokernel $X^*(\mathcal{G}_k)_{\mathbb{Q}}$ of $X^*(\mathcal{S}_k^{\mathrm{der}})_{\mathbb{Q}} \to X^*(S)_{\mathbb{Q}}$ has rank $n - m$, whereas $\mathrm{Pic}(\mathcal{F}\ell_{\mathcal{G}}^{\mathrm{pf}})$ has rank $n + 1 - m$ by Equation (4.10). $\square$

Using that (4.13) is injective and has free cokernel of rank 1 (see Lemma 4.12), we construct a homomorphism

$$\mathrm{Pic}(\mathcal{F}\ell_{\mathcal{G}}^{\mathrm{pf}}) \to \mathbb{Z}[p^{-1}],\ \mathcal{L} \mapsto c_{\mathcal{L}}, \tag{4.15}$$

called the *central charge homomorphism*, uniquely characterized by the following properties: its kernel is $X^*(\mathcal{G}_k)[p^{-1}]$; it factors through $\mathrm{Pic}(\mathcal{F}\ell_{\mathcal{G}}^{\mathrm{pf}}) \to \mathrm{Pic}(\mathcal{F}\ell_{\mathcal{I}}^{\mathrm{pf}})$; for $\mathcal{G} = \mathcal{I}$ the standard $\mathbb{Z}$-lattice $\oplus_s \mathbb{Z} \subset \oplus_s \mathbb{Z}[p^{-1}] \cong \mathrm{Pic}(\mathcal{F}\ell_{\mathcal{I}}^{\mathrm{pf}})$ (see Corollary 4.9) maps onto $\mathbb{Z} \subset \mathbb{Z}[p^{-1}]$, preserving positive degrees. Please note that this map is just the $\mathbb{Z}[p^{-1}]$-linearization of the usual central charge as defined in [Zhu14, Equation (2.2.3)]. The only reason we defined it in the perfect setting is because we have not yet proved Theorem 4.1, so we do not control the Picard group of the $\mathcal{F}\ell_{\mathcal{G}}$, but only of its perfection. We remark that the homomorphism (4.15) is surjective when $\mathcal{G} = \mathcal{I}$ is the Iwahori, but usually not for general parahorics, see [Zhu14, Section 2.2, page 12].

Lemma 4.13. *Let $\omega_i \in X^*(S)$ for $i = 1, \dots, n$ be the dual basis to $a_i^\vee$. Under the isomorphism* (4.10)*, the map* (4.15) *is given by*

$$(\lambda_i) \mapsto \lambda_0 - \sum_{i>0} \langle a_0^\vee, \omega_i \rangle \lambda_i, \tag{4.16}$$

where we use the convention that $\lambda_i = 0$ whenever $s_i \in W_{\mathcal{G}}$. In particular, the coefficients 1 *and $-\langle a_0^\vee, \omega_i \rangle$ are the numbers attached in* [Kac90, Section 6.1] *to the vertices of the dual affine Dynkin diagram of G.*

Proof. The proof of Lemma 4.12 shows that $\mathcal{L}(\omega_i)$ is the image of $\langle a_0^\vee, \omega_i \rangle \epsilon_0 + \epsilon_i$ under the bijection (4.10). Hence, we get $c(\epsilon_i) = -\langle a_0^\vee, \omega_i \rangle c(\epsilon_0)$. So $c(\oplus_s \mathbb{Z}) \subset \mathbb{Z} c(\epsilon_0)$ and by our choice of normalization $c(\epsilon_0) = 1$, thus $c(\epsilon_i) = -\langle a_0^\vee, \omega_i \rangle$ for $i > 0$.

Finally, for the comparison with Kac–Moody theory, this can be seen by inspecting [Kac90, Theorem 4.8, Tables Aff 1-3] or the construction of the central charge for untwisted and twisted Kac–Moody algebras, see [Kac90, Theorems 7.4 and 8.3]. Alternatively, we may observe that these coefficients are combinatorial data that do not really depend on the arithmetic properties of G, so we may assume the latter to be tamely ramified, in which case $\mathcal{F}\ell_{\mathcal{G}}$ identifies with a Kac–Moody flag variety, see [PR08, 9.h and Proposition 10.1] and also [Lou23, Annexe A]. □

Recall that for $\mathcal{G} = \mathrm{GL}_n$ we have an ample line bundle $\mathcal{L}_{\det} = \mathcal{O}(1)$ on $\mathcal{F}\ell_{\mathrm{GL}_n}$ such that $c(\mathcal{L}_{\det}) = 1$. Pulling it back along the adjoint representation $\mathrm{ad}\colon \mathcal{F}\ell_{\mathcal{G}} \to \mathcal{F}\ell_{\mathrm{GL}(\mathrm{Lie}\mathcal{G})}$, we get an ample line bundle $\mathcal{L}_{\mathrm{ad}}$ on $\mathcal{F}\ell_{\mathcal{G}}$ whose central charge can still be determined:

Lemma 4.14. *The central charge $c(\mathcal{L}_{\mathrm{ad}})$ of the adjoint line bundle is equal to $2h^\vee$, where $h^\vee$ is the dual Coxeter number of the split form of G.*

Proof. We invoke [Zhu14, Lemma 4.2] at Iwahori level, which shows that $\mathcal{L}_{\mathrm{ad}}$ has degree 2 when restricted to every S_{s_i}, and which does not use any tameness assumptions. But it is well-known that the sum $1 - \sum \langle \omega_i, a_0^\vee \rangle$ equals the dual Coxeter number. For general parahoric level, there is a reduction step in the remaining paragraphs of the proof of [Zhu14, Proposition 4.1] that follow the Iwahori lemma cited above. □

A key property of (4.15) is its constancy along the fibers of Beilinson–Drinfeld Grassmannians, and we extend the results [Hei10, Lemma 18, Remark 19] and [Zhu14, Proposition 4.1, Corollary 4.3] from tamely ramified groups to general reductive groups as follows. Let $\mathrm{Gr}_{\mathcal{G}} \to \mathrm{Spec}(O)$ be the Beilinson–Drinfeld Grassmannian, see [Ric16, Definition 2.3] and [Ric19, Section 0.3] for a definition independent of auxiliary choices. Then $\mathrm{Gr}_{\mathcal{G}} \to \mathrm{Spec}(O)$ is an ind-projective ind-scheme, its generic fiber $\mathrm{Gr}_{\mathcal{G},K}$ is equivariantly isomorphic to the affine Grassmannian Gr_G formed using an additional formal parameter [Ric19, Section 0.2], whereas its special fiber $\mathrm{Gr}_{\mathcal{G},k}$ is equal to $\mathcal{F}\ell_{\mathcal{G}}$. Looking ahead to the proof of Lemma 4.15 below, we note that the line bundle $\mathcal{L}_{\mathrm{ad}}$ above extends to a line bundle on $\mathrm{Gr}_{\mathcal{G}}$, by the same construction (use [Ric16, §2.5]); we denote the extension also by $\mathcal{L}_{\mathrm{ad}}$. By our assumptions on the group, we can write $G = \mathrm{Res}_{L/K} G_0$ for some finite, separable field extension L/K and some absolutely almost simple, simply connected reductive L-group G_0. Given a scheme X, let $\mathrm{Pic}(X)_{\mathbb{Q}}$ denote the rationalized Picard group of X. For an ind-scheme X, we define $\mathrm{Pic}(X)_{\mathbb{Q}}$ as the limit of the $\mathrm{Pic}(X_i)_{\mathbb{Q}}$ along a presentation (in all cases considered in this paper, this will match the $\mathbb{Q}$-localization of $\mathrm{Pic}(X)$).

Lemma 4.15. *The following properties hold:*

(1) *The map* $\mathrm{Pic}(\mathrm{Gr}_{\mathcal{G}})_{\mathbb{Q}} \to \mathrm{Pic}(\mathcal{F}\ell_{\mathcal{G}})_{\mathbb{Q}}$ *is surjective.*
(2) *Every* $\mathcal{L} \in \mathrm{Pic}(\mathrm{Gr}_{\mathcal{G}})_{\mathbb{Q}}$ *has geometric generic fiber isomorphic to* $\mathcal{O}(c_{\mathcal{L}_k})$, *the* $c_{\mathcal{L}_k}$*-th tensor power of* $\boxtimes_{[L:K]}\mathcal{O}(1)$ *on* $\mathrm{Gr}_{G,\bar{K}} \cong \prod_{[L:K]} \mathrm{Gr}_{G_0,\bar{K}}$.

Proof. There is a natural map $\mathrm{Gr}_{\mathcal{G}} \to [\mathrm{Spec}\, k/\mathcal{G}_k]$ to the classifying stack of $\mathcal{G}_k$-bundles over k, given by forgetting the modification and then restricting the torsor to the subscheme defined by the principal ideal t. This map factors the map $\mathcal{F}\ell_{\mathcal{G}} \to [\mathrm{Spec}\, k/\mathcal{G}_k]$ (compare Equation (4.11)) under the identification $\mathrm{Gr}_{\mathcal{G},k} = \mathcal{F}\ell_{\mathcal{G}}$. Passing to Picard groups, we get maps $X^*(\mathcal{G}_k) \to \mathrm{Pic}(\mathrm{Gr}_{\mathcal{G}}) \to \mathrm{Pic}(\mathcal{F}\ell_{\mathcal{G}})$ whose composition is (4.11). After rationalizations, the maps are injective. Further, $\mathcal{L}_{\mathrm{ad}}$ and $\ker(c)_{\mathbb{Q}} = X^*(\mathcal{G}_k)_{\mathbb{Q}}$ generate the $\mathbb{Q}$-vector space $\mathrm{Pic}(\mathcal{F}\ell_{\mathcal{G}})_{\mathbb{Q}}$ by Corollary 4.9. This shows (1).

For (2), we start by noticing that its conclusion is satisfied by the image of $X^*(\mathcal{G}_k)_{\mathbb{Q}} \to \mathrm{Pic}(\mathrm{Gr}_{\mathcal{G}})_{\mathbb{Q}}$. Indeed, the map $\mathrm{Gr}_{G,\bar{K}} \to [\mathrm{Spec}\, k/\mathcal{G}_k]$ factors through the trivial torsor by Beauville–Laszlo gluing, so it must induce the zero map on the rationalized Picard group. Moreover, the conclusion holds as well for $\mathcal{L}_{\mathrm{ad}}$ defined over $\mathrm{Gr}_{\mathcal{G}}$ again by pulling back $\mathcal{L}_{\det}$ along the adjoint map to the Lie algebra. Indeed, on the geometric generic fiber $\mathrm{Gr}_{G,\bar{K}} \cong \prod_{[L:K]} \mathrm{Gr}_{G_0,\bar{K}}$, the line bundle $\mathcal{L}_{\mathrm{ad}}$ becomes isomorphic to $\mathcal{O}(2h^\vee)$, where $h^\vee$ is the dual Coxeter number of G_0, by Lemma 4.14 applied to each of the factors G_0. On the special fiber $\mathcal{F}\ell_{\mathcal{G}}$, we also know by Lemma 4.14 that $c_{\mathcal{L}_{\mathrm{ad}}} = 2h^\vee$.

Since the previous explicitly given rationalized line bundles generate $\mathrm{Pic}(\mathcal{F}\ell_{\mathcal{G}})_{\mathbb{Q}}$ as seen already, we may and do assume that our abstract rationalized line bundle $\mathcal{L}$ on $\mathrm{Gr}_{\mathcal{G}}$ has trivial special fiber $\mathcal{L}_k = \mathcal{O}$. Let μ be a conjugacy class of cocharacters in $G_{\bar{K}}$ with reflex field $E \supset K$. Let $M_{\mathcal{G},\mu}$ be the orbit closure of $S_{G,\mu}$ over O_E, see Definition 5.1, and suppose that μ is supported on exactly one almost simple factor of $G_{\bar{K}}$. Then, $\mathrm{Pic}(S_{G,\mu,\bar{K}})_{\mathbb{Q}}$ is 1-dimensional by Lemma 4.8. Assume for the sake of contradiction that $\mathcal{L}_{\bar{K}}$ is anti-ample on $S_{G,\mu,\bar{K}}$ (if not, take its inverse). It is therefore equal to the restriction of $\mathcal{L}_{\mathrm{ad},\bar{K}}^{-q}$ for some $q \in \mathbb{Q}_{>0}$. Replacing $\mathcal{L}$ by its product with $\mathcal{L}_{\mathrm{ad}}^q$, we may now ensure that $\mathcal{L}_k$ is ample and $\mathcal{L}_{\bar{K}}$ is trivial on $\mathrm{Gr}_{G,\mu,\bar{K}}$. This contradicts openness of the ample locus of $\mathcal{L}$ on $M_{\mathcal{G},\mu}$, see [Gro66, Corollaire 9.6.4]. In particular, we conclude that $\mathcal{L}_{\bar{K}}$ must be trivial on $S_{G,\mu,\bar{K}}$. Letting μ run over all coweights with irreducible support, we deduce from Corollary 4.9 that $\mathcal{L}_{\bar{K}}$ is trivial. □

Suppose we are given a map $f\colon \mathcal{G}_1 \to \mathcal{G}_2$ of parahoric O-models of simply connected, almost simple K-groups G_1 and G_2. We have an induced pull-back map

$$f^*\colon \mathrm{Pic}(\mathcal{F}\ell_{\mathcal{G}_2}^{\mathrm{pf}}) \to \mathrm{Pic}(\mathcal{F}\ell_{\mathcal{G}_1}^{\mathrm{pf}}) \tag{4.17}$$

that sends equivariant line bundles with respect to LG_2^{pf} to those with respect to LG_1^{pf}. In particular, we get a homomorphism of cokernels defined by their central charges and it follows that $c_1(f^*\mathcal{L}) = d(f)c_2(\mathcal{L})$ where $d(f) \in \mathbb{Z}_{\geq 0}$ is independent of $\mathcal{L}$ and c_i denote the central charges of the respective Picard groups. Here, the non-negativity of $d(f)$ holds because pullback preserves semi-ampleness, and $d(f)$ is an integer because the map of Picard groups also exists on the non-perfected affine flag varieties.

From the constancy of the central charge we draw the following consequence:

Corollary 4.16. *Let L/K be a finite separable extension and consider the natural map*

$$f\colon \mathcal{G} \to \mathrm{Res}_{O_L/O_K}(\tilde{\mathcal{G}}), \tag{4.18}$$

extending the unit of adjunction for $\mathrm{Res}_{L/K}$*, where* $\tilde{\mathcal{G}}$ *is the associated parahoric* O_L*-model of* G_L *induced by the map* $\mathscr{B}(G,K) \to \mathscr{B}(G,L)$*. Then* $d(f) = [L:K]$*.*

Proof. Thanks to Lemma 4.15, we can read off the integer $d(f)$ from the map of affine Grassmannians $\mathrm{Gr}_G \to \mathrm{Gr}_{\mathrm{Res}_{L/K}G_L}$ after base changing to $\bar{K}$. But then $\mathrm{Res}_{L/K}G_L$ splits over $\bar{K}$ as a product of $[L:K]$-many copies of $G_{\bar{K}}$, so $\mathcal{O}(1) = \boxtimes_{[L:K]}\mathcal{O}(1)$ pulls back to $\mathcal{O}([L:K])$ as desired. □

4.1.4. *The Demazure variety is stably compatibly* F*-split.* In order to finish the proof of Theorem 4.1, it remains to show that the assumption of Lemma 4.5 holds, that is, the Demazure variety $D_{\dot{w}}$ is stably compatibly F-split with $D_{\dot{v}}$ for all $\dot{v}$ of colength 1 in $\dot{w}$. By Remark 4.6, we may and do assume that G is simply connected, absolutely almost simple and that $\mathcal{G} = \mathcal{I}$ is the Iwahori group scheme. As in [PR08, Section 8] (for proving F-splitness) and [Cas22, Section 5] (for proving stable F-splitness), we aim to apply the Mehta–Ramanathan splitting criterion, see [BS13, Theorem 5.3.1] and [BK05, Proposition 1.3.11], to $D_{\dot{w}}$ together with its divisors $D_{\dot{v}}$. We need the following result for this.

Lemma 4.17. *There exists a unique line bundle* $\mathcal{L}_{\mathrm{crit}} \in \mathrm{Pic}(\widetilde{\mathcal{F}\ell}_{\mathcal{G}})$ *such that* $\mathcal{L}_{\mathrm{crit}}^{\otimes 2} \simeq \mathcal{L}_{\mathrm{ad}}$*.*

Proof. We first prove uniqueness. Note that $f^*\colon \mathrm{Pic}(\widetilde{S}_w) \to \mathrm{Pic}(D_{\dot{w}})$ is injective as $f_*f^*\mathcal{L} = \mathcal{L}$ by the projection formula and by $f_*\mathcal{O}_{D_{\dot{w}}} = \mathcal{O}_{\widetilde{S}_w}$, see (4.5) for the latter. By Lemma 4.7, we see that $\mathrm{Pic}(\widetilde{S}_w)$ is torsion-free for all $w \in W_{\mathrm{af}}$ and so is $\mathrm{Pic}(\widetilde{\mathcal{F}\ell}_{\mathcal{G}})$. Hence, the map $\pi^*\colon \mathrm{Pic}(\widetilde{\mathcal{F}\ell}_{\mathcal{G}}) \to \mathrm{Pic}(\mathcal{F}\ell_{\mathcal{G}}^{\mathrm{pf}})$ is injective. In particular, $\mathcal{L}_{\mathrm{ad}}$ admits at most one square root.

Next, we prove existence. Recall that $\mathcal{L}_{\mathrm{ad}}$ restricts to $\mathcal{O}(2)$ on every S_s, so it admits a square root inside $\mathrm{Pic}(\mathcal{F}\ell_{\mathcal{G}}^{\mathrm{pf}})$ of central charge equal to $h^\vee$, see Corollary 4.9 and Lemma 4.14. There are inclusions of $\mathbb{Z}$-lattices

$$\mathrm{Pic}(\widetilde{\mathcal{F}\ell}_{\mathcal{G}}) \subset \oplus_s\mathbb{Z} \subset \oplus_s\mathbb{Z}[p^{-1}] = \mathrm{Pic}(\mathcal{F}\ell_{\mathcal{G}}^{\mathrm{pf}}). \tag{4.19}$$

(Note that the lattices are in fact equal, which only follows after finishing the proof of Theorem 4.1.) The cokernel of the inclusion is p-power torsion. If there were no square root $\mathcal{L}_{\mathrm{crit}}$ on $\widetilde{\mathcal{F}\ell}_{\mathcal{G}}$, then the element $(1,\dots,1) \in \oplus_s\mathbb{Z}$ would be a non-trivial 2-torsion point of the cokernel $\mathrm{Pic}(\mathcal{F}\ell_{\mathcal{G}}^{\mathrm{pf}})/\mathrm{Pic}(\widetilde{\mathcal{F}\ell}_{\mathcal{G}})$, which yields a contradiction unless $p = 2$. So, $\mathcal{L}_{\mathrm{crit}}$ exists for whenever $p > 2$.

Now, let $p = 2$. Informally speaking, we aim to show that $\mathrm{Pic}(\widetilde{\mathcal{F}\ell}_{\mathcal{G}})$ is large enough as follows. Lemma 4.10 and Lemma 4.12 implies that $\ker(c) \cap \oplus_s\mathbb{Z}$ already lies in $\mathrm{Pic}(\widetilde{\mathcal{F}\ell}_{\mathcal{G}})$. Since $c(1,\dots,1) = h^\vee$, it is enough to prove the inclusion

$$c(\mathrm{Pic}(\widetilde{\mathcal{F}\ell}_{\mathcal{G}})) \supset h^\vee\mathbb{Z}, \tag{4.20}$$

where we recall the normalization of c from (4.15). In order to verify (4.20), let $e \leq 3$ denote the degree of the smallest extension L/K whose Galois hull $\widetilde{L}/K$ splits G. The flag variety of the corresponding Iwahori model $\widetilde{\mathcal{G}}$ over $O_{\widetilde{L}}$ admits a line bundle with central charge 1 by [Fal03, Theorem 7]. By Corollary 4.16, we obtain the inclusion $e!\mathbb{Z} \subset c(\mathrm{Pic}(\widetilde{\mathcal{F}\ell}_{\mathcal{G}}))$. Looking at the classification of [Bou68, Planches], we see that $e!$ always divides $h^\vee$, unless $G = \mathrm{SU}_{2n+1}$ is an odd-dimensional unitary group.

Finally, if $G = \mathrm{SU}(V,q)$ is a unitary group, where V is a L-vector space and $q\colon V \to L$ is a semi-regular L-hermitian form, we follow the implicit argument that had already been covered in [Zhu14, Lemma 8.3] for $p > 2$, but now for all primes. Namely, in Lemma 4.18 below, we will consider the natural map of K-groups

$$\mathrm{SU}(V,q) \to \mathrm{SL}({}_KV) \tag{4.21}$$

where ${}_K V$ is V regarded as a K-vector space, and construct a certain non-degenerate quadratic form $r\colon {}_K V \to K$ such that the above map factors through $\mathrm{SO}({}_K V, r)$. Notice this solves our problem of constructing a line bundle $\mathcal{L}$ satisfying $c(\mathcal{L}) = 1$, since the determinant has a square root given by the Pfaffian, see [BD91, Section 4.2] and especially [BD91, Section 4.2.16] when $p = 2$. □

The following lemma is used towards the end in the proof of Lemma 4.17.

Lemma 4.18. *Let L/K be a quadratic extension, V a L-vector space and $q\colon V \to L$ a semi-regular L-hermitian form. There is a non-degenerate quadratic form $r\colon {}_K V \to K$ such that $\mathrm{SU}(V, q)$ lies inside $\mathrm{SO}({}_K V, r)$.*

Proof. If $p > 2$, this is a well-known result in the theory of L-sesquilinear and K-bilinear forms, see [PR09, Section 1.2.2], so from now on we assume $p = 2$.

Decomposing into orthogonal summands, we may assume either

$$(V, q) = (L, x \mapsto N(x)) \tag{4.22}$$

is one-dimensional semi-regular, or

$$(V, q) = (L^2, (x, y) \mapsto x\sigma(y) + \sigma(x)y) \tag{4.23}$$

is a two-dimensional regular hermitian hyperbolic plane.

In the first case, taking

$$r = \mathrm{tr}(\lambda q)\colon (x_1, x_2) \mapsto x_1^2 + x_1 x_2 + N(\lambda) x_2^2, \tag{4.24}$$

where $\mathrm{tr}(\lambda) = 1$, gives us a regular symmetric K-hyperbolic plane, as $1 - 4N(\lambda) = 1 \neq 0$ as $p = 2$. As for the second case, the quadratic form

$$r = \mathrm{tr}(\lambda q)\colon (x_1, x_2, y_1, y_2) \mapsto x_2 y_1 + x_1 y_2 \tag{4.25}$$

clearly decomposes into the orthogonal sum of two regular symmetric K-hyperbolic planes. □

Remark 4.19. The construction of $\mathcal{L}_{\mathrm{crit}}$ on the *seminormalized* affine flag variety is used in order to apply the Mehta–Ramanathan criterion. It would be interesting to find a uniform proof for all G and p. Recall that [Fal03, Theorem 7] provides a construction for split G, which is extended in [Lou23, Corollary 4.3.10] for tame G, using negative loops groups that seem, however, not to exist for wildly ramified G. Also, the work [PR08] refers to a construction in [Gör01, Proposition 3.19] for $G = \mathrm{GL}_n$, which we were not able to generalize to other groups.

Now, we are ready to finish the proof of Theorem 4.1. Let $f\colon D_{\dot{w}} \to \widetilde{S}_w$ be the Demazure resolution, compare (4.5). The anti-canonical line bundle admits the formula

$$\omega_{D_{\dot{w}}}^{-1} = \mathcal{O}(\partial D_{\dot{w}}) \otimes f^* \mathcal{L}_{\mathrm{crit}} \tag{4.26}$$

by the argument of [BK05, Proposition 2.2.2], and the fact that $\mathcal{L}_{\mathrm{crit}}$ has degree 1 on every projective line S_s. To apply the Mehta–Ramanathan criterion, see [BS13, Theorem 5.3.1] and [Cas22, Proof of Theorem 5.8], we must produce a section of the $(q-1)$-th power of $\mathcal{L}_{\mathrm{crit}}$ (for some power q of p) avoiding the origin (i.e., the intersection of all the divisors $D_{\dot{v}}$). Note that $\mathcal{L}_{\mathrm{crit}}$ is an ample line bundle, because so is its square $\mathcal{L}_{\mathrm{ad}}$. We deduce that any sufficiently large power of $\mathcal{L}_{\mathrm{crit}}$ is very ample on $\widetilde{S}_w$, and therefore $f^* \mathcal{L}_{\mathrm{crit}}^{q-1}$ will be basepoint free for some sufficiently large power $q \gg 0$ of p.

4.1.5. *Picard groups of seminormalized Schubert varieties.* Using the already proven Theorem 4.1, we can actually upgrade the previous results on Picard groups to seminormalized Schubert varieties.

Lemma 4.20. *There is an isomorphism*

$$\mathrm{Pic}(\widetilde{S}_w) \xrightarrow{\cong} \bigoplus_s \mathbb{Z}, \quad \mathcal{L} \mapsto (\deg(\mathcal{L}|_{S_s}))_s \tag{4.27}$$

where the sum runs over all $s \in \{s_1, \ldots, s_d\}$ with $s \notin W_{\mathcal{G}}$.

Proof. Recall the notation $f\colon D_{\dot{w}} \to \widetilde{S}_w$ for the Demazure resolution from (4.5) and the computation of $\mathrm{Pic}(D_{\dot{w}})$ from Lemma 4.7. As explained in Lemma 4.8, the pullback map $\mathrm{Pic}(\widetilde{S}_w) \to \bigoplus_s \mathbb{Z}$ is injective. For surjectivity, let $(\lambda_s) \in \oplus_s \mathbb{Z}$ and denote by $\mathcal{D} = \mathcal{D}(\lambda_s)$ the corresponding line bundle on $D_{\dot{w}}$.

We show that $\mathcal{L} := f_*\mathcal{D}$ is a line bundle, and that the canonical map $f^*\mathcal{L} \to \mathcal{D}$ is an isomorphism. As in the proof of Lemma 4.5 we factor f into successive partial Demazure resolutions, each having fibers of dimension at most 1. By induction we replace f by one of those maps $g\colon S_s \tilde{\times} \widetilde{S}_v \to \widetilde{S}_w$. By the proof of Lemma 4.8, we already know that the restriction of $\mathcal{D}$ to the fibers of g is trivial after passing to perfections. By the proof of Lemma 4.5, we know that the fibers of g are either $\mathrm{Spec}(\kappa(x))$ or $\mathbb{P}^1_{\kappa(x)}$, so their Picard groups are torsion-free and $\mathcal{D}$ has trivial restriction to all fibers of g. By Theorem 4.1, our varieties have rational singularities[1], so [Lip69, Theorem 12.1 (i)] applies to show that $\mathcal{D}$ is Zariski locally trivial on the base. Using rational singularities again shows $g_*\mathcal{D}$ is a line bundle, and that $g^*\mathcal{L} \to \mathcal{D}$ is an isomorphism. □

Corollary 4.21. *There is an isomorphism*

$$\mathrm{Pic}(\widetilde{\mathcal{F}\ell}^0_{\mathcal{G}}) \xrightarrow{\cong} \bigoplus_s \mathbb{Z}, \quad \mathcal{L} \mapsto (\deg(\mathcal{L}|_{S_s}))_s \tag{4.28}$$

where $\widetilde{\mathcal{F}\ell}^0_{\mathcal{G}}$ denotes the neutral component and the sum runs over all simple reflections $s \in W_{\mathrm{af}} \backslash W_{\mathcal{G}}$.

Proof. This is immediate from Lemma 4.20, as $\mathrm{Pic}(\widetilde{S}_w)$ is again independent of w for sufficiently large lengths by (4.27). □

Lemma 4.20 admits the following slight generalization (see Proposition 4.23) which is used in Section 5. We first need an elementary lemma:

Lemma 4.22. *Finite unions of seminormalized Schubert varieties in $\widetilde{\mathcal{F}\ell}_{\mathcal{G}}$ are seminormal and stable under finite intersections.*

Proof. Due to the compatible F-splitting of seminormalized Schubert varieties from Theorem 4.1, their finite union (and, finite intersection) is again F-split, hence F-injective (and reduced) and therefore seminormal by [Sch09, Theorem 4.7]. In particular, if $S_{w_1}, \ldots, S_{w_n} \subset \mathcal{F}\ell_{\mathcal{G}}$ are Schubert varieties, then the maps $\cup_{i=1}^n \widetilde{S}_{w_i} \to \cup_{i=1}^n S_{w_i}$ and $\cap_{i=1}^n \widetilde{S}_{w_i} \to \cap_{i=1}^n S_{w_i}$ are universal homeomorphisms and induce isomorphisms on all residue fields, and so identify the respective sources as the seminormalizations of their targets. The lemma follows. □

Proposition 4.23. *Let $w_1, \ldots, w_n \in \widetilde{W}$ be right $W_{\mathcal{G}}$-minimal. There is an isomorphism*

$$\mathrm{Pic}\left(\bigcup_{i=1}^n \widetilde{S}_{w_i}\right) \xrightarrow{\cong} \bigoplus_s \mathbb{Z}, \quad \mathcal{L} \mapsto (\deg(\mathcal{L}|_{S_s}))_s \tag{4.29}$$

[1] Strictly speaking, Theorem 4.1 only refers to the $\widetilde{S}_w$ and not their partial Demazure resolutions, but the proof given in the previous section proceeds by descent from $D_{\dot{w}}$, so those also have rational singularities.

where the sum runs over all $s \in \widetilde{W}\backslash W_{\mathcal{G}}$ of length 1 such that $s \leq w_i$ for some $i = 1, \dots, n$.

Proof. Without loss of generality, we may and do assume that $\bigcup_{i=1}^{n} \widetilde{S}_{w_i}$ is connected and contained in the neutral component $\widetilde{\mathcal{F}\ell}_{\mathcal{G}}^{0}$. Next, we proceed by induction on $n \geq 1$. For $n = 1$, this is Lemma 4.20. For the induction step, let $X = \cup_{i=1}^{n-1} \widetilde{S}_{w_i}$ and $Y = \widetilde{S}_{w_n}$ viewed as closed subschemes of $\widetilde{\mathcal{F}\ell}_{\mathcal{G}}$. The sequence of sheaves of abelian groups on $\widetilde{\mathcal{F}\ell}_{\mathcal{G}}$

$$1 \longrightarrow \iota_{X\cup Y,*}\mathcal{O}_{X\cup Y}^{\times} \longrightarrow \iota_{X,*}\mathcal{O}_{X}^{\times} \times \iota_{Y,*}\mathcal{O}_{Y}^{\times} \overset{(a,b)\mapsto ab^{-1}}{\longrightarrow} \iota_{X\cap Y,*}\mathcal{O}_{X\cap Y}^{\times} \longrightarrow 1 \tag{4.30}$$

is exact as is easily checked on stalks, where $\iota_{(\text{-})}$ denotes the respective closed immersion into $\widetilde{\mathcal{F}\ell}_{\mathcal{G}}$. Since $X \cap Y$ is reduced (because seminormal) by Lemma 4.22, we see $\mathrm{H}^0(X \cap Y, \mathcal{O}_{X\cap Y}^{\times}) = k^{\times}$ by connectedness and projectivity of $X \cap Y$. Hence, the long exact (Zariski) cohomology sequence associated with (4.30) identifies $\mathrm{Pic}(X \cup Y) = \mathrm{H}^1(X \cup Y, \mathcal{O}_{X\cup Y}^{\times})$ with $\mathrm{Pic}(X) \times_{\mathrm{Pic}(X\cap Y)} \mathrm{Pic}(Y)$. One easily deduces (4.29) which finishes the induction step. □

4.1.6. *Vanishing of higher coherent cohomology of seminormalized Schubert varieties.* Another consequence of Theorem 4.1 is the following result, to be used in Section 5 below:

Lemma 4.24. *Let $w_1, \dots, w_n \in W_{\mathrm{af}}$ be right $W_{\mathcal{G}}$-minimal, and consider $X = \cup_{i=1}^{n} \widetilde{S}_{w_i}$. Then $\mathrm{H}^j(X, \mathcal{O}_X) = 0$ for all $j \geq 1$.*

Proof. By Lemma 4.22 finite unions of seminormalized Schubert varieties are stable under intersections. Hence, a Mayer–Vietoris argument similar to that in Proposition 4.23 reduces the claim to the case $n = 1$. Consider the Demazure resolution $f\colon D_{\dot{w}} \to \widetilde{S}_w$ from (4.5). Now, $\widetilde{S}_w$ has rational singularities by Theorem 4.1, so $\mathrm{H}^j(\widetilde{S}_w, \mathcal{O}_{\widetilde{S}_w}) = \mathrm{H}^j(D_{\dot{w}}, \mathcal{O}_{D_{\dot{w}}})$ using $Rf_*\mathcal{O}_{D_{\dot{w}}} = \mathcal{O}_{\widetilde{S}_w}$. Since $D_{\dot{w}}$ is an iterated $\mathbb{P}^1_k$-bundle, the vanishing of higher cohomology follows by a straightforward induction argument. □

4.2. **Normality of Schubert varieties.** In this subsection, we extend the normality theorem for Schubert varieties to some wildly ramified groups. Previously, this was proved by Faltings for split groups, see [Fal03, Theorem 8], and by Pappas–Rapoport for Weil-restricted tame groups, see [PR08, Theorem 8.4]. These results were inspired by similar ones in Kac–Moody theory found in [Mat89], but we stress that wildly ramified groups are in principle unrelated to that theory, compare with [Lou23, Annexe A]. The prime-to-p hypothesis on the order of $\pi_1(G_{\mathrm{der}})$ is essential, due to [HLR24, Theorem 2.5].

Theorem 4.25. *Under Hypothesis 2.1, all Schubert varieties S_w are normal if and only if p does not divide the order of $\pi_1(G_{\mathrm{der}})$.*

We need the following auxiliary lemma:

Lemma 4.26. *If G is simply connected and satisfies Hypothesis 2.1, then $\mathcal{F}\ell_{\mathcal{G}}$ is reduced.*

Proof. This is proven in [PR08, Proposition 9.9] for tamely ramified groups and extends to wildly ramified groups under Hypothesis 2.1. We recall the proof for convenience, following closely [PR08, Proposition 9.9].

By [HLR24, Lemma 8.6], it is enough to show that every R-valued point x of $\mathcal{F}\ell_{\mathcal{G}}$, with R being Artinian and strictly Henselian, factors through the reduced locus. By the Bruhat decomposition and formal smoothness of $L^+\mathcal{G}$, we can translate x such that it is supported at the origin $e \in \mathcal{F}\ell_{\mathcal{G}}(k)$. After extending scalars, we may assume that the residue field of R equals k. Moreover, we can use formal smoothness of $L^+\mathcal{G}$ and the fact that R is strictly Henselian to lift x to an R-valued point $\tilde{x}$ of LG supported at the identity. This corresponds to an $R(\!(t)\!)$-valued point of G supported at the identity, so it factors through the big cell $C = U^- \times T \times U^+$. We claim that $\tilde{x}$ is in the subgroup generated by $LU^{\pm}(R)$. Since the ind-schemes $LU^{\pm}$ are reduced, they map to $(\mathcal{F}\ell_{\mathcal{G}})_{\mathrm{red}}$. Hence, we may and do assume that $\tilde{x} \in LT$. But T factors as a product of induced tori indexed by its relative coroots, and thus we can further reduce to the case when G has rank 1. Supressing the wildly ramified restrictions of scalars, then either $G = \mathrm{SL}_2$ or SU_3 and $p \neq 2$ and the needed generation property is explicitly calculated in the proof of [PR08, Proposition 9.3]. So x lies in the reduced locus, and the lemma follows. □

Proof of Theorem 4.25. The seminormalization $\widetilde{S}_w \to S_w$ is proper and surjective, hence an isomorphism if and only if it is a monomorphism (as S_w is reduced). So all Schubert varieties S_w are seminormal (hence normal by Lemma 4.4) if and only if the morphism of ind-schemes

$$\widetilde{\mathcal{F}\ell}_{\mathcal{G}} = \operatorname{colim} \widetilde{S}_w \to \operatorname{colim} S_w = (\mathcal{F}\ell_{\mathcal{G}})_{\mathrm{red}} \subset \mathcal{F}\ell_{\mathcal{G}} \tag{4.31}$$

is a monomorphism, or equivalently, its restriction to the neutral components is so. Using this we prove the theorem as follows.

For the if clause, by [PR08, Section 6.a] we may and do assume that G is simply connected, absolutely almost simple and $\mathcal{G}$ is an Iwahori model. In this case, we claim that (4.31) is an isomorphism. Now observe that by Proposition 2.6, we can find a smooth affine $W(k)[\![t]\!]$-group $\underline{\mathcal{G}}$ with connected fibers lifting $\mathcal{G}$, such that it becomes parahoric as well over $K_0[\![t]\!]$ with $K_0 = W(k)[p^{-1}]$. Hence, (4.31) lifts to a morphism of $W(k)$-ind-schemes

$$\widetilde{\mathcal{F}\ell}_{\underline{\mathcal{G}}} \to \mathcal{F}\ell_{\underline{\mathcal{G}}}, \tag{4.32}$$

where the left side is the ind-normalization of the right side. Indeed, that this commutes with base change to k is a consequence of Theorem 4.1 thanks to the vanishing of higher coherent cohomology of the Demazure resolution, by an application of cohomology and base change, compare with [Fal03, page 52] and [Gör03, Proposition 3.13]. Over K_0, we get an isomorphism by Kac–Moody theory, see [PR08, Section 9.f]. Integrally, we show that the map is formally smooth around the origin, by virtue of an analogue of [Fal03, Lemma 10] or [PR08, Proposition 9.3]. This implies the claim by [Fal03, page 53] or [PR08, Section 9.g].

The only if part follows from the argument in [HLR24, Section 2], because if p divides the order of $\pi_1(G_{\mathrm{der}})$ then the kernel of $G_{\mathrm{sc}} \to G$ is not étale. Hence, the induced morphism $\mathcal{F}\ell_{\mathcal{G}_{\mathrm{sc}}} \to \mathcal{F}\ell_{\mathcal{G}}^0 \subset \mathcal{F}\ell_{\mathcal{G}}$ is not a monomorphism, where $\mathcal{G}_{\mathrm{sc}}$ denotes the parahoric O-model of G_{sc} induced by $\mathcal{G}$. By Lemma 4.26, $\mathcal{F}\ell_{\mathcal{G}_{\mathrm{sc}}}$ is reduced, so (4.31) factors on neutral components as $\widetilde{\mathcal{F}\ell}_{\mathcal{G}}^0 \xrightarrow{\sim} \mathcal{F}\ell_{\mathcal{G}_{\mathrm{sc}}} \to (\mathcal{F}\ell_{\mathcal{G}})^0_{\mathrm{red}}$. Now, if (4.31) were a monomorphism, then $\mathcal{F}\ell_{\mathcal{G}_{\mathrm{sc}}} \to \mathcal{F}\ell_{\mathcal{G}}^0$ would be a monomorphism, which is a contradiction. □

4.3. **Central extensions of line bundles.** In the theory of loop groups and their flag varieties, one is usually faced with the obstacle that not every line bundle on $\mathcal{F}\ell_{\mathcal{G}}$ is LG-equivariant. However, this can partially remedied by considering a certain universal central extension of LG that acts on every line bundle of $\mathcal{F}\ell_{\mathcal{G}}$. This is a recurrent theme in Kac–Moody theory, see [Fal03, page 54], [PR08, Remark 10.2] and [Lou23, Corollary 4.3.11], and also admits an incarnation for the Witt vector Grassmannian by [BS17, Proposition 10.3]. In order to properly explain it, we need to use the geometric results of the previous subsections.

Given a line bundle $\mathcal{L}$ on $\mathcal{F}\ell_{\mathcal{G}}$, we form the group functor on the category of k-algebras R defined by

$$LG\{\mathcal{L}\}(R) \;=\; \{(g, \alpha) \,|\, g \in LG(R),\ \alpha\colon \mathcal{L} \cong g^*\mathcal{L}\}. \tag{4.33}$$

We can now prove the following lemma:

Lemma 4.27. *Suppose G is an almost simple, simply connected K-group satisfying Hypothesis 2.1. Then, the pre-sheaf $LG\{\mathcal{L}\}$ defines a central extension of LG by $\mathbb{G}_{m,k}$ in the category of* *ind-affine k-group ind-schemes. The association $\mathcal{L} \mapsto LG\{\mathcal{L}\}$ induces a group homomorphism*

$$\mathrm{Pic}(\mathcal{F}\ell_{\mathcal{G}}) \to \mathrm{Ext}_{\mathrm{cent}}(LG, \mathbb{G}_{m,k}). \tag{4.34}$$

with the same kernel as (4.15) restricted to $\mathrm{Pic}(\mathcal{F}\ell_{\mathcal{G}})$.

Proof. Note that $LG\{\mathcal{L}\}(R)$ carries a natural group structure via $(g_1, \alpha_1)\cdot(g_2, \alpha_2) = (g_1 g_2, g_2^* \alpha_1 \circ \alpha_2)$, thus having $\mathbb{G}_{m,k}(R) = \{(1, c) \,|\, c \in R^\times\}$ as a central subgroup. We claim moreover that $\mathbb{G}_{m,k}(R) \subset LG\{\mathcal{L}\}(R)$ is the kernel of the natural projection to $LG(R)$. In other words, we claim that the automorphism group $\mathrm{Aut}(\mathcal{L}_R)$ as a line bundle on $\mathcal{F}\ell_{\mathcal{G},R}$ equals $R^\times$. After tensoring with $\mathcal{L}^{-1}$, we may and do assume that $\mathcal{L} = \mathcal{O}$. Thus, it suffices to show that $H^0(\mathcal{F}\ell_{\mathcal{G},R}, \mathcal{O}) = R$ which is implied by Lemma 4.26.

Next, we study the action of $LG(R)$ on the Picard groups. Note that $\mathrm{Pic}(\mathcal{F}\ell_{\mathcal{G},R})$ is the direct sum of $\mathrm{Pic}(R)$ and $\mathrm{Pic}(\mathcal{F}\ell_{\mathcal{G}})$, since the Picard functor of the flag variety is constant étale due to [Kle05, Corollary 5.13] using Lemma 4.24. The action of $LG(R)$ on $\mathrm{Pic}(R)$ is trivial, and we claim that the same holds for the quotient $\mathrm{Pic}(\mathcal{F}\ell_{\mathcal{G},R})/\mathrm{Pic}(R)$. By Theorem 4.25 and (4.28), that quotient is torsion-free and we may check triviality of the $LG(R)$-action on generators of the associated $\mathbb{Q}$-vector space. A set of generators is given by LG-equivariant line bundles, see Lemma 4.10, and the adjoint line bundle. For an LG-equivariant line bundle, the claim is trivial and we even see directly that $LG\{\mathcal{L}\} \to LG$ splits and thus is the trivial extension. For the adjoint line bundle, one sees that the difference

$$\mathcal{L}_{\mathrm{det}}^{-1} \cdot g^* \mathcal{L}_{\mathrm{det}} = \det(t^{-a} R[\![t]\!]^n / g R[\![t]\!]^n) \in \mathrm{Pic}(\mathcal{F}\ell_{\mathrm{SL}_n, R}) \tag{4.35}$$

for $a \gg 0$ is in the image of $\mathrm{Pic}(R)$, compare [Fal03, page 43], so the same remains true after pulling back to $\mathcal{F}\ell_{\mathcal{G},R}$.

We can use the previous paragraph to show that any R-valued point of LG lifts along the map $LG\{\mathcal{L}\} \to LG$ after we replace $\mathrm{Spec}\, R$ by a finite union of affine opens. Indeed, we saw above that $\mathcal{L}$ and $g^*\mathcal{L}$ differ by an element of $\mathrm{Pic}(R)$ which can be trivialized over an affine open $\mathrm{Spec}\, S \subset \mathrm{Spec}\, R$. Replacing R by S, we may assume the existence of an isomorphism $\alpha\colon \mathcal{L} \cong g^*\mathcal{L}$, thereby producing a lift in $LG\{\mathcal{L}\}(S)$. Letting $\mathrm{Spec}\, R$ run over sufficiently small affine opens of a presentation of LG, the existence of lifts shows that $LG\{\mathcal{L}\}$ is representable by an ind-affine k-group ind-scheme and that it is an extension of LG by $\mathbb{G}_{m,k}$. Finally, it is clear that the kernel of (4.34) consists of those $\mathcal{L}$ that admit an LG-equivariant structure, hence coincides with the kernel of (4.15) after restricting the latter to $\mathrm{Pic}(\mathcal{F}\ell_{\mathcal{G}})$ thanks to Lemma 4.10. □

The lemma implies that the image of (4.34) is a free $\mathbb{Z}$-module of rank 1, see Lemma 4.12. Identify the image with $\mathbb{Z}$ via the unique isomorphism sending ample line bundles to positive integers.

Corollary 4.28. *For any $\mathcal{L} \in \mathrm{Pic}(\mathcal{F}\ell_{\mathcal{G}})$ with $c_{\mathcal{L}} = 1$, the resulting central extension $\widehat{LG} := LG\{\mathcal{L}\}$ has the property that every line bundle on $\mathcal{F}\ell_{\mathcal{G}}$ admits a $\widehat{LG}$-equivariant structure which is unique up to multiplication by $k^\times$.*

Proof. Using Lemma 4.10, the central charge induces a short exact sequence $0 \to \mathrm{Pic}([LG \backslash \mathcal{F}\ell_{\mathcal{G}}]) \to \mathrm{Pic}(\mathcal{F}\ell_{\mathcal{G}}) \xrightarrow{c} \mathbb{Z} \to 0$. The choice of $\mathcal{L}$ provides a splitting. So the corollary follows from the equality $\mathrm{Aut}(\mathcal{M}) = k^\times$ for any line bundle $\mathcal{M}$ on $\mathcal{F}\ell_{\mathcal{G}}$, see the proof of Lemma 4.27. □

5. Local models

In this final section, let O be a complete discretely valued ring with fraction field K and perfect residue field k of characteristic $p > 0$. Let G be a reductive K-group, μ a (not necessarily minuscule) geometric conjugacy class of cocharacters in G and $\mathcal{G}$ a parahoric O-model of G. The reflex field E of μ is a finite separable field extension of K with ring of integers O_E and residue field k_E.

Let $\breve{O}$ be the completed strict Henselisation of O with fraction field $\breve{K}$ and algebraically closed residue field $\bar{k}$. Let T be the centralizer of some maximal $\breve{K}$-split torus S which is defined over K and contains a maximal K-split torus with apartment containing the facet associated with $\mathcal{G}$, see [BT84, Corollaire 5.1.12]. The connected Néron model $\mathcal{T}$ of T is a closed subgroup scheme of $\mathcal{G}$.

5.1. **Equicharacteristic local models.** Assume $K = k(\!(t)\!)$ is a Laurent series field with ring of integers $O = k[\![t]\!]$. Let us recall the definition of local models in equicharacteristic, which only depend on the pair $(\mathcal{G}, \mu)$ and not on additional auxiliary choices. Recall that we have defined the Beilinson–Drinfeld Grassmannian $\mathrm{Gr}_{\mathcal{G}} \to \operatorname{Spec} O$ before Lemma 4.15. Its generic fiber is equivariantly isomorphic to the affine Grassmannian $\mathrm{Gr}_G \to \operatorname{Spec} K$ whereas its special fiber is equal to the affine flag variety $\mathcal{F}\ell_{\mathcal{G}} \to \operatorname{Spec} k$. Let $S_{G,\mu} \subset \mathrm{Gr}_G \times_{\operatorname{Spec} K} \operatorname{Spec} E$ be the Schubert variety attached to μ.

Definition 5.1. Let $M_{\mathcal{G},\mu}$ denote the flat closure of $S_{G,\mu}$ inside the Beilinson–Drinfeld Grassmannian $\mathrm{Gr}_{\mathcal{G},O_E} := \mathrm{Gr}_{\mathcal{G}} \times_{\operatorname{Spec} O} \operatorname{Spec} O_E$. We denote by $\widetilde{M}_{\mathcal{G},\mu}$ its seminormalization [Sta23, 0EUK].

Remark 5.2. The formation of orbit closures and their seminormalizations are functorial in the following sense. A morphisms of pairs $(\mathcal{G}, \mu) \to (\widetilde{\mathcal{G}}, \widetilde{\mu})$ is a map of O-group schemes $\mathcal{G} \to \widetilde{\mathcal{G}}$ which maps μ into $\widetilde{\mu}$ under the induced map of reductive K-groups $G \to \widetilde{G}$ in the generic fiber. Any such map of pairs induces a map $M_{\mathcal{G},\mu} \to M_{\widetilde{\mathcal{G}},\widetilde{\mu}}$ commuting over $\operatorname{Spec} O_E \to \operatorname{Spec} O_{\widetilde{E}}$ where $\widetilde{E}$ denotes the reflex field of $\widetilde{\mu}$. By functoriality of seminormalizations [Sta23, Tag 0EUS], we get a map $\widetilde{M}_{\mathcal{G},\mu} \to \widetilde{M}_{\widetilde{\mathcal{G}},\widetilde{\mu}}$ commuting over the map of orbit closures.

In order to describe the special fiber of the schemes from Definition 5.1, we recall the admissible locus [PRS13, Section 4.3]. The Kottwitz homomorphism induces an isomorphism $X_*(T)_I \cong T(\breve{K})/\mathcal{T}(\breve{O}), \bar{\lambda} \mapsto \bar{\lambda}(t)$ where the source denotes the coinvariants of the cocharacter lattice $X_*(T)$ under the inertia subgroup I of the absolute Galois group of K. Note that the isomorphism does not depend on the choice of uniformizer t.

Definition 5.3. The admissible locus $A_{\mathcal{G},\mu}$ is the reduced k_E-subscheme of $\mathcal{F}\ell_{\mathcal{G},k_E}$ given by the k_E-descent of the union of $\bar{k}$-Schubert varieties $S_{\bar{\lambda}(t)}$, where $\lambda \in X_*(T)$ runs through the (finitely many) representatives of μ and where $\bar{\lambda} \in X_*(T)_I$ denotes its image in the coinvariants under I. We denote by $\widetilde{A}_{\mathcal{G},\mu}$ its seminormalization.

Note that $A_{\mathcal{G},\mu}$ does not depend on the choice of the maximal torus T as above. Further, $A_{\mathcal{G},\mu}$ is geometrically connected and, by [Hai18, Theorem 4.2], its irreducible $\bar{k}$-components are the Schubert varieties $S_{\bar{\lambda}(t)}$ where λ runs through the $\breve{K}$-rational representatives of μ in $X_*(T)$.

Let us now discuss finer geometric properties. It was shown in [HR21, Theorem 6.12] that the reduced special fiber of $M_{\mathcal{G},\mu}$ coincides with $A_{\mathcal{G},\mu}$, but we shall only need to use the inclusion of $A_{\mathcal{G},\mu}$ in the reduced special fiber, already proved in [Ric16, Lemma 3.12]. Note that $(\widetilde{A}_{\mathcal{G},\mu})_{\bar{k}} = \cup_\lambda \widetilde{S}_{\bar{\lambda}(t)}$ by Lemma 4.22, where λ ranges over the $\breve{K}$-rational representatives of μ in $X_*(T)$. Since the F-split property for proper schemes can be descended from $\bar{k}$ to k_E, $\widetilde{A}_{\mathcal{G},\mu}$ is F-split. It identifies moreover with the admissible locus $A_{\widetilde{\mathcal{G}},\widetilde{\mu}}$ associated with any z-extension $\widetilde{G}$ of G with simply connected derived group, and any lift $\widetilde{\mu}$ of μ, by Theorem 4.25, at least when Hypothesis 2.1 holds.

Now, we may state our main result on the singularities of local models.

Theorem 5.4. *Under Hypothesis 2.1, the local model $\widetilde{M}_{\mathcal{G},\mu}$ is Cohen–Macaulay, has F-rational singularities (and thus is pseudo-rational), and has reduced special fiber equal to the seminormalized admissible locus $\widetilde{A}_{\mathcal{G},\mu}$.*

Proof. The key step of the proof is showing that the special fiber is reduced and equal to $\widetilde{A}_{\mathcal{G},\mu}$ from which the other properties follow by using the F-splitness of $\widetilde{A}_{\mathcal{G},\mu}$; in fact, we shall prove that $\widetilde{M}_{\mathcal{G},\mu}$ has F-rational singularities. This part of the proof essentially follows from [Zhu14, Section 4.2], relying on Theorem 4.25 for wildly ramified groups. Here is an outline. By using faithfully flat descent of F-rationality [DM20, Proposition A.5] we may reduce to the case $O = \breve{O}$, so G is quasi-split.

First, we show that for any finite field extension $\widetilde{E}/E$ the base change $\widetilde{M}_{\mathcal{G},\mu} \otimes_{\mathcal{O}_E} \mathcal{O}_{\widetilde{E}}$ is normal with reduced special fiber equal to $\widetilde{A}_{\mathcal{G},\mu}$ as follows. Passage to the adjoint group induces a map of pairs $(\mathcal{G},\mu) \to (\mathcal{G}_{\mathrm{ad}},\mu_{\mathrm{ad}})$ where $\mathcal{G}_{\mathrm{ad}}$ is the parahoric associated with G_{ad} and μ_{ad} is induced by μ under $G \to G_{\mathrm{ad}}$. The corresponding map $\widetilde{M}_{\mathcal{G},\mu} \to \widetilde{M}_{\mathcal{G}_{\mathrm{ad}},\mu_{\mathrm{ad}}} \otimes_{O_{E_{\mathrm{ad}}}} O_E$ is a universal homeomorphism inducing isomorphisms on residue fields by [HR22, Corollary 2.3 and its proof], thus an isomorphism if the target is (semi-)normal. Without loss of generality, we reduce to the case where G is adjoint. A similar argument shows that the formation of $\widetilde{M}_{\mathcal{G},\mu}$ commutes with products in G, so we first assume that G is adjoint and simple, so $G = \mathrm{Res}_{L/K}(G_0)$ for a finite separable field extension L/K (necessarily totally ramified) and an absolutely simple L-group G_0.

The simply connected cover $G_{\mathrm{sc}} \to G$ induces a universally closed and universally injective morphism $\iota\colon \mathrm{Gr}_{\mathcal{G}_{\mathrm{sc}}} \to \mathrm{Gr}_{\mathcal{G}}$ which gives on generic fibers the universal homeomorphism $\mathrm{Gr}_{G_{\mathrm{sc}}} \to \mathrm{Gr}_G^0$ onto the neutral component. We consider the translate $t_\mu^{-1} M_{\mathcal{G},\mu} \subset \iota(\mathrm{Gr}_{\mathcal{G}_{\mathrm{sc}},O_E})$, where t_μ is an O_E-valued point of $L\mathcal{T}$ lifting the corresponding section of $\mathrm{Gr}_{\mathcal{T}}$, and consider the unique reduced closed subscheme $M_{\mathcal{G}_{\mathrm{sc}},\mu} \subset \mathrm{Gr}_{\mathcal{G}_{\mathrm{sc}},O_E}$ with the topological space $\iota(M_{\mathcal{G}_{\mathrm{sc}},\mu})$ being the same as the translation. Likewise, we denote by $A_{\mathcal{G}_{\mathrm{sc}},\mu}$ (respectively, $S_{G_{\mathrm{sc}},\mu}$) the t_μ-translated admissible locus inside $\mathcal{F}\ell_{\mathcal{G}_{\mathrm{sc}}}$ (respectively, $\mathrm{Gr}_{G_{\mathrm{sc}},E}$). These are also unions of translates of Schubert varieties for some choice of Iwahori group scheme. The induced finite universal homeomorphism $M_{\mathcal{G}_{\mathrm{sc}},\mu} \to M_{\mathcal{G},\mu}$ factors on generic fibers as $S_{G_{\mathrm{sc}},\mu} \cong \widetilde{S}_{G,\mu} \to S_{G,\mu}$ (hence is birational).

We will prove that for all $n \geq 1$, we have

$$\dim_k H^0(A_{\mathcal{G}_{\mathrm{sc}},\mu}, \mathcal{L}_{\mathrm{ad}}^{\otimes n}) = \dim_E H^0(S_{G_{\mathrm{sc}},\mu}, \mathcal{L}_{\mathrm{ad}}^{\otimes n}), \tag{5.1}$$

where $\mathcal{L}_{\mathrm{ad}}$ denotes the pullback of the determinant line bundle along the adjoint representation, compare Lemma 4.14. But before we do so let us explain how it implies that $\widetilde{M}_{\mathcal{G},\mu}$ has special fiber equal to $\widetilde{A}_{\mathcal{G},\mu}$. By [Ric16, Lemma 3.12], we have an inclusion of $A_{\mathcal{G}_{\mathrm{sc}},\mu}$ in the reduced special fiber of $M_{\mathcal{G}_{\mathrm{sc}},\mu}$. Since $\mathcal{L}_{\mathrm{ad}}$ is a relatively ample line bundle on $M_{\mathcal{G}_{\mathrm{sc}},\mu}$, (5.1) implies that the special fiber of $M_{\mathcal{G}_{\mathrm{sc}},\mu}$ is reduced and equal to $A_{\mathcal{G}_{\mathrm{sc}},\mu}$. By Serre's criterion (see [PZ13, Proposition 9.2]) it follows that $M_{\mathcal{G}_{\mathrm{sc}},\mu}$ is normal. Consequently, as the map $M_{\mathcal{G}_{\mathrm{sc}},\mu} \to M_{\mathcal{G},\mu}$ induces an isomorphism on every residue field, it identifies with the seminormalization, so induces an isomorphism $M_{\mathcal{G}_{\mathrm{sc}},\mu} \cong \widetilde{M}_{\mathcal{G},\mu}$. Using the normality of Schubert varieties for simply connected groups in Theorem 4.25 we then see that the special fiber of $\widetilde{M}_{\mathcal{G},\mu}$ is $\widetilde{A}_{\mathcal{G},\mu}$.

It remains to prove (5.1). For this, consider the $W(k)[\![t]\!]$-lift $\underline{\mathcal{G}_{\mathrm{sc}}}$ of $\mathcal{G}_{\mathrm{sc}}$ provided by Proposition 2.6 under our Hypothesis 2.1, which holds for $\Phi_{G_{\mathrm{sc}}}$. Consider the affine flag scheme $\mathcal{F}\ell_{\underline{\mathcal{G}_{\mathrm{sc}}}}$ over $W(k)$. It admits the flat, closed subscheme $A_{\underline{\mathcal{G}_{\mathrm{sc}}},\mu}$ whose generic fiber is $A_{\mathcal{G}'_{\mathrm{sc}},\mu'}$ with $\mathcal{G}'_{\mathrm{sc}} = \underline{\mathcal{G}_{\mathrm{sc}}} \otimes K_0[\![t]\!]$ and μ' corresponding to μ using (2.11), and whose special fiber contains $A_{\mathcal{G}_{\mathrm{sc}},\mu}$. As explained in the last paragraph of the proof of [Lou23, Théorème 5.2.1], one deduces from the combinatorics of Schubert varieties and their compatible F-splitness an equality

$$\dim_k H^0(A_{\mathcal{G}_{\mathrm{sc}},\mu}, \mathcal{L}_{\mathrm{ad}}^{\otimes n}) = \dim_{K_0} H^0(A_{\mathcal{G}'_{\mathrm{sc}},\mu'}, \mathcal{L}_{\mathrm{ad}}^{\otimes n}), \tag{5.2}$$

for all $n \geq 1$. Note that (5.2) uses again the normality of Iwahori Schubert varieties for simply connected groups (Theorem 4.25) to deduce their F-splitness (Theorem 4.1). Likewise, the analogue of (5.2) also holds for $S_{G_{\mathrm{sc}},\mu}$ versus $S_{G'_{\mathrm{sc}},\mu'}$ with $G'_{\mathrm{sc}} = \mathcal{G}'_{\mathrm{sc}} \otimes K_0(\!(t)\!)$. Appealing now to the coherence theorem of [Zhu14] for the group G'_{sc} in characteristic 0 (those are always tamely ramified) finishes the proof of (5.1). Thus, $\widetilde{M}_{\mathcal{G},\mu}$ is normal and has reduced special fiber which is equal to $\widetilde{A}_{\mathcal{G},\mu}$, and the same holds for the base change $\widetilde{M}_{\mathcal{G},\mu} \otimes_{O_E} O_{\widetilde{E}}$ by an application of Serre's criterion as the generic fiber is geometrically normal.

Since, as noted above, the formation of $\widetilde{M}_{\mathcal{G},\mu}$ commutes with products in G, it follows that for general G the special fiber of $\widetilde{M}_{\mathcal{G},\mu}$ is reduced and is equal to $\widetilde{A}_{\mathcal{G},\mu}$. We now prove the other parts of the theorem by using results from the theory of F-singularities, see Section 3. Since $\widetilde{S}_{G,\mu,\bar{K}} \cong S^{\mathrm{sc}}_{G,\mu,\bar{K}}$ is an Iwahori Schubert variety for the simply connected, split reductive group $G_{\mathrm{sc},\bar{K}}$, it is Cohen–Macaulay and even F-rational by [Cas22, Theorem 1.4]. (Alternatively, these properties of $\widetilde{S}_{G,\mu,\bar{K}}$ also follow directly from Theorem 4.1.) Hence, so is $\widetilde{S}_{G,\mu}$ by faithfully flat descent of [DM20, Proposition A.5]. We already know that $\widetilde{A}_{\mathcal{G},\mu}$ is F-split by Theorem 4.1, so it is F-injective in particular. We also note that all rings and schemes involved in our argument are F-finite since k is algebraically closed. Then Lemma 3.2 implies that $\widetilde{M}_{\mathcal{G},\mu}$ is F-rational, so pseudo-rational by Lemma 3.4 and in particular Cohen–Macaulay. □

Remark 5.5. There is an equality $\widetilde{M}_{\mathcal{G},\mu} = M_{\mathcal{G},\mu}$ if and only if $\widetilde{A}_{\mathcal{G},\mu} = A_{\mathcal{G},\mu}$ and $\widetilde{S}_{G,\mu} = S_{G,\mu}$. This is ensured, for instance, when $p \nmid |\pi_1(G_{\mathrm{der}})|$. If $p \mid |\pi_1(G_{\mathrm{der}})|$, then the equality still holds when $\bar{\mu} \in X_*(T)_I$ is minuscule with respect to the échelonnage roots and the closure of $\mathbf{f}$ contains a special vertex; see the proof of [HLR24, Proposition 9.1]. Otherwise the equality is false for infinitely many values of μ, see [HLR24, Corollary 9.2].

Remark 5.6. Cass has proved somewhat stronger properties of the singularities of $\widetilde{M}_{\mathcal{G},\mu}$ when the group G is a constant split reductive group and $p > 2$, see [Cas21, Theorem 1.6].

Remark 5.7. There is an alternative proof for the reducedness of the special fiber of $\widetilde{M}_{\mathcal{G},\mu}$ via perfectoid geometry, see [GL24, Lemma 1.2, Theorem 1.3], without the need for Hypothesis 2.1. We stress that it does not directly imply that the special fiber is seminormal and F-split as in Theorem 5.4, upon which the last sentence of [GL24, Corollary 1.4] actually relies. On the other hand, combining the results of [GL24] with Theorem 4.25 immediately yields an identification between $\widetilde{A}_{\mathcal{G},\mu}$ and the special fiber of $\widetilde{M}_{\mathcal{G},\mu}$, compare with the proof of [GL24, Theorem 2.1] or the discussion surrounding [AGLR22, Conjecture 7.25].

We can also deduce the following facts on the Picard group of the local models.

Corollary 5.8. *Under Hypothesis 2.1, the following properties hold:*

(1) *The restriction map* $\mathrm{Pic}(\widetilde{M}_{\mathcal{G},\mu}) \to \mathrm{Pic}(\widetilde{A}_{\mathcal{G},\mu})$ *is an isomorphism.*
(2) *Let G_i for $i = 1, \ldots, m$ be an enumeration of the simple factors of G_{ad} such that the image $\bar{\mu}_i$ of μ in the group $X_*(T_i)_I$ attached to G_i is non-zero. Then the restriction map*

$$\prod_{i=1}^{m} \mathrm{Pic}(\widetilde{\mathcal{F}\ell}^{\tau_i}_{\mathcal{G}_i}) \to \mathrm{Pic}(\widetilde{A}_{\mathcal{G},\mu}) \tag{5.3}$$

is an isomorphism, where $\mathcal{G}_i$ is the associated parahoric O-model of $\mathcal{G}_i$ and the superscript τ_i indicates the connected component attached to μ_i.

(3) *There is a commutative diagram:*

$$\begin{tikzcd}
\mathrm{Pic}(\widetilde{M}_{\mathcal{G},\mu}) \arrow[rr] \arrow[d, "\sim"] & & \mathrm{Pic}(\widetilde{S}_{G,\mu}) \arrow[d, "\sim"] \\
\mathrm{Pic}(\widetilde{A}_{\mathcal{G},\mu}) & & \prod_{i=1}^m \mathrm{Pic}(\widetilde{S}_{G_i,\mu_i}) \arrow[d, "\prod_{i=1}^m \deg_i"] \\
\prod_{i=1}^m \mathrm{Pic}(\widetilde{\mathcal{F}\ell}_{\mathcal{G}_i}^{\tau_i}) \arrow[u, "\sim"] \arrow[rr, "\prod_{i=1}^m c_i"] & & \mathbb{Z}^m,
\end{tikzcd} \tag{5.4}$$

where the maps of Picard groups are induced by functoriality, $\deg_i$ *denotes the degree homomorphism, and the* c_i *are the central charge homomorphisms for* $\mathcal{F}\ell_{\mathcal{G}_{i,\mathrm{sc}}}$ *translated to the respective connected components.*

Proof. By Theorem 5.4, the special fiber of $\widetilde{M}_{\mathcal{G},\mu}$ is equal to $\widetilde{A}_{\mathcal{G},\mu} = \cup_\lambda \widetilde{S}_{\bar{\lambda}(t)}$, see Definition 5.3. For (1), it is enough to prove that every line bundle on $\widetilde{A}_{\mathcal{G},\mu}$ lifts uniquely to $\widetilde{M}_{\mathcal{G},\mu}$, or equivalently to the formal scheme $\widetilde{M}_{\mathcal{G},\mu} \times_{\mathrm{Spec}(O_E)} \mathrm{Spf}(O_E)$ by Grothendieck's formal GAGA. Since $\mathrm{H}^j(\widetilde{A}_{\mathcal{G},\mu}, \mathcal{O}_{\widetilde{A}_{\mathcal{G},\mu}}) = 0$ for $j = 1, 2$ by Lemma 4.24, obstruction theory (compare [Kle05, Proposition 5.19]) shows the existence and uniqueness of such lifts.

For (2), we may and do assume that k is algebraically closed by étale descent. We use Proposition 4.23 which calculates $\mathrm{Pic}(\widetilde{A}_{\mathcal{G},\mu})$ as $\oplus_s \mathbb{Z}$ where the sum runs over all $s \in \widetilde{W} \setminus W_{\mathcal{G}}$ with $l(s) = 1$ and $s \leq \bar{\lambda}(t)$ for some rational representative λ of μ in $X_*(T)$. In order to finish the proof of the second part, we may and do assume that G is simple and μ is non-zero. We have to show that the map $\mathrm{Pic}(\widetilde{\mathcal{F}\ell}_{\mathcal{G}}^{\tau_\mu}) \to \mathrm{Pic}(\widetilde{A}_{\mathcal{G},\mu})$ is an isomorphism where τ_μ denotes the unique length 0 element in the admissible set $\mathrm{Adm}(\mu) \subset \widetilde{W}$. It is enough to show that every simple reflection $s \in W_{\mathrm{af}}$ appears in $\tau_\mu^{-1}\mathrm{Adm}(\mu)$, see Corollary 4.21. Assume the contrary. Then the subgroup generated by the simple reflections which do appear is a finite Coxeter group, say, W' containing $\tau_\mu^{-1}\mathrm{Adm}(\mu)$. Therefore, W' (hence $\tau_\mu^{-1}\mathrm{Adm}(\mu)$) contains at most one representative for each coset in the finite Weyl group $W_0 = W_{\mathrm{af}}/X_*(T_{\mathrm{sc}})_I$: if there were two representatives, their difference would be a non-trivial translation, so W' would not be finite. However, this contradicts the fact that $\mathrm{Adm}(\mu)$ contains always at least two different translations $t_{\bar{\mu}}$ and $t_{w_0(\bar{\mu})}$ because $\bar{\mu} \neq 0$.

Part (3) is verified as follows. Since the groups involved are all torsion-free, we only need to check commutativity after tensoring with $\mathbb{Q}$. But then Lemma 4.15 applied to each of the simple factors provides rationalized line bundles on $\widetilde{M}_{\mathcal{G},\mu}$ whose generic fiber is given by $\mathcal{O}(c_{\mathcal{L}_k})$, exactly as claimed. □

Recall that Pappas–Rapoport's coherence conjecture in [PR08], as corrected by Zhu in [Zhu14], gives an equality of dimensions of certain cohomology groups, which we can now formulate and prove in greater generality.

Corollary 5.9. *Let* $\mathcal{L}$ *be an ample line bundle on* $\widetilde{A}_{\mathcal{G},\mu}$. *Under Hypothesis 2.1, there is an equality*

$$\dim_k \mathrm{H}^0(\widetilde{A}_{\mathcal{G},\mu}, \mathcal{L}) = \dim_{\bar{K}} \mathrm{H}^0(\widetilde{S}_{G,\mu,\bar{K}}, \mathcal{O}(c_{\mathcal{L}})), \tag{5.5}$$

where $\mathcal{O}(c_{\mathcal{L}}) := \boxtimes_i \mathcal{O}(c_i(\mathcal{L}))$ *and the* c_i *are the central charge homomorphisms of the simple factors of* G_{ad}, *compare with Corollary 5.8.*

Proof. Note that given a flat proper scheme X over a discrete valuation ring with F-split special fiber, and an ample line bundle $\mathcal{L}$ on X, the dimension of the global sections of $\mathcal{L}$ on X_s and X_η agree by the vanishing of higher cohomology (and constancy of the Euler characteristic). Therefore, the statement follows directly from Theorem 4.1, Theorem 5.4, and Corollary 5.8. Indeed, by Corollary 5.8 (1), $\mathcal{L}$ lifts uniquely to an ample line bundle over $\widetilde{M}_{\mathcal{G},\mu}$ with geometric generic fiber equal to $\mathcal{O}(c_{\mathcal{L}})$ by Corollary 5.8 (3) (note that the integers $c_i(\mathcal{L})$ are well defined by Corollary 5.8 (2)). $\square$

5.2. **Mixed characteristic.** In this subsection, we assume $K/\mathbb{Q}_p$ is of characteristic 0, and fix a uniformizer $\pi \in K$. Further, G is assumed to be adjoint, quasi-split and to satisfy Hypothesis 2.1. Then G is a product of K-simple groups compatibly with the tori $S \subset T$, and we fix the data in (2.1) for each factor. The resulting $O[\![t]\!]$-group lift $\underline{\mathcal{G}}$ of its parahoric model $\mathcal{G}$ is defined as the product of the lifts from Proposition 2.8 of each simple factor. We denote $\mathcal{G}' := \underline{\mathcal{G}} \otimes k[\![t]\!]$.

Let us recall the basic properties of the Beilinson–Drinfeld Grassmannian $\mathrm{Gr}_{\underline{\mathcal{G}}} \to \operatorname{Spec} O$, where the power series variable is $z = t - \pi$, compare with [PZ13, Section 6].

Proposition 5.10. *The O-functor $\mathrm{Gr}_{\underline{\mathcal{G}}}$ is representable by an ind-projective ind-scheme. Its generic fiber is isomorphic to Gr_G, whereas the special fiber is identified with $\mathcal{F}\ell_{\mathcal{G}'}$.*

Proof. Representability by an ind-quasi-projective ind-scheme follows from [PZ13, Proposition 11.7], thanks to Proposition 2.6. Its special fiber is the affine flag variety associated to the $k[\![t]\!]$-group scheme $\mathcal{G}'$, that is, $\mathrm{Gr}_{\underline{\mathcal{G}},k} = \mathcal{F}\ell_{\mathcal{G}'}$. As for the generic fiber, we have to find and choose an identification between $\underline{\mathcal{G}} \otimes K[\![z]\!]$ and $G \otimes K[\![z]\!]$. But the former group scheme is reductive, so such an isomorphism exists by [Ric19, Lemma 0.2], which says that every reductive group scheme over $K[\![z]\!]$ is constant.

Finally, we show projectivity by the same argument of [PZ13, Proposition 6.5]: it is enough to verify the valuative criterion for $\mathrm{Gr}_{\underline{\mathcal{T}}}$. Since $\underline{\mathcal{T}}$ is a product of restrictions of scalars of the multiplicative group along maps of the smooth O-curves in (2.18), this is a consequence of [HR20, Corollary 3.6, Lemma 3.8]. $\square$

Just as in [PZ13, Section 7], we introduce local models in mixed characteristic.

Definition 5.11. Let $M_{\underline{\mathcal{G}},\mu}$ denote the flat closure of $S_{G,\mu}$ inside $\mathrm{Gr}_{\underline{\mathcal{G}},O_E}$. We denote by $\widetilde{M}_{\underline{\mathcal{G}},\mu}$ its seminormalization.

The reader is referred to Remark 5.18 for the extension to not necessarily adjoint groups and to Lemma 5.23 for the relation to the (modified) local models from [HPR20, Section 2.6]. In the following paragraphs, we single out some important properties of the local models.

Lemma 5.12. *The reduced special fiber of $M_{\underline{\mathcal{G}},\mu}$ contains the μ'-admissible locus $A_{\mathcal{G}',\mu'}$ in equicharacteristic, where μ' is the corresponding dominant absolute coweight of G'.*

Proof. The proof is the same as the proof of [Ric16, Lemma 3.12]. This depends on [Ric16, Lemma 2.21] which is formulated in an equicharacteristic setting, but the proof extends to the mixed characteristic setting using that $\underline{\mathcal{T}}$ is induced. $\square$

Remark 5.13. Since our group lifts seldom coincide with the corresponding constructions of [Lev16], we do not know how to compare our $M_{\underline{\mathcal{G}},\mu}$ and $\widetilde{M}_{\underline{\mathcal{G}},\mu}$ with the local models from [Lev16], when μ is non-minuscule. However, our arguments and results below still hold for both objects. For minuscule μ, both constructions do coincide by [AGLR22, Section 7].

Now, we may state our main result on the singularities of local models.

Theorem 5.14. *Under Hypothesis 2.1, the local model $\widetilde{M}_{\underline{\mathcal{G}},\mu}$ is Cohen–Macaulay, and has a reduced special fiber equal to $\widetilde{A}_{\mathcal{G}',\mu'}$. If the admissible locus is irreducible, then $\widetilde{M}_{\underline{\mathcal{G}},\mu}$ has pseudo-rational singularities.*

Proof. As in the proof of Theorem 5.4, we reduce to the case $O = \breve{O}$, G simple and note that $M_{\underline{\mathcal{G}},\mu}$ has a finite, birational, universally homeomorphic cover $M_{\underline{\mathcal{G}_{\mathrm{sc}}},\mu}$ isomorphic to a subscheme of the Grassmannian $\mathrm{Gr}_{\underline{\mathcal{G}_{\mathrm{sc}}}}$ associated to the simply connected cover $G_{\mathrm{sc}} \to G$. In particular, by Theorem 4.25, its generic fiber is isomorphic to $S_{G,\mu} \cong \widetilde{S}_{G,\mu}$ (Schubert varieties in characteristic 0 are normal) and by Lemma 5.12 the special fiber contains $\widetilde{A}_{\mathcal{G}',\mu'}$.

Let $\mathcal{L}_{\mathrm{ad}}$ be the line bundle on $\mathrm{Gr}_{\underline{\mathcal{G}_{\mathrm{sc}}}}$ given by pullback of the determinant line bundle under the adjoint representation. Its restriction to $M_{\underline{\mathcal{G}_{\mathrm{sc}}},\mu}$ is ample, by Proposition 5.10. By (5.1), we get an equality

$$\dim_k H^0(\widetilde{A}_{\mathcal{G}'_{\mathrm{sc}},\mu'}, \mathcal{L}_{\mathrm{ad}}^{\otimes n}) = \dim_E H^0(S_{G_{\mathrm{sc}},\mu}, \mathcal{L}_{\mathrm{ad}}^{\otimes n}). \tag{5.6}$$

This implies that $\widetilde{M}_{\underline{\mathcal{G}},\mu}$ is normal and its special fiber is reduced and equal to $\widetilde{A}_{\mathcal{G}',\mu'}$, compare with the proof of Theorem 5.4. The Cohen–Macaulayness follows from flatness and that of $\widetilde{A}_{\mathcal{G}',\mu'}$ proven in Theorem 5.4, see [HR22, Lemma 5.7]. Moreover, if $\widetilde{A}_{\mathcal{G}',\mu'} = \widetilde{S}_{\mathcal{G}',\mu'}$ is irreducible, then it has F-rational singularities by Theorem 4.1, so pseudo-rationality follows by Lemma 3.5. □

Remark 5.15. Again, there is an equality $\widetilde{M}_{\underline{\mathcal{G}},\mu} = M_{\underline{\mathcal{G}},\mu}$ if and only if $\widetilde{A}_{\mathcal{G}',\mu'} = A_{\mathcal{G}',\mu'}$. (Note that $\widetilde{S}_{G,\mu} = S_{G,\mu}$ because Schubert varieties in characteristic 0 are normal.) This is ensured, for instance, when $p \nmid |\pi_1(G)|$, and may otherwise very well fail, see [HLR24, Corollary 9.2].

Remark 5.16. We note that, for μ minuscule, the $\mathcal{G}_k$-scheme $\widetilde{A}_{\mathcal{G}',\mu'}$ is related to the Witt vector affine Grassmannian of $\mathcal{G}$, see [AGLR22, Section 3].

Remark 5.17. If $\mathcal{G}$ is special parahoric, then the admissible locus is irreducible, so $\widetilde{M}_{\underline{\mathcal{G}},\mu}$ has (pseudo-)rational singularities. For a complete list of triples $(G, \mu, \mathcal{G})$ with G absolutely simple and μ minuscule such that the associated admissible locus is irreducible, the reader is referred to [HPR20, Theorem 7.1 (1)].

Remark 5.18. The local models constructed in [AGLR22] are invariant under passing to the adjoint group. So, if G is not necessarily adjoint, we may define following [HR20, Section 7.1] the local model as $\widetilde{M}_{\underline{\mathcal{G}_{\mathrm{ad}}},\mu_{\mathrm{ad}}} \otimes_{O_{E_{\mathrm{ad}}}} O_E$ where μ_{ad} is induced by μ under $G \to G_{\mathrm{ad}}$ and $E_{\mathrm{ad}} \subset E$ denotes its reflex field. Then Theorem 5.14 holds for this more general definition: this is clear if E/E_{ad} is unramified, and else follows from the method of proof.

We also get a complete description of the Picard group of the local model in mixed characteristic.

Corollary 5.19. *Under Hypothesis 2.1, the following properties hold:*

1. *The restriction map $\mathrm{Pic}(\widetilde{M}_{\underline{\mathcal{G}},\mu}) \to \mathrm{Pic}(\widetilde{A}_{\mathcal{G}',\mu'})$ is an isomorphism.*
2. *Let G_i for $i = 1, \ldots, m$ be an enumeration of the simple factors of G such that the image $\bar{\mu}_i$ of μ in the group $X_*(T_i)_I$ attached to G_i is non-zero. Then the restriction map*

$$\prod_{i=1}^{m} \mathrm{Pic}(\widetilde{\mathcal{F}\ell}^{\tau_i}_{\mathcal{G}'_i}) \to \mathrm{Pic}(\widetilde{A}_{\mathcal{G}',\mu'}) \tag{5.7}$$

is an isomorphism, where $\mathcal{G}'_i$ is the associated parahoric $k[\![t]\!]$-model of G'_i and the superscript τ_i indicates the connected component attached to μ'_i.

(3) *There is a commutative diagram:*

$$
\begin{tikzcd}
\mathrm{Pic}(\widetilde{M}_{\underline{\mathcal{G}},\mu}) \arrow[rr] \arrow[d, "\sim"] & & \mathrm{Pic}(\widetilde{S}_{G,\mu}) \arrow[d, "\sim"] \\
\mathrm{Pic}(\widetilde{A}_{\mathcal{G}',\mu'}) & & \prod_{i=1}^m \mathrm{Pic}(\widetilde{S}_{G_i,\mu_i}) \arrow[d, "\prod_{i=1}^m \deg_i"] \\
\prod_{i=1}^m \mathrm{Pic}(\widetilde{\mathcal{F}\ell}^{\tau_i}_{\mathcal{G}'_i}) \arrow[u, "\sim"] \arrow[rr, "\prod_{i=1}^m c'_i"] & & \mathbb{Z}^m,
\end{tikzcd}
\tag{5.8}
$$

where the maps of Picard groups are induced by functoriality, $\deg_i$ *denotes the degree homomorphism, and the* c'_i *are the central charge homomorphisms for* $\mathcal{F}\ell_{\mathcal{G}'_{i,\mathrm{sc}}}$ *translated to the other components.*

Proof. The proof is the same as in Corollary 5.8, and we briefly explain the necessary changes. For (1), we use Theorem 5.14 to know that $\widetilde{A}_{\mathcal{G}',\mu'}$ equals the special fiber of $\widetilde{M}_{\underline{\mathcal{G}},\mu}$. The structure sheaf has vanishing higher cohomology by Lemma 4.24, so line bundles lift uniquely.

Part (2) follows directly from Corollary 5.8 (2).

For (3), we need to produce enough line bundles on the mixed characteristic local model $\widetilde{M}_{\underline{\mathcal{G}},\mu}$, compare the proof of Lemma 4.15. We have already seen how to construct the adjoint line bundle during Theorem 5.14. As for the kernel of the central charge, we define a map $\mathrm{Gr}_{\underline{\mathcal{G}}} \to [\operatorname{Spec} O/\underline{\mathcal{G}}_{t=0}]$ by reducing torsors to the subscheme defined by the principal ideal t, where $\underline{\mathcal{G}}_{t=0}$ denotes the reduction of the $O[\![t]\!]$-group scheme $\underline{\mathcal{G}}$ to O via $t \mapsto 0$. Pulling back line bundles of $[\operatorname{Spec} O/\underline{\mathcal{G}}_{t=0}]$ to $\widetilde{M}_{\underline{\mathcal{G}},\mu}$ yields the desired lifts of $\ker c$ with trivial generic fiber. ☐

In the equicharacteristic case, we have seen in Theorem 5.4 that local models have rational singularities. Together with Theorem 5.14 at special level, this provides some motivation for the following:

Conjecture 5.20. *The local model* $\widetilde{M}_{\underline{\mathcal{G}},\mu}$ *has pseudo-rational singularities.*

This would follow from Conjecture 3.6. For the purpose of proving Conjecture 5.20 for minuscule μ, that is, the case relevant to Shimura varieties, it would suffice (by Theorem 5.14) to also assume in Conjecture 3.6 that R is Cohen–Macaulay and $R[\pi^{-1}]$ is regular (as in [FW89, Proposition 2.13]), and F-injective can be replaced by F-split.

5.3. **Functoriality of local models.** In this subsection, we discuss the behavior of our local models under certain maps of parahoric group schemes. This is not used elsewhere in the paper, but plays an important role in [AGLR22, Section 7] for proving a comparison theorem between the power series approach of the present paper and the perfectoid approach in [AGLR22].

In both equal and mixed characteristic, a morphism of pairs $(\mathcal{G},\mu) \to (\widetilde{\mathcal{G}},\widetilde{\mu})$ is a map of O-group schemes $\mathcal{G} \to \widetilde{\mathcal{G}}$ which maps μ into $\widetilde{\mu}$ under the induced map of reductive K-groups $G \to \widetilde{G}$ in the generic fiber, compare Remark 5.2. In order to study functoriality properties, it is useful to base change the local model to the absolute integral closure $\bar{O}$ of O with fraction field denoted $\bar{K}$.

In equicharacteristic the formation of local models is functorial in the following sense:

Lemma 5.21. *In equicharacteristic (Section 5.1), the association* $(\mathcal{G},\mu) \mapsto \widetilde{M}_{\mathcal{G},\mu} \otimes_{O_E} \bar{O}$ *from the category of pairs as above to the category of* $\bar{O}$*-schemes is functorial. Under Hypothesis 2.1,* *it commutes with finite products, and the map $\mathcal{G} \to \mathcal{G}_{\mathrm{ad}}$ induces an isomorphism of O_E-schemes*

$$\widetilde{M}_{\mathcal{G},\mu} \cong \widetilde{M}_{\mathcal{G}_{\mathrm{ad}},\mu_{\mathrm{ad}},O_E}, \tag{5.9}$$

where $\mathcal{G} \to \mathcal{G}_{\mathrm{ad}}$ is the map of parahoric O-models extending $G \to G_{\mathrm{ad}}$ and μ_{ad} is the composite of μ with $G_{\bar{K}} \to G_{\mathrm{ad},\bar{K}}$.

Proof. This was proven in the course of Theorem 5.4, see especially the reduction in the beginning of its proof. Recall that for the isomorphism (5.9) and the commutation with finite products, the key fact is that $\widetilde{M}_{\mathcal{G},\mu} \otimes_{O_E} O_{\widetilde{E}}$ is normal for every finite field extension $\widetilde{E} \supset E$. $\square$

Remark 5.22. Using Remark 5.7, the special fiber of $\widetilde{M}_{\mathcal{G},\mu} \otimes_{O_E} \bar{O}$ is always reduced, so the base changed local model is normal and Lemma 5.21 holds without assuming Hypothesis 2.1.

In mixed characteristic (Section 5.2), functoriality of $(\mathcal{G},\mu) \mapsto M_{\underline{\mathcal{G}},\mu}$ (or, its base change to $\bar{O}$) is subtle due to the auxiliary choices involved in the construction of the $O[\![t]\!]$-group lift $\underline{\mathcal{G}}$. Here we point out two particularly interesting cases of functoriality: canonical z-extensions, making the connection to [HPR20, Section 2.6], and embeddings into the Weil restriction of the split form, used in [AGLR22, Section 7].

5.3.1. *Canonical z-extensions following* [Lou23, Section 2.4]. Assume $K/\mathbb{Q}_p$ is of characteristic 0 and use the notation introduced in Section 5.2. In particular, G satisfies Hypothesis 2.1, is adjoint, quasi-split and equipped with a quasi-pinning. We lift the quasi-pinning along the simply connected cover $G_{\mathrm{sc}} \to G$. This induces a map $\underline{\mathcal{G}_{\mathrm{sc}}} \to \underline{\mathcal{G}}$ on the $O[\![t]\!]$-lifts by functoriality of extending birational group laws, compare Proposition 2.8. The maximal torus T acts by inner automorphisms on G_{sc}, so we may form $\widetilde{G} := G_{\mathrm{sc}} \rtimes T$. By [Lou23, Lemme 2.4.2], there is the z-extension

$$1 \to T_{\mathrm{sc}} \xrightarrow{t\mapsto(t,t^{-1})} \widetilde{G} \xrightarrow{(g,t)\mapsto gt} G \to 1 \tag{5.10}$$

with $\widetilde{G}_{\mathrm{der}} = G_{\mathrm{sc}}$ and $T \hookrightarrow \widetilde{G}, t \mapsto (1,t)$ being a maximal torus. By functoriality of extensions of birational group laws, the connected Néron model $\underline{\mathcal{T}}$ acts on $\underline{\mathcal{G}_{\mathrm{sc}}}$ by inner automorphisms. This allows us to define the $O[\![t]\!]$-group scheme $\underline{\widetilde{\mathcal{G}}} := \underline{\mathcal{G}_{\mathrm{sc}}} \rtimes \underline{\mathcal{T}}$, which equals the model birationally glued from $(\underline{\mathcal{T}_{\mathrm{sc}}} \times \underline{\mathcal{T}}, (\underline{\mathcal{U}_a})_{a\in\Phi_G^{\mathrm{nd}}})$ as in Proposition 2.8. Moreover, it fits in a short exact sequence of $O[\![t]\!]$-group schemes

$$1 \to \underline{\mathcal{T}_{\mathrm{sc}}} \to \underline{\widetilde{\mathcal{G}}} \to \underline{\mathcal{G}} \to 1, \tag{5.11}$$

as can be seen by showing that $\underline{\mathcal{G}}$ and the fppf quotient $\underline{\widetilde{\mathcal{G}}}/\underline{\mathcal{T}_{\mathrm{sc}}}$ are solutions to the same birational group law, hence are isomorphic. The extension (5.11) is called the *canonical z-extension of $\underline{\mathcal{G}}$.*

The following lemma relates $\widetilde{M}_{\underline{\mathcal{G}},\mu}$ to the construction of local models via z-extensions as in [HPR20, Section 2.6]. Here we view μ as a geometric cocharacter of T.

Lemma 5.23. *Under Hypothesis 2.1, the map $\underline{\widetilde{\mathcal{G}}} \to \underline{\mathcal{G}}$ from (5.10) induces an isomorphism of O_E-schemes*

$$M_{\underline{\widetilde{\mathcal{G}}},\widetilde{\mu}} \xrightarrow{\cong} \widetilde{M}_{\underline{\mathcal{G}},\mu}, \tag{5.12}$$

where $\tilde{\mu} = (1,\mu)$ is viewed as a geometric cocharacter of $\widetilde{G} = G_{\mathrm{sc}} \rtimes T$.

Proof. Firstly, as T is a maximal torus in both G and $\widetilde{G}$, the cocharacters $\mu, \widetilde{\mu}$ have the same reflex field E. Thus, $\underline{\widetilde{\mathcal{G}}} \to \underline{\mathcal{G}}$ induces a finite birational universal homeomorphism on orbit closures

$$M_{\underline{\widetilde{\mathcal{G}}},\widetilde{\mu}} \to M_{\underline{\mathcal{G}},\mu}, \tag{5.13}$$

which is an isomorphism on residue fields, see [HR22, Corollary 2.3 and its proof]. As $\widetilde{G}_{\mathrm{der}} = G_{\mathrm{sc}}$, the orbit closure $M_{\underline{\widetilde{\mathcal{G}}},\widetilde{\mu}}$ is normal by the proof of Theorem 5.14. So the map (5.13) induces $M_{\underline{\widetilde{\mathcal{G}}},\widetilde{\mu}} \cong \widetilde{M}_{\underline{\mathcal{G}},\mu}$ because the latter is normal by Theorem 5.14. $\square$

5.3.2. *Embedding into the Weil restriction of the split form.* We record the following result concerning the functoriality of the construction $\mathcal{G} \mapsto \underline{\mathcal{G}}$, used in [AGLR22]. Recall the notation from Section 2 and consider the adjunction morphism

$$G = \mathrm{Res}_{L/K}(G_0) \to \mathrm{Res}_{L/K}\mathrm{Res}_{\widetilde{K}/L}(H_0 \otimes_{\mathbb{Z}} K) = \mathrm{Res}_{\widetilde{K}/K}(H_0 \otimes_{\mathbb{Z}} K) =: \widetilde{G}, \tag{5.14}$$

where $\widetilde{K}$ contains the Galois hull of M/L and $H_0/\mathbb{Z}$ is the split form of G_0 induced by (2.1). We assume the following:

Hypothesis 5.24. If $p = 3$, then $G_0 \otimes_K \breve{K}$ is not a triality form of type D_4.

Recall the $O(\!(t)\!)$-group lifts $\underline{G}$ from Lemma 2.7. We equip $\widetilde{G}$ with the quasi-pinning induced from the pinning of $H_0 \otimes_{\mathbb{Z}} K$, leading to the $O(\!(t)\!)$-group lift $\underline{\widetilde{G}}$.

Lemma 5.25. *Under Hypothesis 5.24, the map (5.14) lifts to a locally closed immersion of $O(\!(t)\!)$-group schemes*

$$\underline{G} \to \underline{\widetilde{G}}, \tag{5.15}$$

compatibly with reduction to $\kappa(\!(t)\!)$ for $\kappa = k, K$.

Proof. As the formation of $\underline{G}$ is compatible with restriction of scalars, we assume without loss of generality that $G = G_0$, so $L = K$. Hypothesis 5.24 ensures that the Galois hull of the fraction fields of the ring extension $O[\![t]\!] \to O_{M^{\mathrm{nr}}}[\![v]\!]$ is given by the fraction field of $O_{\widetilde{K}^{\mathrm{nr}}}[\![v]\!]$. The map (5.15) exists by definition of $\underline{G}$ over the étale locus U of $O(\!(t)\!) \to O_{M^{\mathrm{nr}}}(\!(v)\!)$, compare with Lemma 2.3. It can be further extended to $\mathrm{Spec}\, O(\!(t)\!)$ by taking the obvious inclusions for the models of the root groups (2.21), respectively the connected Néron models of tori, and by applying functoriality of solutions to birational group laws, compare with [Lou23, Proposition 3.3.9]. This constructs (5.15), which is a locally closed immersion by [BT84, Proposition 2.2.10]. □

Let $\widetilde{S} \subset \widetilde{G}$ be the maximal split subtorus contained in $\mathrm{Res}_{\widetilde{K}/K}(S)$. The inclusion of apartments

$$\mathscr{A}(G, S, K) \subset \mathscr{A}(\widetilde{G}, \widetilde{S}, K) \tag{5.16}$$

is also compatible with the isomorphism (2.20). For a point $x \in \mathscr{A}(G, S, K)$, we denote its image by $\widetilde{x} \in \mathscr{A}(\widetilde{G}, \widetilde{S}, K)$.

Corollary 5.26. *For $x \in \mathscr{A}(G, S, K)$, the map (5.15) extends to a locally closed immersion of the $O[\![t]\!]$-group schemes*

$$\underline{\mathcal{G}_x} \to \underline{\widetilde{\mathcal{G}}_{\widetilde{x}}}, \tag{5.17}$$

constructed in Proposition 2.8. The map (5.17) reduces to the canonical map of parahoric group schemes over O and $\kappa[\![t]\!]$ with $\kappa = k, K$.

Proof. Applying functoriality of solutions to birational group laws, it suffices to construct the maps between the models of roots groups and of tori, following [Lou23, Proposition 3.4.8]. The resulting map is again a locally closed immersion by [BT84, Proposition 2.2.10]. That its reduction over O, respectively $\kappa[\![t]\!]$, is the expected map on parahoric group schemes is clear from the construction, compare Proposition 2.8. □

Let us briefly return to the situation illustrated in (5.14) of the closed embedding of G into the associated Weil-restricted split form $\widetilde{G}$. We denote by $\widetilde{\mu}$ the geometric conjugacy class of cocharacters of $\widetilde{G}$ obtained as the image of μ. The following compatibility at the level of local models plays a role in the proof of [AGLR22, Theorem 7.23].

Lemma 5.27. *Under Hypothesis 2.1 and Hypothesis 5.24, the map* $\underline{\mathcal{G}} := \underline{\mathcal{G}}_x \to \underline{\widetilde{\mathcal{G}}}_{\widetilde{x}} := \underline{\widetilde{\mathcal{G}}}$ *from* (5.17) *induces a finite morphism*

$$\widetilde{M}_{\underline{\mathcal{G}},\mu} \to \widetilde{M}_{\underline{\widetilde{\mathcal{G}}},\widetilde{\mu}} \tag{5.18}$$

factoring uniquely through its scheme-theoretic image via a universal homeomorphism.

Proof. By naturality of the Beilinson–Drinfeld Grassmannian, we obtain a map between the orbit closures, and hence the map (5.18) by functoriality of seminormalizations [Sta23, Tag 0EUS]. By projectivity of local models, it is enough to show that (5.18) is injective on geometric points, which in turn can be tested on orbit closures.

In the generic fiber, the map $S_{G,\mu} \to S_{\widetilde{G},\widetilde{\mu}}$ of Schubert varieties is a closed immersion because (5.14) is so. In the reduced special fibers, the map is given by $\mathcal{A}_{\mathcal{G}',\mu'} \to \mathcal{A}_{\widetilde{\mathcal{G}}',\widetilde{\mu}'}$ on the respective admissible loci and is induced from $\mathcal{F}\ell_{\mathcal{G}'} \to \mathcal{F}\ell_{\widetilde{\mathcal{G}}'}$.

It may happen that $\mathcal{F}\ell_{\mathcal{G}'} \to \mathcal{F}\ell_{\widetilde{\mathcal{G}}'}$ is not a monomorphism, because $\mathcal{G}' \to \widetilde{\mathcal{G}}'$ is a *locally* closed immersion. But this difference amounts to passing to a finite étale quotient of $\mathcal{F}\ell_{\mathcal{G}'}$ with isomorphic connected components (given by the affine flag variety of the flat closure of the immersion), which embeds into $\mathcal{F}\ell_{\widetilde{\mathcal{G}}'}$. Since $\mathcal{A}_{\mathcal{G}',\mu'}$ is connected, this is enough to deduce injectivity of $\mathcal{A}_{\mathcal{G}',\mu'} \to \mathcal{A}_{\widetilde{\mathcal{G}}',\widetilde{\mu}'}$ on geometric points. □

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ON THE CONNECTEDNESS OF p-ADIC PERIOD DOMAINS.

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Abstract. We prove that all p-adic period domains (and their non-minuscule analogues) are geometrically connected. This answers a question of Hartl [Har13] and has consequences to the geometry of Shimura and local Shimura varieties.

Contents

1. Introduction

In the study of Shimura varieties and p-adic Hodge theory, p-adic period domains and their geometric properties are recurring themes. These period domains arise as p-adic analytic open subsets of flag varieties attached to reductive groups. These open subsets arise as the open image of the Grothendieck–Messing period morphism, which stems from the theory of p-divisible groups. The first appearance of p-adic period domains in the literature is due to Drinfeld [Dri76], who introduced the Drinfeld upper half-space Ω_n. This was later complemented by Gross–Hopkins [HG94], who treated the period morphism for the Lubin–Tate tower. However, the first rigorous definition of p-adic period domains in terms of weakly admissible and admissible loci was given in the seminal book of Rapoport–Zink [RZ96], which initiated their systematic study. Since then, additional significant contributions to their study include the works of Hartl [Har08], Rapoport–Viehmann [RV14], Scholze–Weinstein [SW13, SW20], Chen–Fargues–Shen [CFS21] among others. We refer to the book of Dat–Orlik–Rapoport [DOR10] for a detailed introduction to the subject replete with examples.

The purpose of this article is to prove that p-adic period domains are geometrically connected. Our result confirms in complete generality a conjecture of Hartl, see [Har13, Conjecture 6.5]. It is also a key ingredient in understanding p-adic uniformization of Newton strata on Shimura varieties, compare also with the work of the first author and Lim–Xu [GLX22]. All of our work is done within Scholze's framework of diamonds [Sch17] which allows us to formulate and prove a more general statement. Namely, we prove that the b-admissible loci of the B^+_{dR}-affine Grassmannians (not to be confused with the μ-admissible loci in Witt flag varieties!) are geometrically connected.

1.1. **The main theorem.** Let us formulate precisely our main result. We consider a p-adic shtuka datum (G, b, μ) in the sense of Rapoport–Viehmann [RV14, Definition 5.1] but dropping the minuscule assumption on μ, compare with [SW20, Definition 23.1.1]. This consists of a reductive group G over $\mathbb{Q}_p$, an element b of the Kottwitz set $B(G) = G(\breve{\mathbb{Q}}_p)/\mathrm{ad}_\varphi(G(\breve{\mathbb{Q}}_p))$ in the sense of Kottwitz [Kot85, Kot97], and a geometric conjugacy class of (not necessarily minuscule) cocharacters $\mu \in \mathrm{Hom}(\mathbb{G}_m, G_{\bar{\mathbb{Q}}_p})/\mathrm{ad}(G(\bar{\mathbb{Q}}_p))$, such that $b \in B(G,\mu)$ as in [Kot97, §6]. Let E over $\mathbb{Q}_p$ be the reflex field of μ, i.e., the finite field extension over which the conjugacy class of μ is defined. We let $\mathbb{C}_p$ be a completed algebraic closure of $\mathbb{Q}_p$, $\breve{E} \subset \mathbb{C}_p$ the completion of the maximal unramified extension of E, and Γ denote the absolute Galois group of $\mathbb{Q}_p$.

Given $b \in B(G)$ and a characteristic p perfectoid space S, one can construct a G-bundle $\mathcal{E}_b$ over the relative Fargues–Fontaine curve $X_{\mathrm{FF},S}$ functorially in S. Attached to (G,μ), we have the spatial diamond $\mathrm{Gr}_{G,\mu}$ over $\mathrm{Spd}\,\breve{E}$ that parametrizes B^+_{dR}-lattices with G-structure that are bounded by μ in the Bruhat order [SW20, §§19-22]. Moreover, using Beauville–Laszlo glueing one can identify $\mathrm{Gr}_{G,\mu}$ with the moduli space of G-bundle modifications of $\mathcal{E}_b$

$$\mathrm{Gr}_{G,\mu}(S) = \{(\mathcal{E}, f) \mid f : \mathcal{E} \dashrightarrow \mathcal{E}_b,\ \mathrm{rel}(f) \leq \mu\}/\cong$$

whose relative position is bounded by μ (see [FS21, §III.3] for the case $b = 1$). This gives a Beauville–Laszlo uniformization map:

$$\mathcal{BL}_b : \mathrm{Gr}_{G,\mu} \to \mathrm{Bun}_G,\ (\mathcal{E}, f) \mapsto \mathcal{E},$$

that is analogous to that of [FS21, Proposition III.3.1]. Here, Bun_G denotes the small v-stack of G-bundles on the Fargues–Fontaine curve as in the book of Fargues–Scholze [FS21, §III]. Let Bun^1_G denote the sub-v-stack of Bun_G of those G-bundles that are fiberwise trivial [FS21, §III.2.3]. By [SW20, Corollary 22.5.1, Proposition 24.1.2], the b-admissible locus,[1] $\mathrm{Gr}^b_{G,\mu} := \mathcal{BL}_b^{-1}(\mathrm{Bun}^1_G)$, is non-empty and open in $\mathrm{Gr}_{G,\mu}$. Our main theorem is the following:

Theorem 1.1. *The map $\mathrm{Gr}^b_{G,\mu} \to \mathrm{Spd}\,\breve{E}$ has connected geometric fibers. Moreover, $\mathrm{Gr}^b_{G,\mu} \subset \mathrm{Gr}_{G,\mu}$ remains a dense open after base change along geometric points $\mathrm{Spd}(C, C^+) \to \mathrm{Spd}\,\breve{E}$.*

Remark 1.2. When μ is minuscule and G is quasi-split we have an identification $\mathrm{Gr}_{G,\mu} = (G/P_\mu)^\diamondsuit$, where P_μ is the parabolic subgroup of G defined as the $\mathbb{G}_m$-attractor of $-\mu$. In this case, $\mathrm{Gr}_{G,\mu}$ is (the diamond attached to) a classical flag variety (see [AGLR22, §2.2] for a discussion of the diamond functor). Moreover, we also have a formula:

$$\mathrm{Gr}^b_{G,\mu} = \pi_{\mathrm{GM}}(\mathcal{M}^\diamondsuit_{(G,b,\mu)})$$

where $\mathcal{M}_{(G,b,\mu)}$ is the local Shimura variety attached to (G,b,μ) and π_{GM} is the Grothendieck–Messing period morphism, compare with [RV14, Section 5.2] and [SW20, Definition 24.1.3]. By [Sch17, Lemma 15.6], $\mathrm{Gr}^b_{G,\mu}$ is the diamond associated with a unique open subset $\mathcal{F}(G,b,\mu)^a$ of the rigid-analytic space attached to G/P_μ. This open subset is the p-adic period domain associated to (G,b,μ), and $\mathrm{Gr}^b_{G,\mu} = \mathcal{F}(G,b,\mu)^{a,\diamondsuit}$. In particular, Theorem 1.1 shows that $\mathcal{F}(G,b,\mu)^a$ has connected geometric fibers.

Let us put Theorem 1.1 in context. In [Kis17], Kisin uses in an essential way the connected components of affine Deligne–Lusztig varieties (ADLVs) to study the Langlands–Rapoport conjecture for integral models of Shimura varieties, see [LR87]. On the other hand, Chen [Che14] uses the connected components of ADLVs to derive her main results on connected components of local Shimura varieties (LSVs). These two works motivated Chen–Kisin–Viehmann [CKV15] to compute the connected components of ADLVs at hyperspecial parahoric level building on previous work of Viehmann [Vie08]. Since then, several authors have pushed the strategy of [CKV15] to compute connected components of ADLVs deriving as corollaries results on the geometry of integral models of Shimura varieties (see the following results of Nie [Nie18, Theorem 1.1], He–Zhou [HZ20, Theorem 0.1], Hamacher [Ham20, Theorem 1.1(3)], Nie [Nie23, Theorem 0.2]).

[1] We warn the reader that in the literature $\mathrm{Gr}^b_{G,\mu}$ often stands for $\mathcal{BL}_1^{-1}(\mathrm{Bun}^b_G)$ instead.

Our result is an essential stepping stone in finishing the computation of connected components of mixed characteristic closed ADLVs in full generality, i.e., for all $(\mathcal{I},\mu,b)$ with $\mathcal{I}$ being an Iwahori group $\mathbb{Z}_p$-model of G in the sense of Bruhat–Tits [BT84]. For this reason, it carries decisive implications to the geometry of integral models of Shimura varieties and local Shimura varieties. In order to explain this, we still need to fix some notation. Let φ denote the canonical lift of arithmetic Frobenius to $\breve{\mathbb{Z}}_p$. Let $\mathrm{Adm}(\mu)\subset \mathcal{I}(\breve{\mathbb{Z}}_p)\backslash G(\breve{\mathbb{Q}}_p)/\mathcal{I}(\breve{\mathbb{Z}}_p)$ denote the μ-admissible set of Kottwitz–Rapoport [KR00] (again, this bears no relation to b-admissibility). We consider the closed affine Deligne–Lusztig variety attached to $(\mathcal{I},b,\mu)$, given by the formula

$$X_{\mathcal{I},\mu}(b)=\{g\mathcal{I}(\breve{\mathbb{Z}}_p)\mid g^{-1}b\varphi(g)\in \mathcal{I}(\breve{\mathbb{Z}}_p)\,\mathrm{Adm}(\mu)\mathcal{I}(\breve{\mathbb{Z}}_p)\}. \tag{1.1}$$

It admits the structure of a perfect scheme locally perfectly of finite presentation by the representability result of Bhatt–Scholze [BS17] on the Witt flag varieties defined by Zhu [Zhu17]. Let $\kappa_G: G(\breve{\mathbb{Q}}_p)\to \pi_1(G)_I$ denote the Kottwitz map in the sense of [Kot97, 7.4]. The map κ_G induces a map $\omega_G:\pi_0(X_{\mathcal{I},\mu}(b))\to\pi_1(G)_I$ that factors through a unique coset $c_{b,\mu}\pi_1(G)_I^{\varphi}\in \pi_1(G)_I/\pi_1(G)_I^{\varphi}$. The following is a consequence of our main theorem.

Corollary 1.3. *The Kottwitz map induces a bijection*

$$\omega_G:\pi_0(X_{\mathcal{I},\mu}(b))\xrightarrow{\simeq} c_{b,\mu}\pi_1(G)_I^{\varphi}, \tag{1.2}$$

whenever (b,μ) is Hodge–Newton irreducible.

Indeed, in work of the first author with Lim–Xu [GLX22], it is explained how to deduce Corollary 1.3 from Theorem 1.1. This work, together with [GLX22], finishes the problem of computing the connected components of closed ADLVs in mixed characteristic.

1.2. **Sketch of the proof.** Let us briefly explain the proof of Theorem 1.1 in the case where G is quasi-split. Fix a Borel $B\subseteq G$. When b is basic, Theorem 1.1 can be proved directly, and it is an unpublished result of Hansen–Weinstein. Suppose that b is not basic and let $P\subseteq G$ be the parabolic subgroup generated by B and the centralizer of ν_b.

To prove connectedness, we may and do replace $\mathrm{Gr}_{G,\mu}$ by its dense open subset $L^+P\cdot\xi^\mu$. Now, by Beauville–Laszlo gluing, $L^+P\cdot\xi^\mu$ gets identified with the space of modifications of $\mathcal{E}_p$, where $\mathcal{E}_p$ is the Harder–Narasimhan P-reduction of $\mathcal{E}_b$. Moreover, on this open subset we have a factorization:

$$\mathcal{BL}_b: L^+P\cdot\xi^\mu\to \mathrm{Bun}_P\xrightarrow{p}\mathrm{Bun}_G \tag{1.3}$$

with the first map being the analogous Beauville-Laszlo map $\mathcal{BL}_p$ for the P-torsor $\mathcal{E}_p$. Recall the following general fact. Let X be a connected locally spatial diamond that is smooth and partially proper over $\mathrm{Spa}\,\mathbb{C}_p$. Suppose that we have an open immersion $j:U\to X$ and a complementary closed immersion $i:Z\to X$. For U to be connected, it suffices that $\dim(Z)<\dim(X)$ by [Han21, Corollary 4.11]. In our case, roughly $X=L^+P\cdot\xi^\mu$ and $U=L^+P\cdot\xi^\mu\cap\mathrm{Gr}^b_{G,\mu}$ and we have left to show that the dimension of $X\setminus U$ drops, i.e., $\dim(X\setminus U)<\dim(X)$. An important observation is that the non-empty fibers of $\mathcal{BL}_p$ are torsors under the group of unipotent filtered automorphisms of $\mathcal{E}_b$, see Lemma 3.3 for a precise statement. In particular, they all have the same dimension. Also, $\mathcal{BL}_p$ factors through one connected component $\mathrm{Bun}_P^\kappa\subseteq\mathrm{Bun}_P$ determined by $\mu-\nu_b$.

Let $Y=\mathrm{Bun}_P^\kappa\setminus p^{-1}(\mathrm{Bun}_G^1)$, with p as in 1.3. The second key point is that $\dim(Y)<\dim(\mathrm{Bun}_P^\kappa)$. To prove this, we study the following diagram,

$$\begin{tikzcd} \mathrm{Bun}_P^m \arrow[r] \arrow[d] & \mathrm{Bun}_P \arrow[r] \arrow[d] & \mathrm{Bun}_G \\ \mathrm{Bun}_M^m \arrow[r] & \mathrm{Bun}_M & \end{tikzcd} \tag{1.4}$$

where M is the Levi quotient of P, $m \in B(M)$ and the square is Cartesian. When m is basic and ν_m is anti-dominant as a coweight of G, $\mathrm{Bun}_P^m \to \mathrm{Bun}_G$ is smooth of a dimension that can be made explicit. In our situation of interest, the assumption $b \in B(G,\mu)$ yields the nonpositivity condition $\mu^\diamond - \nu_b \in \mathbb{Q}_{\geq 0}\Phi_G^+$ and the relevant basic element $m \in B(M)$ satisfies that ν_m (which is related to $\nu_b - \mu^\diamond$) is in $\mathbb{Q}_{\leq 0}\Phi_G^+$. Therefore, we perform an inductive argument reducing the non-positive to the anti-dominant case via a sequence of carefully chosen parabolics and their Levis.

1.3. **Organization of the paper.** We now explain the organization of this article. We start § with some cohomological considerations to formally deal with dimension. Then, we make some preparations explaining the combinatorics involving the induction process that reduces the non-positive case to the anti-dominant case. Afterwards, we bound dimensions of Newton strata that arise from the diagram 1.4. Finally, § is dedicated to proving Theorem 1.1.

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2. Bounding dimensions of Newton strata.

2.1. **Dimension for stacky maps.** In the following sections we bound the dimensions of certain Artin v-stacks. Since we do not intend to develop foundations, we will work with an ad-hoc notion of dimension. Let $f : X \to Y$ be a *fine* morphism of Artin v-stacks in the sense of [GHW22, Definition 1.3] and let $n \in \mathbb{N}$. Let $S \to Y$ be a map with S a spatial diamond, let $f_S : X_S \to S$ denote the base change, and let $\mathcal{F} \in D^{\leq 0}_{\text{ét}}(X_S, \mathbb{F}_\ell)$.

Definition 2.1. We say that the *ℓ-cohomological dimension of f is bounded by n*, which we abbreviate as $\dim_\ell(f) \leq n$ if, for all $S \to Y$ and $\mathcal{F}$ as above,

$$f_{S,!}\mathcal{F} \in D^{\leq 2n}_{\text{ét}}(S, \mathbb{F}_\ell), \tag{2.1}$$

and we write $\dim_\ell(X) \leq n$ when $Y = *$.

Convention 2.2. From now on we will only consider maps of Artin v-stacks that are fine and we will not include this adjective in our statements.

Actually, the stacky morphisms used in this article are all obtained as compositions of smooth maps and locally closed immersions which are all fine morphisms.

Lemma 2.3. *Let $f : X \to Y$ and $g : Y \to Z$ be map of Artin v-stacks such that $\dim_\ell(f) \leq n$ and $\dim_\ell(g) \leq m$. Then $\dim_\ell(g \circ f) \leq m + n$.*

Proof. Let $S \to Z$ be a map and denote by X_S and Y_S the base changes. Let $\mathcal{F} \in D^{\leq 0}_{\text{ét}}(X_S, \mathbb{F}_\ell)$. Observe that $f_{S,!}\mathcal{F}[2n] \in D^{\leq 0}_{\text{ét}}(Y_S, \mathbb{F}_\ell)$, which implies that $g_{!,S} f_{S,!}\mathcal{F}[2n] \in D^{\leq 2m}_{\text{ét}}(S, \mathbb{F}_\ell)$. It follows that $\dim_\ell(g \circ f) \leq n + m$. $\square$

Lemma 2.4. *Let $f : X \to Y$ be a map of Artin v-stacks. Suppose that for any $s : \mathrm{Spa}(C, C^+) \to Y$ the fibers satisfy $\dim_\ell(X_s) \leq n$. Then $\dim_\ell(f) \leq n$.*

Proof. This follows from [Sch17, Theorem 1.9.(2)], [GHW22, Theorem 1.4.(4)], since $\mathcal{F} \in D^{\leq 2n}_{\text{ét}}(S, \mathbb{F}_\ell)$ can be checked on geometric point. □

The next lemma carries some weight in our paper and gives a cancelation property for ℓ-dimension in the presence of a smooth cover.

Lemma 2.5. *Let $f : X \to Y$ be a surjective ℓ-cohomologically smooth map of Artin v-stacks with constant ℓ-dimension d. Let $g : Y \to Z$ be a map of Artin v-stacks. Then $\dim_\ell(g) \leq n$ if and only if $\dim_\ell(g \circ f) \leq n + d$.*

Proof. Assume first that $\dim_\ell(g) \leq n$. In order to bound $\dim_\ell(g \circ f)$, it suffices by Lemma 2.3 to prove $\dim_\ell(f) \leq d$. It suffices to prove that $\mathrm{RHom}(f_{S,!}\mathcal{F}, \mathcal{G}) = 0$ for every map $S \to Y$, every object $\mathcal{G} \in D^{\geq 2d+1}_{\text{ét}}(S, \mathbb{F}_\ell)$ and every object $\mathcal{F} \in D^{\leq 0}_{\text{ét}}(X, \mathbb{F}_\ell)$. By adjunction, we may prove $\mathrm{RHom}(\mathcal{F}, f^!_S\mathcal{G}) = 0$ instead. Now, by ℓ-cohomological smoothness $f^!\mathcal{G} = f^*\mathcal{G} \otimes f^!\mathbb{F}_\ell$ and $f^!\mathbb{F}_\ell$ is an invertible object in $D_{\text{ét}}(X, \mathbb{F}_\ell)$ concentrated in degree $-2d$. In particular, $f^!_S\mathcal{G} \in D^{\geq 1}_{\text{ét}}(X_S, \mathbb{F}_\ell)$ while $\mathcal{F} \in D^{\leq 0}_{\text{ét}}(X_S, \mathbb{F}_\ell)$, so the required vanishing follows by the corresponding property for the natural t-structure.

For the converse, we have to show that $\dim_\ell(g) \leq n$. Let $S \to Z$ a map with S a spatial diamond, let $\mathcal{F} \in D^{\leq 0}_{\text{ét}}(Y_S, \mathbb{F}_\ell)$ and let $\mathcal{G} \in D^{\geq 2n+1}_{\text{ét}}(S, \mathbb{F}_\ell)$. As above, it suffices to prove:

$$\mathrm{RHom}(\mathcal{F}, g^!_S\mathcal{G}) = 0 \tag{2.2}$$

In other words, we wish to prove that $g^!_S\mathcal{G} \in D^{\geq 1}_{\text{ét}}(Y_S, \mathbb{F}_\ell)$, for all $\mathcal{G} \in D^{\geq 2n+1}_{\text{ét}}(S, \mathbb{F}_\ell)$. This can be verified on geometric points so we may show

$$f^*_S g^!_S\mathcal{G} \in D^{\geq 1}_{\text{ét}}(X_S, \mathbb{F}_\ell) \tag{2.3}$$

instead, since f_S is surjective. By smoothness, $f^!_S\mathbb{F}_\ell \in D^{-2d}_{\text{ét}}(X_S, \mathbb{F}_\ell)$ is an invertible object and $f^*_S g^!_S\mathcal{G} = f^!_S g^!_S\mathcal{G} \otimes (f^!_S\mathbb{F}_\ell)^{-1}$. On the other hand, it follows from the bound $\dim_\ell(g \circ f) \leq n + d$ that $f^!_S g^!_S\mathcal{G} \in D^{\geq 1-2d}_{\text{ét}}(X_S, \mathbb{F}_\ell)$ by testing RHom against the natural t-structure and passing to adjoints. In particular, we can verify that (2.3) holds. □

Lemma 2.6. *Let $f : X \to Y$ be a map of Artin v-stacks. Let $i : Z \to X$ be a closed immersion and let $j : U \to X$ denote the complementary open immersion. Suppose that $\dim_\ell(i \circ f) \leq n$ and that $\dim_\ell(j \circ f) \leq n$, then $\dim_\ell(f) \leq n$. Conversely if $\dim_\ell(f) \leq n$ then $\dim_\ell(i \circ f) \leq n$ and $\dim_\ell(j \circ f) \leq n$.*

Proof. Notice that the fibers of j and i are 0-dimensional. By Lemma 2.3, the second claim follows. For the first claim, let $\mathcal{F} \in D^{\leq 0}_{\text{ét}}(X, \mathbb{F}_\ell)$, and consider the following distinguished triangle

$$f_! j_! j^* \mathcal{F} \to f_! \mathcal{F} \to f_! i_* i^* \mathcal{F} \tag{2.4}$$

in the derived category. We may pass to geometric fibers, where one of the terms vanish. □

2.2. **Averages of coweights.** Let G be a quasi-split reductive group over $\mathbb{Q}_p$ and let $T \subset B \subset G$ be a pair consisting of a maximal torus that is maximally $\mathbb{Q}_p$-split and a Borel both defined over $\mathbb{Q}_p$. Let Φ_G be the absolute root system of G with respect to T and Δ_G the basis of positive simple absolute roots with respect to B. We let $X_*(T)$ denote the set of geometric cocharacters and denote by $X_*(T)_\mathbb{Q}$ the resulting rational vector space. We use the symbol M to denote a standard Levi of G defined over $\mathbb{Q}_p$, and by Δ_M the induced base of positive simple roots.

Definition 2.7. We say that $\nu \in X_*(T)_\mathbb{Q}$ is *M-dominant* (resp. *M-central*) if $\langle \alpha, \nu \rangle \geq 0$ (resp. $\langle \alpha, \nu \rangle = 0$) for all $\alpha \in \Delta_M$ and denote by $X_*(T)^{+_M}_\mathbb{Q}$ the convex set of M-dominant vectors in $X_*(T)_\mathbb{Q}$.

Following Schremmer [Sch22], we now define the so called M*-average* of ν:

$$\mathrm{av}_M(\nu) = \frac{1}{|W_M|} \sum_{w \in W_M} w\nu \tag{2.5}$$

where W_M denotes the absolute Weyl group of M.

Lemma 2.8. *The M-average $\mathrm{av}_M(\nu)$ is the unique M-central $\mu \in X_*(T)_{\mathbb{Q}}$ whose difference $\mu - \nu$ lies in the $\mathbb{Q}$-vector space spanned by the M-coroots $\Delta_M^\vee$.*

Proof. Notice that $\mathrm{av}_M(\nu)$ is W_M-invariant by definition. Also, a vector is W_M-invariant if and only if it is M-central. To see that the diference is spanned by $\Delta_M^\vee$, it is enough to check that the M-coroots evaluate to 0 under av_M. This is clear by summing left $\{1, s_\alpha\}$-cosets in W_M first, since $s_\alpha(\alpha^\vee) = -\alpha^\vee$. □

Thinking in terms of fundamental weights reveals that $2\rho_G - 2\rho_M$ pairs to 0 with every $\alpha^\vee \in \Delta_M^\vee$. Thus, it follows that $\langle 2\rho_G - 2\rho_M, \nu\rangle = \langle 2\rho_G - 2\rho_M, \mathrm{av}_M(\nu)\rangle$. We study how averaging interacts with the notion of non-positivity presented below.

Definition 2.9. We say that $\nu \in X_*(T)_{\mathbb{Q}}$ is *non-positive* (resp. *non-negative*) if it belongs to the convex hull of $X_*(Z_G)_{\mathbb{Q}}$ and $\mathbb{Q}_{\leq 0}\alpha^\vee$ (resp. $\mathbb{Q}_{\geq 0}\alpha^\vee$), where Z_G is the center of G and α runs over Δ_G. The convex set of non-positive vectors is denoted by $X_*(T)_{\mathbb{Q}}^{\leq 0}$.

In our definition above, ν is non-positive if and only if the inequality $\nu_{\mathrm{ad}} \leq 0$ holds in the rational Bruhat order of $X_*(T_{\mathrm{ad}})_{\mathbb{Q}}$. Here T_{ad} denotes the image of T in the adjoint group G_{ad} of G. An anti-dominant vector is necessarily non-positive, but for most groups the converse doesn't hold. In the following, we note that averaging preserves non-negativity, compare with [Sch22, Lemma 3.1].

Proposition 2.10. *The function av_M preserves $X_*(T)_{\mathbb{Q}}^{\leq 0}$.*

Proof. It suffices to see that it preserves $X_*(Z_G)_{\mathbb{Q}}$ and $\mathbb{Q}_{\leq 0}\alpha^\vee$. This is clear for $X_*(Z_G)_{\mathbb{Q}}$. If $\alpha^\vee \in \Delta_M$ then we already know that $\mathrm{av}_M(\alpha^\vee) = 0$, so it suffices to consider $\mathrm{av}_M(\alpha^\vee)$ for $\alpha \in \Delta_G \setminus \Delta_M$. In this case $w\alpha^\vee$ is a positive coroot for all $w \in W_M$, being a coroot of the unipotent radical of the associated standard parabolic P, and thereby finishing the proof. □

Remark 2.11. If $G = \mathrm{GL}_n$, we may interpret ν as a polygon and its non-positivity as meaning the polygon lies below the straight line connecting its extremities and never crosses it. The vector $\mathrm{av}_M(\nu)$ corresponds to connecting vertices according to a partition of n. In this case, it is visually clear that this partial average polygon lies below the total average polygon, since we started with a non-positive one.

As a corollary, we get the following technical result that is relevant in the next subsection:

Lemma 2.12. *Let $\nu \in X_*(T)_{\mathbb{Q}}^{\leq 0}$ be invariant under Γ and M-central. There is a sequence of standard Levi subgroups $M = M_0 \subset \cdots \subset M_i \subset \cdots \subset M_k = G$ defined over $\mathbb{Q}_p$ and also of Γ-invariant vectors $\nu = \nu_0, \ldots, \nu_i, \ldots \nu_k = \mathrm{av}_G(\nu)$ in $X_*(T)_{\mathbb{Q}}^{\leq 0}$ such that the following properties hold*

(1) $\nu_j = \mathrm{av}_{M_j}(\nu_i)$ *for* $j \geq i$.
(2) ν_i *is* M_{i+1}*-anti-dominant.*

Proof. Suppose $\langle \alpha, \nu\rangle \geq 0$ for all $\alpha \in \Delta_G \setminus \Delta_M$. Since $\langle \alpha, \nu\rangle = 0$ for $\alpha \in \Delta_M$ by hypothesis, we also get $\langle \rho_G, \nu\rangle \geq 0$. On the other hand, the convex hull of $X_*(Z_G)_{\mathbb{Q}}$ and $\mathbb{Q}_{\leq 0}\alpha^\vee$ for all $\alpha \in \Delta_G$ pairs non-positively with the strictly dominant weight ρ_G, and it vanishes exactly on G-central elements. Therefore, the only possibility would be $M = G$, in which case $k = 0$ and ν is G-central.

Otherwise, there exists some $\alpha \in \Delta_G \setminus \Delta_M$ such that $\langle \alpha, \nu \rangle < 0$. By Γ-invariance, this holds for its entire Γ-orbit. Now let L be the standard Levi defined over $\mathbb{Q}_p$ with $\Delta_L = \Delta_M \cup \Gamma\alpha$. Clearly ν is L-anti-dominant. Moreover, by Proposition 2.10 $\mathrm{av}_L(\nu)$ is non-positive and L-central, which finishes the proof of the lemma by induction on the cardinality of $\Delta_G \setminus \Delta_M$. □

2.3. **Parabolic Newton strata.** In this subsection, we continue to work under the same assumptions and use similar notations. Pick $b \in B(M)$ and write $\nu_b \in X_*(T)^\Gamma_{\mathbb{Q}}$ for the M-dominant Newton point of b. We warn the reader that we follow the opposite sign convention to [FS21, pages 59 and 90], so that the slope 1 isocrystal $(\breve{\mathbb{Q}}_p, \pi\varphi)$ is sent to the line bundle $\mathcal{O}(1)$ on the Fargues–Fontaine curve: this will lead to sign changes everywhere compared to many of our sources below. We have notions of dominance and positivity for elements of $B(M)$.

Definition 2.13. We say that $b \in B(M)$ is *dominant* (respectively *anti-dominant*) if the M-dominant Newton point ν_b is G-dominant (respectively G-anti-dominant). We say that it is *non-positive* if the M-dominant Newton point ν_b is G-non-positive.

From now on, we assume that $\nu := \nu_b$ is non-positive. Consider a sequence $M = M_0 \subset \cdots \subset M_i \subset \cdots \subset M_k = G$ of standard Levi subgroups defined over $\mathbb{Q}_p$ and of Γ-invariant vectors ν_i as in Lemma 2.12. We inductively choose basic elements $b_i \in B(M_i)$ sharing the same image under the Kottwitz map, i.e., with $\kappa_{M_i}(b_i) = \kappa_{M_i}(b)$. It follows immediately by construction that the Newton point of b_i coincides with ν_i. For $i \leq j$, we still write b_i for its image in $B(M_j)$ under the natural map $B(M_i) \to B(M_j)$. In particular, b_i is non-positive in M_j for all $j \geq i$ and even anti-dominant if $j = i+1$. For simplicity, we also use the shorthand $\rho_i := \rho_{M_i}$ and $\rho_{ij} := \rho_j - \rho_i$. Note that $\langle 2\rho_{ij}, -\nu_i \rangle = \langle 2\rho_{ij}, -\nu_j \rangle$ for all $i \leq j$.

For any $j \geq i$, let $P_{ij} \subset M_{i+1}$ be the standard parabolic whose standard Levi is M_j. We now define the locally closed stratum $\mathrm{Bun}_{P_{ij}}^{b_i}$ by demanding that the square in the following commutative diagram

$$\begin{tikzcd}
\mathrm{Bun}_{P_{ij}}^{b_i} \arrow[r] \arrow[d] & \mathrm{Bun}_{P_{ij}} \arrow[r] \arrow[d] & \mathrm{Bun}_{M_j} \\
\mathrm{Bun}_{M_i}^{b_i} \arrow[r] & \mathrm{Bun}_{M_i} &
\end{tikzcd} \tag{2.6}$$

is Cartesian. The following theorem was shown in [AB21] and further refined in [Ham22].

Theorem 2.14. *The map of Artin v-stacks $\mathrm{Bun}_{P_{ij}} \to \mathrm{Bun}_{M_i}$ is ℓ-cohomologically smooth and has connected geometric fibers. Over the b_i-stratum, it is of relative ℓ-dimension $\langle 2\rho_{ij}, -\nu_i \rangle$.*

Proof. This follows from [Ham22, Proposition 3.16, Proposition 4.7]. □

In the example below, we see concrete examples of these parabolic strata in the case of GL_2 and taking our sign convention into account.

Example 2.15. Let $G = \mathrm{GL}_2$, let $T \subseteq G$ denote the diagonal torus and let $B \subseteq G$ denote the upper diagonal matrices. In our sign convention, we consider $\mathcal{O}(1)$ to have slope 1. Consider the polygons $b_1 = (2, 1)$ and $b_2 = (1, 2)$. For our sign convention, b_1 is G-dominant and corresponds to $\mathcal{O}(2) \oplus \mathcal{O}(1)$ as a T-torsor. On the other hand, b_2 is G-anti-dominant and corresponds to the T-torsor $\mathcal{O}(1) \oplus \mathcal{O}(2)$. In this example, $\mathrm{Bun}_B^{b_1}$ classifies extensions of the form

$$0 \to \mathcal{O}(2) \to \mathcal{E} \to \mathcal{O}(1) \to 0. \tag{2.7}$$

These extensions are trivial and the automorphism group is 1-dimensional. Overall, $\dim(\mathrm{Bun}_B^{b_1}) = -1$ which agrees with the formula $\langle \rho_G, -\nu_{b_1} \rangle$.

Remark 2.16. Suppose $\nu := \nu_0$ is already anti-dominant, then we may take $k = 1$ and $M_k = G$. Then, the b-stratum of Bun_P identifies with the Fargues–Scholze chart $\mathcal{M}_b$ attached to $b \in B(G)$, see [FS21, Example V.3.4] (note the change of sign convention here). If, on the other hand, we worked with dominant coweights, the b-stratum of Bun_P would identify with that of Bun_G by the Harder–Narasimhan filtration.

We want to study the geometry of the natural map $\mathrm{Bun}_P^b \to \mathrm{Bun}_G$ for non-positive basic $b \in B(M)$. We will proceed by induction with the help of our sequence of standard Levis in order to bootstrap a (somewhat weaker) statement from the anti-dominant case. We get the following commutative diagram with Cartesian square

$$\begin{tikzcd}
\mathrm{Bun}_{P_{ik}} \arrow[r] \arrow[d] & \mathrm{Bun}_{P_{jk}} \arrow[r] \arrow[d] & \mathrm{Bun}_{M_k} \\
\mathrm{Bun}_{P_{ij}} \arrow[r] \arrow[d] & \mathrm{Bun}_{M_j} & \\
\mathrm{Bun}_{M_i} & &
\end{tikzcd} \tag{2.8}$$

with P_{ij} denoting the standard parabolic of M_j with standard Levi equal to M_i. In particular, we get a natural map $\Delta_{ijk} \colon \mathrm{Bun}_{P_{ik}} \to \mathrm{Bun}_{M_i} \times \mathrm{Bun}_{M_j}$, and we define the (b_i, c_j)-strata as the pullback

$$\mathrm{Bun}_{P_{ik}}^{(b_i,c_j)} := \Delta_{ijk}^{-1}(\mathrm{Bun}_{M_i}^{b_i} \times \mathrm{Bun}_{M_j}^{c_j}) \subset \mathrm{Bun}_{P_{ik}} \tag{2.9}$$

for some $b_i \in B(M_i)$ and $c_j \in B(M_j)$.

Proposition 2.17. *Let $b_i \in B(M_i)$ be a sequence of non-positive basic elements with anti-dominant steps. For every $j \geq i$, the b_i-stratum of $\mathrm{Bun}_{P_{ij}}$ contains an open subspace $\mathcal{T}_{ij}$ such that the induced map $f_{b_i} : \mathcal{T}_{ij} \to \mathrm{Bun}_{M_j}$ satisfies the following:*

(1) *f_{b_i} factors through the b_j-stratum of Bun_{M_j} and it is ℓ-cohomologically smooth of relative dimension $\langle 2\rho_j - 2\rho_i, -\nu_i \rangle$.*
(2) *The dimension of the closed complement drops, i.e., $\dim_\ell(\mathrm{Bun}_{P_{ij}}^{b_i} \setminus \mathcal{T}_{ij}) < \langle 2\rho_j - 2\rho_i, -\nu_i \rangle$.*

Proof. We proceed to construct the open subspace $\mathcal{T}_{ij}$ recursively and show inductively that the natural map towards Bun_{M_j} is ℓ-cohomologically smooth. We do this by induction on $j - i$. If it equals 1, then b_i is M_j-anti-dominant, so the natural map $\mathrm{Bun}_{P_{ij}} \to \mathrm{Bun}_{M_j}$ restricts to an ℓ-cohomologically smooth map over the b_i-stratum by [FS21, Theorem V.3.7]. If we set $\mathcal{T}_i$ as the (b_i, b_j)-stratum of $\mathrm{Bun}_{P_{ij}}$ following (2.9) (here we put $j = k$), then we immediately get the desired properties.

In the general case, we consider $i < j < k$. By the inductive hypothesis, we may and do assume that the result is known for the intermediate pairs (i, j) and (j, k). In fact, we may even assume that b_i is M_j-anti-dominant by passing to the immediate step $j = i + 1$ if necessary. Pulling back the diagram along the b_i- and b_j-strata yields the following commutative diagram with Cartesian squares:

$$\begin{tikzcd}
\mathrm{Bun}_{P_{ik}}^{(b_i,b_j)} \arrow[r] \arrow[d] & \mathrm{Bun}_{P_{ik}}^{b_i} \arrow[r] \arrow[d] & \mathrm{Bun}_{P_{ij}}^{b_i} \arrow[d] \\
\mathrm{Bun}_{P_{jk}}^{b_j} \arrow[r] & \mathrm{Bun}_{P_{jk}} \arrow[r] & \mathrm{Bun}_{M_j}.
\end{tikzcd} \tag{2.10}$$

By induction, we obtain a map $\mathcal{T}_{jk} \to \mathrm{Bun}_{P_{jk}}$ factoring over the b_j-stratum, i.e., the lower left corner in the above diagram. We now define $\mathcal{T}_{ik} \to \mathrm{Bun}_{P_{ik}}$ as the pullback of this new arrow along the left vertical one in the diagram above, thereby completing it with a new Cartesian square.

$$\begin{tikzcd}
\mathcal{T}_{ik} \arrow[r] \arrow[d] & \mathrm{Bun}_{P_{ik}}^{(b_i,b_j)} \arrow[r] \arrow[d] & \mathrm{Bun}_{P_{ik}}^{b_i} \arrow[r] \arrow[d] & \mathrm{Bun}_{P_{ij}}^{b_i} \arrow[d] \\
\mathcal{T}_{jk} \arrow[r] & \mathrm{Bun}_{P_{jk}}^{b_j} \arrow[r] & \mathrm{Bun}_{P_{jk}} \arrow[r] & \mathrm{Bun}_{M_j}.
\end{tikzcd} \tag{2.11}$$

By induction, the map $\mathcal{T}_{jk} \to \mathrm{Bun}_{M_k}$ is ℓ-cohomologically smooth and since b_i is M_j-anti-dominant so is also $\mathcal{T}_{ik} \to \mathcal{T}_{jk}$, implying ℓ-cohomological smoothness of the composition. The dimension claim follows since $\mathrm{Bun}_{M_k} \to *$ is ℓ-smooth of dimension 0 by [FS21, Theorem I.4.1.(vii)], while $\mathrm{Bun}_{P_{ik}}^{b_i} \to *$ is ℓ-cohomologically smooth of dimension $\langle 2\rho_{ik}, -\nu_i\rangle$ by Theorem 2.14.

Next, we handle the second claim for the above definition of $\mathcal{T}_{ik}$. Pick $c_j \in B(M_j)$ in the image of $\mathrm{Bun}_{P_{ij}}^{b_i} \to \mathrm{Bun}_{M_j}$ and let μ_j be its M_j-dominant Newton point. We get an ℓ-cohomologically smooth map $\mathrm{Bun}_{P_{ik}}^{(b_i,c_j)} \to \mathrm{Bun}_{P_{jk}}^{c_j}$ of dimension $\langle 2\rho_{ij}, -\nu_i\rangle$. By Theorem 2.14, the map $\mathrm{Bun}_{P_{jk}}^{c_j} \to \mathrm{Bun}_{M_j}$ is ℓ-cohomologically smooth of dimension $\langle 2\rho_{ik}, -\mu_j\rangle$. Since the b_i- and the c_j-strata belong to the same connected component of Bun_{M_j}, we see that $\nu_i - \mu_j$ lies in the $\mathbb{Q}$-span of $\Delta_{M_j}^\vee$ and so it is orthogonal to ρ_{jk}. We conclude that the composition

$$\mathrm{Bun}_{P_{ik}}^{(b_i,c_j)} \to \mathrm{Bun}_{P_{jk}}^{c_j} \to \mathrm{Bun}_{M_j}^{c_j} \tag{2.12}$$

is ℓ-cohomologically smooth of relative dimension $\langle 2\rho_{ik}, -\nu_i\rangle$. The term on the right has strictly negative dimension as soon as $c_j \neq b_j$, so by Lemma 2.3 and Lemma 2.6, it follows that the dimension of the complement of the (b_i, b_j)-stratum of $\mathrm{Bun}_{P_{ik}}$ drops. On the other hand, we have a Cartesian diagram by definition:

$$\begin{tikzcd}
\mathrm{Bun}_{P_{ik}}^{(b_i,b_j)} \setminus \mathcal{T}_{ik} \arrow[r] \arrow[d] & \mathrm{Bun}_{P_{ik}}^{(b_i,b_j)} \arrow[r] \arrow[d] & \mathrm{Bun}_{P_{ik}}^{b_i} \arrow[d] \\
\mathrm{Bun}_{P_{jk}}^{b_j} \setminus \mathcal{T}_{jk} \arrow[r] & \mathrm{Bun}_{P_{jk}}^{b_j} \arrow[r] & \mathrm{Bun}_{P_{jk}}
\end{tikzcd} \tag{2.13}$$

and we know by induction that the dimension drops along the left bottom horizontal arrow. Since the vertical maps are ℓ-cohomologically smooth of dimension $\langle 2\rho_{ij}, -\nu_i\rangle$, we conclude that dimension also drops along the left upper horizontal arrow. By Lemma 2.6, we may now combine these two dimension drops to reach our desired conclusion.

□

3. $\mathrm{Gr}_{G,\mu}^b$ IS CONNECTED

In contrast with the previous section, we will momentarily not assume that G is quasi-split. Fix C an arbitrary algebraically closed non-archimedean field extension of $\breve{E}$ and consider the Beauville–Laszlo map defined over $\mathrm{Spd}\, C$,

$$\mathcal{BL}_b : \mathrm{Gr}_{G,\mu} \to \mathrm{Bun}_G \times \mathrm{Spd}\, C \to \mathrm{Spd}\, C. \tag{3.1}$$

We fix our sign convention for modifications when $G = \mathbb{G}_m$, and extend the convention by functoriality to all other groups. We consider the inclusion of the ideal sheaf $\mathcal{O}(-1) \subseteq \mathcal{O}$ to be a modification of $\mathcal{O}$ of type $\mu = 1 \in \mathbb{Z} \simeq X_*(\mathbb{G}_m)$. We observe that under our sign convention, $\mathcal{BL}_b$ factors through the unique connected component of Bun_G parametrized by $\kappa_G(b) - \mu^\natural \in \pi_1(G)_\Gamma$, compare with [FS21, Corollary IV.1.23]. We formulate Theorem 1.1 as follows:

Theorem 3.1. *If $b \in B(G,\mu)$, then $\mathrm{Gr}_{G,\mu}^b$ is connected and it is dense in $\mathrm{Gr}_{G,\mu}$.*

Without loss of generality we may assume that G is adjoint (see the proof of [PR24, Proposition 3.1.1]). Moreover, when G is adjoint it follows that if G^* denotes its quasi-split inner form, then G^* is a pure inner form of G. In particular, we have an identification $\mathrm{Bun}_{G^*} \simeq \mathrm{Bun}_G$. This allow us to assume that G is quasi-split, at the expense of proving the more general Theorem 3.2 below.

In order to do this, we will fix additional notation. From now on we assume again that G is quasi-split and we fix $T \subset B \subset G$ as in the previous section. We define an element $\mu^\diamond \in X_*(T)^\Gamma_{\mathbb{Q}}$ given by the formula:

$$\mu^\diamond := \frac{1}{[\Gamma : \Gamma_\mu]} \sum_{\gamma \in \Gamma/\Gamma_\mu} \gamma(\mu), \tag{3.2}$$

where Γ_μ denotes the stabilizer of μ for the Γ-action. Notice that $\langle 2\rho_G, \mu^\diamond \rangle = \langle 2\rho_G, \mu \rangle$, because ρ_G is Γ-invariant.

Let $A_Z(G,\mu) \subset B(G)$ be the set of acceptable elements modulo center, i.e. for which $\mu^\diamond - \nu_b$ is non-negative as in Definition 2.9. This is related to the notion of acceptable elements $A(G,\mu)$ of [RV14, Definition 2.3], in the sense that $A_Z(G,\mu)$ equals the pre-image of $A(G_{\mathrm{ad}}, \mu_{\mathrm{ad}})$ along $B(G) \to B(G_{\mathrm{ad}})$. If $m \in B(M)$, we let m_μ denote the unique basic element in $B(M)$ such that $\kappa_M(m_\mu) = \kappa_M(m) - \mu^\natural$. Let $d = \dim_\ell(\mathrm{Gr}_{G,\mu}) = \langle 2\rho_G, \mu \rangle$ and define the (c,b)-admissible locus $\mathrm{Gr}_{G,\mu}^{(c,b)} := \mathcal{BL}_b^{-1}(\mathrm{Bun}_G^c) \subset \mathrm{Gr}_{G,\mu}$. If $c = 1$, this recovers our usual b-admissible locus.

Theorem 3.2. *If $b \in A_Z(G,\mu)$, then $\mathrm{Gr}_{G,\mu}^{(b_\mu,b)}$ is dense in $\mathrm{Gr}_{G,\mu}$ and connected.*

Proof. To prove that $\mathrm{Gr}_{G,\mu}^{(b_\mu,b)}$ is dense and connected, it suffices to show that $\dim_\ell(\mathrm{Gr}_{G,\mu}^{(c,b)}) < d$ for all $c \in B(G)$ with $c \neq b_\mu$. Since the dimension drops on the complement of the Schubert cell $\mathrm{Gr}_{G,\mu}^\circ \subset \mathrm{Gr}_{G,\mu}$, it suffices to compute the dimension of their intersection $\mathrm{Gr}_{G,\mu}^{\circ,(c,b)}$.

If b is basic, $\mathcal{BL}_b : [\underline{G(\mathbb{Q}_p)}\backslash \mathrm{Gr}_{G,\mu}^\circ] \to \mathrm{Bun}_G$ is smooth of relative dimension d by [FS21, Proposition IV.1.18, Theorem IV.1.19]. In particular, $\dim_\ell(\mathrm{Gr}_{G,\mu}^{\circ,(g,b)}) = d + \dim_\ell(\mathrm{Bun}_G^g)$. Now, b_μ is the unique basic element in the image of $\mathcal{BL}_b$ and for non-basic elements $\dim_\ell(\mathrm{Bun}_G^g) < 0$ by [FS21, IV.1.22]. This finishes the proof in this case.

Suppose now that b is not basic. Let M denote the centralizer of ν_b, and m denote the unique basic element in $B(M)$ lifting b whose Newton point is G-dominant, i.e., $\nu_m = \nu_b$. Now, $\mathrm{Bun}_P^m \simeq \mathrm{Bun}_G^b$ by our choice of m, and we let $\mathcal{E}_p$ denote the unique P-reduction of $\mathcal{E}_b$ on this strata. This is the so-called Harder–Narasimhan reduction of $\mathcal{E}_b$.

The space of P-modifications of $\mathcal{E}_p$ gets identified with $\mathrm{Gr}_P \subset \mathrm{Gr}_G$. We consider $\mathrm{Gr}_{P,\mu}^\circ$, defined as the intersection of $\mathrm{Gr}_{G,\mu}$ with the connected component of Gr_P attached to the G-dominant representative of μ. We can also write this as $\mathrm{Gr}_{P,\mu}^\circ := L^+P \cdot \xi^\mu \cdot L^+P/L^+P$, where $\xi \in B_{\mathrm{dR}}^+$ is a uniformizer and $L^+P = P(B_{\mathrm{dR}}^+)$. By our choice of μ, we get an open immersion, $\mathrm{Gr}_{P,\mu}^\circ \subset \mathrm{Gr}_{G,\mu}^\circ$, and a smooth map $\mathrm{Gr}_{P,\mu}^\circ \to \mathrm{Gr}_{M,\mu}^\circ$ of relative dimension $\langle 2\rho_{M,G}, \mu \rangle$, where we set $\rho_{M,G} = \rho_G - \rho_M$. We can also describe $\mathrm{Gr}_{M,\mu}^\circ$ as the quotient $L^+M \cdot \xi^\mu \cdot L^+M/L^+M$ compatibly with the parabolic description and the smooth cover. Moreover, we have commutative diagrams

$$\begin{tikzcd}
\mathrm{Gr}_{P,\mu}^{m_\mu} \arrow[r] \arrow[d] & \mathrm{Gr}_{P,\mu}^\circ \arrow[r] \arrow[d, "\mathcal{BL}_p"] & \mathrm{Gr}_{G,\mu}^\circ \arrow[d, "\mathcal{BL}_b"] \\
\mathrm{Bun}_P^{m_\mu} \arrow[r] \arrow[d] & \mathrm{Bun}_P \arrow[r] \arrow[d] & \mathrm{Bun}_G \\
\mathrm{Bun}_M^{m_\mu} \arrow[r] & \mathrm{Bun}_M. &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{Gr}_{P,\mu}^\circ \arrow[r] \arrow[d, "\mathcal{BL}_p"] & \mathrm{Gr}_{M,\mu}^\circ \arrow[d, "\mathcal{BL}_m"] \\
\mathrm{Bun}_P \arrow[r] & \mathrm{Bun}_M
\end{tikzcd} \tag{3.3}$$

with the first row of vertical arrows being Beauville–Laszlo uniformization maps and the top corner $\mathrm{Gr}_{P,\mu}^{m_\mu}$ making the left upper square Cartesian. In particular, the map $\mathrm{Gr}_{P,\mu}^{m_\mu} \to \mathrm{Gr}_{P,\mu}^{\circ}$ is a non-empty open immersion, so the left side has dimension d. Moreover, the map $\mathrm{Gr}_{P,\mu}^{\circ} \to \mathrm{Bun}_M$ is the composition of maps that are either ℓ-cohomologically smooth or pro-étale. Using this and the fact that $m_\mu \in B(M)$ is basic, it follows that the dimension drops on the complement of $\mathrm{Gr}_{P,\mu}^{m_\mu} \to \mathrm{Gr}_{P,\mu}^{\circ}$. We are reduced to showing that for $c \neq b_\mu$ the following inequality holds

$$\dim_\ell(\mathrm{Gr}_{P,\mu}^{m_\mu} \cap \mathrm{Gr}_{G,\mu}^{(c,b)}) < d. \tag{3.4}$$

We claim that for our choice of m, the element $m_\mu \in B(M)$ is non-positive. Recall that by our sign conventions $\kappa_M(m_\mu) = \kappa_M(m) - (\mu^\diamond)^\natural$ in $\pi_1(M)_\Gamma$. This corresponds to the unique connected component in $\pi_0(\mathrm{Bun}_M)$ through which $\mathcal{BL}_m$ factors. It follows that $\nu_{m_\mu} = \mathrm{av}_M(\nu_b - \mu^\diamond)$ since M-central elements in $X_*(T)^\Gamma_{\mathbb{Q}}$ are determined by their image in $\pi_1(M)^\Gamma_{\mathbb{Q}} \simeq \pi_1(M)_{\mathbb{Q},\Gamma}$. Using our assumption that $b \in A_Z(G,\mu)$ and Proposition 2.10 it follows that ν_{m_μ} is non-positive.

An application of Proposition 2.17 shows that

$$\dim_\ell(\mathrm{Bun}_P^{(m_\mu,c)}) < \langle 2\rho_{M,G}, -\nu_{m_\mu}\rangle = \langle 2\rho_{M,G}, \mathrm{av}_M(\mu^\diamond - \nu_b)\rangle. \tag{3.5}$$

By Lemma 3.3 below, the geometric fibers of

$$\mathrm{Gr}_{P,\mu}^{m_\mu} \to \mathrm{Gr}_{M,\mu}^{\circ} \times_{\mathrm{Bun}_M} \mathrm{Bun}_P^{m_\mu} \tag{3.6}$$

have all dimension bounded by $\langle 2\rho_{M,G}, \nu_b\rangle$. Consequently, Lemma 2.4 shows that (3.4) holds. Indeed, $\dim_\ell(\mathrm{Gr}_{P,\mu}^{m_\mu} \cap \mathrm{Gr}_{G,\mu}^{(c,b)})$ is bounded by the dimension of $\mathrm{Gr}_{M,\mu}^{\circ} \times_{\mathrm{Bun}_M} \mathrm{Bun}_P^{(m_\mu,c)}$ plus the dimension of the fibers. The former is strictly smaller than $\langle 2\rho_M, \mu\rangle + \langle 2\rho_{M,G}, \mathrm{av}_M(\mu^\diamond - \nu_b)\rangle$ and the latter is bounded by $\langle 2\rho_{M,G}, \nu_b\rangle$. Moreover, we have equalities

$$\langle 2\rho_{M,G}, \mathrm{av}_M(\mu^\diamond - \nu_b)\rangle = \langle 2\rho_{M,G}, \mu^\diamond - \nu_b\rangle = \langle 2\rho_{M,G}, \mu - \nu_b\rangle, \tag{3.7}$$

from which the bound follows. $\square$

Lemma 3.3. *The geometric fibers of (3.6) are either empty or torsors under the kernel of* $\mathrm{Aut}(\mathcal{E}_p) \to \mathrm{Aut}(\mathcal{E}_m)$*. In the former case, their dimension is* $\langle 2\rho_{M,G}, \nu_b\rangle$*.*

Remark 3.4. The quasi-torsor assertion does not use any special properties of $b \in B(G)$ or of our chosen P-reduction $\mathcal{E}_p \in \mathrm{Bun}_P(C)$. Nevertheless, the precise dimension of this group of automorphisms only holds for the Harder–Narasimhan reduction $\mathcal{E}_p$ of b.

Proof. We begin by observing that the geometric fibers of the Beauville–Laszlo map

$$\mathcal{BL}_{\mathcal{E}_p} : \mathrm{Gr}_P \to \mathrm{Bun}_P \tag{3.8}$$

are torsors on the left for the group $A^{-1}P(B_e)A$ where $A \in P(B_{\mathrm{dR}})$ is the Beauville–Laszlo gluing data for the P-torsor $\mathcal{E}_p$, see [SW20, Theorem 13.5.3.(2)]. Indeed, this is the group of modifications of the form

$$f : \mathcal{E}_p \dashrightarrow \mathcal{E}_p$$

and given a fixed modification $\alpha : \mathcal{E} \dashrightarrow \mathcal{E}_p$, every other modification having $\mathcal{E}$ as its source and $\mathcal{E}_p$ as its target can be obtained by composing α with some f as above. Similarly, the geometric fibers of $\mathrm{Gr}_M \to \mathrm{Bun}_M$ are $A^{-1}M(B_e)A$-torsors. We deduce that the geometric fibers of $\mathrm{Gr}_P \to \mathrm{Bun}_P \times_{\mathrm{Bun}_M} \mathrm{Gr}_M$ are torsors under the group $A^{-1}U(B_e)A$. In other words, the fibers are torsors under the group of modifications $\mathcal{E}_p \dashrightarrow \mathcal{E}_p$ that induce the identity on $\mathcal{E}_m$.

Recall that every $t \in P(B_{\mathrm{dR}})$ has a unique expression $t = u_t \cdot m_t$ with $u \in U(B_{\mathrm{dR}})$ and $m \in M(B_{\mathrm{dR}})$. We claim that if $t \in P(B_{\mathrm{dR}}^+)\xi^\mu P(B_{\mathrm{dR}}^+)$ then $u_t \in U(B_{\mathrm{dR}}^+)$. This follows from the normality of $U(B_{\mathrm{dR}}^+)$ in $P(B_{\mathrm{dR}}^+)$ and from the inclusion $\xi^\mu U(B_{\mathrm{dR}}^+) \subseteq U(B_{\mathrm{dR}}^+)\xi^\mu$, given the fact that μ is dominant. Consequently, if $u \in U(B_{\mathrm{dR}})$ and $x \in \mathrm{Gr}_{P,\mu}^{\circ}$ are such that $u \cdot x \in \mathrm{Gr}_{P,\mu}^{\circ}$, then necessarily $u \in U(B_{\mathrm{dR}}^+)$.

This implies that the non-empty geometric fibers of our map (3.6) form a torsor under the group $U(B_{\mathrm{dR}}^+) \cap A^{-1}U(B_e)A$, the unipotent part of the automorphism group of $(\mathcal{E}_p)$. By [FS21, Proposition III.5.1], its ℓ-dimension equals $\langle 2\rho_{M,G}, \nu_b \rangle$, and we may conclude the same about the non-empty fibers. □

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FIXED POINTS UNDER PINNING-PRESERVING AUTOMORPHISMS OF REDUCTIVE GROUP SCHEMES

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Abstract. In this paper we determine the scheme-theoretic fixed points of pinned reductive group schemes acted upon by a group of pinning-preserving automorphisms. The results are used in a companion paper to establish a ramified geometric Satake equivalence with integral or modular coefficients.

1. Introduction

1.1. In this paper we study some basic properties (e.g. flatness and smoothness) of the fixed point group scheme for an action of an abstract group on a split reductive group scheme by pinning preserving automorphisms. Our motivation for such a study comes from work on a ramified version of the geometric Satake equivalence; in fact, in the companion paper [ALRR23] we show that such fixed points group schemes (over the spectrum of the ℓ-adic integers $\mathbb{Z}_\ell$ or the finite field $\mathbb{F}_\ell$ for some prime number ℓ) arise as Tannakian groups for appropriate categories of equivariant perverse sheaves on affine Grassmannians attached to parahoric groups associated with special facets of Bruhat–Tits buildings. (An analogous study over $\mathbb{Q}_\ell$ was undertaken by Zhu [Zhu15] and by the third named author [Ric16].) We believe that our study is of independent interest. It is treated here in greater generality and in more detail than what is actually needed in [ALRR23].

1.2. **Statement.** Let S be a nonempty scheme, and consider a pinned reductive group $(\mathbf{G}, \mathbf{T}, M, \mathfrak{R}, \Delta, X)$ over S. In particular, $\mathbf{T} = \mathrm{D}_S(M)$ is a (split) maximal torus in $\mathbf{G}$, $\mathfrak{R}$ is the associated root system, Δ is a basis of $\mathfrak{R}$, and X is a collection of nonvanishing sections of the root subspaces in $\mathscr{L}ie(\mathbf{G})$ attached to simple roots. (In case S is the spectrum of a field or a principal ideal domain, one can take for $\mathbf{G}$ any reductive group scheme admitting a split maximal torus $\mathbf{T}$; then M is the lattice of characters of $\mathbf{T}$, and a pinning X always exists because the weight spaces in $\mathscr{L}ie(\mathbf{G})$ are free over $\mathscr{O}(S)$.) We then have a corresponding group $\mathrm{Aut}(\mathbf{G}, \mathbf{T}, M, \mathfrak{R}, \Delta, X)$ of automorphisms of $\mathbf{G}$ preserving these data (see §3.2 for precise references), which identifies with the group of automorphisms of the associated root datum. We consider an abstract group A and an action of A on $\mathbf{G}$ defined by a homomorphism $A \to \mathrm{Aut}(\mathbf{G}, \mathbf{T}, M, \mathfrak{R}, \Delta, X)$.

Our main result is the following (see Theorem 5.1 and Proposition 5.10). In this statement, M_A denotes the group of coinvariants for the action of A on M (see §2.2).

Theorem 1.1. (1) *The group scheme $\mathbf{G}^A$ is flat over S.*

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(2) *The group scheme* $\mathbf{G}^A$ *has geometrically connected fibers over* S *iff either* M_A *is torsion free or* S *has exactly one residual characteristic* $\ell > 0$ *and the torsion part of* M_A *is an* ℓ*-group.*

(3) *The group scheme* $\mathbf{G}^A$ *is smooth over* S *iff the following conditions hold:*
   (a) *the order of the torsion subgroup of* M_A *is prime to all residual characteristics of* S*;*
   (b) *if* $\mathfrak{R}$ *has an indecomposable component of type* A_{2n} *for some* $n \geq 1$ *whose stabilizer in* A *acts nontrivially on this component, then* 2 *is not a residual characteristic of* S*.*

(4) *If* $S = \mathrm{Spec}(\Bbbk)$ *for some field* $\Bbbk$*, then the reduced neutral component* $(\mathbf{G}^A)^\circ_{\mathrm{red}}$ *is a split reductive group scheme, and if* $\overline{\Bbbk}$ *is an algebraic closure of* $\Bbbk$ *one has*

$$((\overline{\Bbbk} \otimes_\Bbbk \mathbf{G})^A)^\circ_{\mathrm{red}} = \overline{\Bbbk} \otimes_\Bbbk (\mathbf{G}^A)^\circ_{\mathrm{red}}.$$

Let us first discuss some previous occurrences of such results in the literature. Statement (4) was already known. The case when $\Bbbk$ is algebraically closed and A is finite and cyclic (and satisfies a condition weaker than preserving a pinning) was treated by Steinberg [Ste68]; see also [DM94, Théorème 1.8] for a further study. A more general version (not requiring $\mathbf{G}$ to be split, and also replacing the existence of a fixed pinning by a weaker condition) is due to Adler–Lansky, see [AL14, Proposition 3.5]. In case $\Bbbk$ is algebraically closed and A is finite, the same statement as ours appears in work of Haines [Hai15]. All of these proofs are based on [Ste68]. We give a new proof of this statement here, which does not rely on [Ste68] except for elementary claims on root systems.

The smoothness of $\mathbf{G}^A$ (in case $S = \mathrm{Spec}(\mathbb{Z}[\frac{1}{2}])$, $\mathbf{G}$ is semisimple and simply-connected and A is cyclic) is also proved in [DHKM20, Lemma 4.25]; the nonsmoothness in the setting of condition (3b) is mentioned in [DHKM20, Remark 4.26]. In case $S = \mathrm{Spec}(\Bbbk)$ for a field $\Bbbk$, and A is finite of cardinality invertible in $\Bbbk$ (but does not necessarily fix a pinning), the fact that $(\mathbf{G}^A)^\circ$ is reductive (in particular, smooth) is also proved in [PY02, Theorem 2.1].

Let us point out also that if S is the spectrum of a mixed characteristic discrete valuation ring and the coinvariants M_A of the action of A on M are torsion-free, then $\mathbf{G}^A$ is a quasi-reductive S-group scheme in the sense of Prasad–Yu [PY06] (see Theorem 5.1(4)). A particularly interesting example is when $S = \mathrm{Spec}(\mathbb{Z}_2)$ and $\mathbf{G} = \mathrm{SL}_{2n+1,\mathbb{Z}_2}$ for some $n \geq 1$, with the unique nontrivial action of $\mathbb{Z}/2\mathbb{Z}$. In this case, $\mathbf{G}^A \to S$ is nonsmooth by (3), and hence in particular nonreductive.

1.3. **Outline of the proof.** The main step in the proof of Theorem 1.1 consists of an analysis of the fixed points of A on the big cell in $\mathbf{G}$ attached to our given pinning. For this we study separately the fixed points on $\mathbf{T}$ (which is rather straightforward) and on the (positive and negative) "maximal unipotent subgroups" $\mathbf{U}$ and $\mathbf{U}^-$. This part is more subtle, and requires the construction of an appropriate "extension" of X to a Chevalley system compatible (in the appropriate sense) with the action of A. We also analyze the case when S is the spectrum of an algebraically closed field in great detail in §5.3.

Let us note that the groups considered in Theorem 1.1 share many standard properties of reductive group schemes, although they are not reductive in general. In particular:

- to such a group we attach a root system (and even several root data, see §6.1);
- in §4.3 we construct certain "twisted SL_2-maps" associated with positive roots in this root system (whose domain is not necessarily $\mathrm{SL}_{2,S}$);
- in §6.2 we show that the quotient of the normalizer of $\mathbf{T}^A$ by its centralizer is (the constant group scheme attached to) a Coxeter group, which identifies with the Weyl group of the associated root system;
- in §6.3 we study analogues of parabolic and Levi subgroups in $\mathbf{G}^A$.

Here again, in the special case when S is the spectrum of a field, some of these constructions appear in work of Adler–Lansky and Haines, sometimes under weaker assumptions; see in particular [AL19, Hai18] for discussions of root data attached to fixed points.

1.4. **Contents.** In Section 2 we recall the definition of fixed-point schemes, and study some first examples. In Section 3 we recall the definition of pinned reductive group schemes, and some basic results on their structure. In Section 4 we construct our twisted SL_2-maps. In Section 5 we prove Theorem 1.1. Finally, in Section 6 we prove some complements, some of which will be used in [ALRR23].

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2. Fixed points

2.1. **Definition.** Let S be a scheme, and let A be an abstract group. Recall that if $X \to S$ is an S-scheme, endowed with an action of A by S-scheme automorphisms, then the functor of A-fixed points X^A is defined as the functor that sends an S-scheme Y to the set $\mathrm{Hom}_S(Y, X)^A$, where A acts on the set of S-scheme morphisms $\mathrm{Hom}_S(Y, X)$ via its action on X, and the superscript means fixed points in the usual sense. (In other words, X^A is the fixed-points sheaf associated with the natural action of A on the fpqc sheaf on the category of S-schemes represented by X.) If this functor is representable by a scheme, then this scheme will also be denoted X^A. It is clear from the definition that this construction is stable under base change; namely, if $S' \to S$ is a morphism of schemes then we have an identification of functors

$$S' \times_S X^A \xrightarrow{\sim} (S' \times_S X)^A \tag{2.1}$$

where on the right-hand side we consider the natural A-action on $S' \times_S X$ induced by the action on X. In particular, if X^A is representable by a scheme, then so is $(S' \times_S X)^A$. It is also clear that given an open covering $S = \bigcup_{i \in I} S_i$, the functor X^A is representable by a scheme if and only if for any i the functor $(S_i \times_S X)^A$ is representable by a scheme. (In this case we have an open covering $X^A = \bigcup_{i \in I} (S_i \times_S X)^A$.) This construction is functorial in the sense that if $f : X \to Y$ is an A-equivariant morphism of schemes we have a canonical morphism of functors $f^A : X^A \to Y^A$.

The following properties are easily verified from the definitions.

Lemma 2.1. (1) *Let X and Y be S-schemes endowed with actions of A, let $f : X \to Y$ be an A-equivariant monomorphism over S, and let $f^A : X^A \to Y^A$ be the induced morphism. Then the following diagram is cartesian:*

$$\begin{tikzcd} X^A \arrow[r, "f^A"] \arrow[d] & Y^A \arrow[d] \\ X \arrow[r, "f"] & Y. \end{tikzcd}$$

In particular, if Y^A is representable by a scheme then so is X^A, and if f is an open immersion, resp. a closed immersion, resp. an immersion, then so is f^A.

(2) *Let X and Y be S-schemes endowed with actions of A, and consider the diagonal action on $X \times_S Y$. Then we have a canonical identification*

$$(X \times_S Y)^A = X^A \times_S Y^A.$$

For general results on fixed-point schemes the reader is referred to [Fog71]. Below we will only consider the case when the morphism $X \to S$ is affine. Recall that under this assumption X^A is always representable by a scheme, and the natural morphism $X^A \to X$ is a closed immersion. Indeed, passing to an open covering we can assume that S is affine, say $S = \mathrm{Spec}(k)$ for some ring k. Then X is also affine, say $X = \mathrm{Spec}(R)$, where R is a k-algebra. The action of A on X corresponds to an action on R by k-algebra automorphisms. It is easily checked that X^A is represented by the closed subscheme $\mathrm{Spec}(R/I) \subset X$, where $I \subset R$ is the ideal generated by elements of the form $r - a \cdot r$ for $r \in R$ and $a \in A$.

In case X is an affine group scheme over S and A acts by group scheme automorphisms, then of course X^A is a (closed) subgroup scheme of X.

2.2. **The case of diagonalizable groups.** Let us now study the construction of §2.1 for certain actions on diagonalizable group schemes. Recall that for any scheme S and any abelian group M, we have an associated group scheme

$$\mathrm{D}_S(M)$$

representing the group valued functor $\underline{\mathrm{Hom}}_{S\text{-GrpSch}}(\underline{M}_S, \mathbf{G}_{\mathrm{m},S})$ on the category of S-schemes, see e.g. [DG70b, Exp. VIII, Définition 1.1]. Consider a group A and an action of A on M by group automorphisms. We deduce an action of A on the group scheme $\mathrm{D}_S(M)$ over S, by group scheme automorphisms. (By [DG70b, Exp. VIII, Corollaire 1.6], any action of A on $\mathrm{D}_S(M)$ by group scheme automorphisms arises in this way in case M is of finite type and S is connected.) We will denote by M_A the group of coinvariants for this action, i.e. the quotient of M by the subgroup generated by the elements of the form $m - a \cdot m$ for $a \in A$ and $m \in M$.

Lemma 2.2. *There exists a canonical isomorphism of group schemes over S*

$$\mathrm{D}_S(M_A) \xrightarrow{\sim} \big(\mathrm{D}_S(M)\big)^A.$$

Proof. The projection $M \to M_A$ is A-equivariant for the trivial action on the right-hand side; it therefore induces an A-equivariant morphism $\mathrm{D}_S(M_A) \to \mathrm{D}_S(M)$, which necessarily factors through a morphism $\mathrm{D}_S(M_A) \to (\mathrm{D}_S(M))^A$. To prove that this morphism is an isomorphism we can assume that $S = \mathrm{Spec}(k)$ is affine. Then $(\mathrm{D}_S(M))^A$ is the spectrum of the quotient R of the group algebra $k[M]$ by the ideal generated by the elements of the form $x - a \cdot x$ for $a \in A$ and $x \in k[M]$. For any k-algebra R' we have

$$\begin{aligned}\mathrm{Hom}_{k\text{-alg}}(R, R') = \mathrm{Hom}_{k\text{-alg}}(k[M], R')^A &= \mathrm{Hom}_{\mathrm{gps}}(M, (R')^\times)^A \\ &= \mathrm{Hom}_{\mathrm{gps}}(M_A, (R')^\times) = \mathrm{Hom}_{k\text{-alg}}(k[M_A], R')\end{aligned}$$

where k-alg is the category of k-algebras, gps is the category of groups, and in all cases the A-action is induced in the natural way by the action on M. We deduce an identification $R = k[M_A]$, which finishes the proof. □

Lemma 2.2 shows that $(\mathrm{D}_S(M))^A$ is always flat over S. If we assume that M is of finite type, then by [DG70b, Exp. VIII, Proposition 2.1(e)] this group scheme is smooth over S iff the order of the torsion subgroup of M_A is prime to all residual characteristics of S. Still assuming that M is of finite type, $(\mathrm{D}_S(M))^A$ is geometrically connected iff either M_A is torsion free or S has exactly one residual characteristic $\ell > 0$ and the torsion part of M_A is an ℓ-group.

2.3. **The case of** SL_{2n+1}**.** Let $n \geq 1$, and consider the group scheme $\mathrm{SL}_{2n+1,\mathbb{Z}}$ over $\mathrm{Spec}(\mathbb{Z})$. In this subsection, we study the construction of §2.1 for a certain action of the group $A = \mathbb{Z}/2\mathbb{Z}$ on $\mathrm{SL}_{2n+1,\mathbb{Z}}$.

2.3.1. *Action.* Denote by J_{2n+1} the square matrix of size $2n+1$ whose coefficient in position (i,j) is given by:

- 0 if $i+j \neq 2n+2$;
- 1 if $i+j = 2n+2$ and i is even;
- -1 if $i+j = 2n+2$ and i is odd.

That is, J_{2n+1} has entries $(-1,1,-1,\dots,-1,1,-1)$ on the anti-diagonal and 0 else, so $J_{2n+1}^2 = \mathrm{id}$. We then make $A = \mathbb{Z}/2\mathbb{Z}$ act on $\mathrm{SL}_{2n+1,\mathbb{Z}}$ by having the nontrivial element act by

$$M \mapsto J_{2n+1} \cdot {}^{\mathrm{t}}M^{-1} \cdot J_{2n+1}.$$

2.3.2. *The case of* SL_3. First, let us consider the case $n=1$, and denote by $\mathrm{U}_{3,\mathbb{Z}}$ the subgroup scheme of $\mathrm{SL}_{3,\mathbb{Z}}$ consisting of upper triangular unipotent matrices. For any ring R, the action of the nontrivial element of A on $\mathrm{U}_{3,\mathbb{Z}}(R)$ is given by

$$\begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & z & xz-y \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix}.$$

It follows that the fixed-point subscheme $(\mathrm{U}_{3,\mathbb{Z}})^{\mathbb{Z}/2\mathbb{Z}}$ is the closed subgroup scheme defined by the equations

$$x = z, \quad xz - y = y;$$

we therefore have

$$(\mathrm{U}_{3,\mathbb{Z}})^{\mathbb{Z}/2\mathbb{Z}} \cong \mathrm{Spec}\big(\mathbb{Z}[x,y]/(x^2-2y)\big).$$

Since $x^2 - 2y$ is monic as a polynomial in x, we see that $(\mathrm{U}_{3,\mathbb{Z}})^{\mathbb{Z}/2\mathbb{Z}}$ is finite and flat over $\mathrm{Spec}(\mathbb{Z}[y]) = \mathbb{A}^1_{\mathbb{Z}}$, and hence flat over $\mathbb{Z}$. More generally, for an arbitrary nonempty scheme S, considering the group scheme

$$\mathrm{U}_{3,S} := S \times_{\mathrm{Spec}(\mathbb{Z})} \mathrm{U}_{3,\mathbb{Z}},$$

in view of (2.1) we see that

$$(\mathrm{U}_{3,S})^{\mathbb{Z}/2\mathbb{Z}} = S \times_{\mathrm{Spec}(\mathbb{Z})} (\mathrm{U}_{3,\mathbb{Z}})^{\mathbb{Z}/2\mathbb{Z}}$$

is flat over S.

We claim that $(U_{3,S})^{\mathbb{Z}/2\mathbb{Z}}$ is smooth over S if and only if 2 is not a residual characteristic of S. Indeed, if 2 is a residual characteristic, and if $s \in S$ is such that the residue field $\kappa(s)$ has characteristic 2, then

$$\mathrm{Spec}(\kappa(s)) \times_S (\mathrm{U}_{3,S})^{\mathbb{Z}/2\mathbb{Z}} = \mathrm{Spec}(\kappa(s)[x,y]/x^2)$$

is not reduced, and hence not smooth (see [Sta22, Tag 056T]), so $(\mathrm{U}_{3,S})^{\mathbb{Z}/2\mathbb{Z}} \to S$ is not smooth. To prove the converse implication we can assume that S is affine, say $S = \mathrm{Spec}(k)$. If 2 is not a residual characteristic of S, then it is invertible in k, so $k \to k[x,y]/(x^2-2y)$ is a standard smooth ring map in the sense of [Sta22, Tag 00T6] (because $\frac{\partial}{\partial y}(x^2-2y)$ is invertible), and finally $(\mathrm{U}_{3,S})^{\mathbb{Z}/2\mathbb{Z}} \to S$ is smooth by [Sta22, Tag 00T7].

If $S = \mathrm{Spec}(\Bbbk)$ is the spectrum of a field, the reduced subscheme $(\mathrm{U}_{3,\Bbbk})^{\mathbb{Z}/2\mathbb{Z}}_{\mathrm{red}}$ is given by

$$(\mathrm{U}_{3,\Bbbk})^{\mathbb{Z}/2\mathbb{Z}}_{\mathrm{red}} = \left\{ \left(\begin{smallmatrix} 1 & x & x^2/2 \\ 0 & 1 & x \\ 0 & 0 & 1 \end{smallmatrix} \right) \right\} \text{ if } \mathrm{char}(\Bbbk) \neq 2, \qquad \left\{ \left(\begin{smallmatrix} 1 & 0 & y \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix} \right) \right\} \text{ if } \mathrm{char}(\Bbbk) = 2. \tag{2.2}$$

Over $\mathbb{Z}[\frac{1}{2}]$, the map of group schemes $\mathrm{SL}_{2,\mathbb{Z}[\frac{1}{2}]} \to (\mathrm{SL}_{3,\mathbb{Z}[\frac{1}{2}]})^{\mathbb{Z}/2\mathbb{Z}}$ given explicitily by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a^2 & ab & \frac{1}{2}b^2 \\ 2ac & ad+bc & bd \\ 2c^2 & 2cd & d^2 \end{pmatrix}$$

induces a closed immersion of group schemes

$$\xi : \mathrm{PGL}_{2,\mathbb{Z}[\frac{1}{2}]} \to (\mathrm{SL}_{3,\mathbb{Z}[\frac{1}{2}]})^{\mathbb{Z}/2\mathbb{Z}}. \tag{2.3}$$

(This morphism is induced by the adjoint action of $\mathrm{SL}_{2,\mathbb{Z}[\frac{1}{2}]}$ on its Lie algebra using the ordered basis $\left(\begin{smallmatrix} 0 & -2 \\ 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$.) On the other hand, we will see below that $(\mathrm{SL}_{3,\mathbb{F}_2})^{\mathbb{Z}/2\mathbb{Z}}$ is nonreduced; we have a closed immersion of group schemes

$$\xi : \mathrm{SL}_{2,\mathbb{F}_2} \to (\mathrm{SL}_{3,\mathbb{F}_2})^{\mathbb{Z}/2\mathbb{Z}}_{\mathrm{red}} \tag{2.4}$$

where the right-hand side is the reduced group scheme associated with $(\mathrm{SL}_{3,\mathbb{F}_2})^{\mathbb{Z}/2\mathbb{Z}}$, given explicitly by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a & 0 & b \\ 0 & 1 & 0 \\ c & 0 & d \end{pmatrix}.$$

In fact, the maps (2.3) and (2.4) are isomorphisms as we show in Example 5.9(1).

2.3.3. *Some morphisms.* Now, let us return to the case of a general $n \geq 1$. For any ring R, let $e_1, e_2, \ldots, e_{2n+1}$ be the standard basis of the free module $R^{2n+1} = \mathbb{A}^{2n+1}_{\mathbb{Z}}(R)$. For any $i \in \{1, \ldots, n\}$, define an embedding

$$\varphi_i : \mathbb{A}^3_{\mathbb{Z}} \to \mathbb{A}^{2n+1}_{\mathbb{Z}} \qquad \text{by} \qquad \begin{cases} e_1 \mapsto e_i, \\ e_2 \mapsto e_{n+1}, \\ e_3 \mapsto (-1)^{i+n} e_{2n+2-i} \end{cases}$$

Let $M_i = \mathrm{span}_R \, \{e_j : j \notin \{i, n+1, 2n+2-i\}\}$. Then $R^{2n+1} = \mathrm{image}(\varphi_i(R)) \oplus M_i$.

Make $\mathrm{SL}_{3,\mathbb{Z}}(R)$ act on R^{2n+1} by having it act trivially on M_i, and by the natural action on $\mathbb{A}^3_{\mathbb{Z}}(R)$ (transported across φ_i). This action defines a closed immersion of group schemes

$$f_{i,n} : \mathrm{SL}_{3,\mathbb{Z}} \to \mathrm{SL}_{2n+1,\mathbb{Z}}. \tag{2.5}$$

Explicitly, $f_{i.n}$ sends a matrix $M = (m_{rs})_{1 \leq r,s \leq 3}$ in $\mathrm{SL}_{3,\mathbb{Z}}(R)$ to the $(2n+1) \times (2n+1)$ matrix whose entry in position (j,k) is given by the following table:

Position	*Entry*	*Position*	*Entry*
(i,i)	m_{11}	$(2n+2-i,i)$	$(-1)^{i+n}m_{31}$
$(i,n+1)$	m_{12}	$(2n+2-i,n+1)$	$(-1)^{i+n}m_{32}$
$(i,2n+2-i)$	$(-1)^{i+n}m_{13}$	$(2n+2-i,2n+2-i)$	m_{33}
$(n+1,i)$	m_{21}	$(j,j),\ j \notin \{i,n+1,2n+2-i\}$	1
$(n+1,n+1)$	m_{22}	all other entries	0
$(n+1,2n+2-i)$	$(-1)^{i+n}m_{23}$		

One checks by explicit computation that this morphism is equivariant with respect to the actions of $\mathbb{Z}/2\mathbb{Z}$ considered above; it therefore restricts to a morphism of group schemes

$$(\mathrm{SL}_{3,\mathbb{Z}})^{\mathbb{Z}/2\mathbb{Z}} \to (\mathrm{SL}_{2n+1,\mathbb{Z}})^{\mathbb{Z}/2\mathbb{Z}}.$$

3. Pinned reductive group schemes

3.1. **Definition.** Let S be a nonempty scheme, and let $(\mathbf{G}, \mathbf{T}, M, \mathfrak{R}, \Delta, X)$ be a pinned reductive group scheme over S in the sense of [DG11b, Exp. XXIII, Definition 1.1]. In concrete terms:

- $\mathbf{T}$ is a maximal torus of $\mathbf{G}$, M is a free abelian group of finite rank, and we are given an isomorphism of S-group schemes $\mathbf{T} \cong \mathrm{D}_S(M)$;
- $\mathfrak{R} \subset M$ is a root system of $\mathbf{G}$ with respect to $\mathbf{T}$ such that $(M, \mathfrak{R})$ defines a splitting of $\mathbf{G}$ in the sense of [DG11b, Exp. XXII, Définition 1.13];
- $\Delta \subset \mathfrak{R}$ is a system of simple roots;

- $X = (X_\alpha : \alpha \in \Delta)$ is a collection of elements in $\mathscr{L}ie(\mathbf{G})$ such that each X_α is a nowhere vanishing section of the invertible $\mathscr{O}_S$-module $\mathscr{L}ie(\mathbf{G})^\alpha$.

The datum of $\Delta \subset \mathfrak{R}$ determines a subset of positive roots, which will be denoted $\mathfrak{R}_+$.

Let us comment briefly on these data, following [DG11b, Exp. XXII, Proposition 2.2]. Consider a reductive group scheme $\mathbf{G}$ over S with a maximal torus $\mathbf{T}$. Saying that $\mathbf{T}$ is split is the same as saying that there exists a free abelian group M and an isomorphism $\mathbf{T} \cong \mathrm{D}_S(M)$. If S is connected, then M is canonically determined by $\mathbf{T}$, since it identifies with the group of S-scheme morphisms from $\mathbf{T}$ to $\mathbb{G}_{\mathrm{m},S}$.

Next, we have the functor of roots $\mathcal{R}$ of $\mathbf{G}$ with respect to $\mathbf{T}$, which is a locally constant, finite scheme, realized as an open and closed subscheme of the group scheme $\underline{\mathrm{Hom}}_{S\text{-GrpSch}}(\mathbf{T}, \mathbf{G}_{\mathrm{m},S})$, see [DG11b, Exp. XIX, Proposition 3.8]. Given a subset $\mathfrak{R}$ of the group of morphisms of S-group schemes from $\mathbf{T}$ to $\mathbb{G}_{\mathrm{m},S}$, $\mathfrak{R}$ is a root system for $\mathbf{G}$ with respect to $\mathbf{T}$ iff the canonical inclusions of $\mathfrak{R}_S$ and $\mathcal{R}$ into $\underline{\mathrm{Hom}}_{S\text{-GrpSch}}(\mathbf{T}, \mathbf{G}_{\mathrm{m},S})$ induce an isomorphism $\mathfrak{R}_S \to \mathcal{R}$.

We now observe that when $\mathbf{G}$ admits a split maximal torus $\mathbf{T} \cong \mathrm{D}_S(M)$ and S is connected, a root system always exists, and is unique. Indeed, the action of $\mathbf{T}$ on $\mathscr{L}ie(\mathbf{G})$ determines an M-grading on this $\mathscr{O}_S$-module. If $\mathfrak{R} \subset M$ is the set of nonzero weights for this action, for any $\alpha \in \mathfrak{R}$ the corresponding α-weight space $\mathscr{L}ie(\mathbf{G})^\alpha$ is a direct summand in $\mathscr{L}ie(\mathbf{G})$, and hence a locally free sheaf. Since its rank is locally constant (see [Sta22, Tag 01C9]), it has to be constant (and positive), so α is a root for $\mathbf{G}$ by [DG11b, Exp. XIX, Définition 3.2]. In view of [DG11b, Exp. XIX, Définition 3.2], $\mathfrak{R}$ is therefore a root system for $\mathbf{G}$ with respect to $\mathbf{T}$.

Let us continue with the assumptions of the previous paragraph. As explained above, for any $\alpha \in \mathfrak{R}$ the root subspace $\mathscr{L}ie(\mathbf{G})^\alpha$ is a locally free $\mathscr{O}_S$-module of rank one, see [DG11b, Exp. XIX, §3.4]. The pair $(M, \mathfrak{R})$ defines a splitting of $\mathbf{G}$ with respect to $\mathbf{T}$ in the sense of [DG11b, Exp. XXII, Définition 1.13] iff each $\mathscr{L}ie(\mathbf{G})^\alpha$ is free, which is automatic e.g. if $\mathrm{Pic}(S)$ is trivial. Under this assumption, of course a collection $(X_\alpha : \alpha \in \Delta)$ of nowhere vanishing sections of the root subgroups attached to any choice of simple roots exists.

In conclusion, in case S is connected and $\mathrm{Pic}(S)$ is trivial (e.g. if S is the spectrum of a principal ideal domain), the datum of a pinned reductive group scheme over S is equivalent to the datum of a reductive group with a given split maximal torus, a system of simple roots, and a collection of nowhere vanishing sections of the associated simple root subspaces.

Example 3.1. Below we will use the standard pinning of the group scheme $\mathbf{G} = \mathrm{SL}_{2n+1,\mathbb{Z}}$ over $\mathrm{Spec}(\mathbb{Z})$ (for $n \geq 1$). In this case:

- $\mathbf{T}$ is the subgroup of diagonal matrices;
- M is the quotient $\mathbb{Z}^{2n+1}/\Delta\mathbb{Z}$ (where $\Delta\mathbb{Z}$ is the diagonal copy of $\mathbb{Z}$ in $\mathbb{Z}^{2n+1}$);
- $\mathfrak{R} = \{[\varepsilon_i - \varepsilon_j] : i \neq j \in \{1, \ldots, 2n+1\}\}$ (where $(\varepsilon_1, \ldots, \varepsilon_{2n+1})$ is the standard basis of $\mathbb{Z}^{2n+1}$, and $[\lambda]$ is the class of an element $\lambda \in \mathbb{Z}^{2n+1}$ in M);
- $\Delta = \{[\varepsilon_i - \varepsilon_{i+1}] : i \in \{1, \ldots, 2n\}\}$;
- if $\alpha = [\varepsilon_i - \varepsilon_{i+1}]$, then X_α is the matrix whose unique nonzero coefficient is 1 in position $(i, i+1)$.

3.2. **Automorphisms.** Let us now come back to the case of a general base scheme S. From now on in this section we fix a pinned reductive group scheme

$$(\mathbf{G}, \mathbf{T}, M, \mathfrak{R}, \Delta, X) \tag{3.1}$$

over S. We can then consider the group

$$\mathrm{Aut}(\mathbf{G}, \mathbf{T}, M, \mathfrak{R}, \Delta, X)$$

of automorphisms $f : \mathbf{G} \to \mathbf{G}$ of $\mathbf{G}$ that preserve the given pinning, in the sense of [DG11b, Exp. XXIII, Définition 1.3]. By definition (see in particular [DG11b, Exp. XXII, Définition 4.2.1]), any such automorphism f restricts to an automorphism of $\mathbf{T}$ induced by an automorphism of M,[1] which preserves $\mathfrak{R}$ and Δ, and it permutes the collection X according to its action on Δ. In fact, as noted in [DG11b, Exp. XXII, Remarque 4.2.2], f determines an automorphism of the root datum

$$(M, M^\vee, \mathfrak{R}, \mathfrak{R}^\vee)$$

attached to $\mathbf{G}$, and by [DG11b, Exp. XXIII, Théorème 4.1] this procedure identifies $\mathrm{Aut}(\mathbf{G}, \mathbf{T}, M, \mathfrak{R}, \Delta, X)$ with the group of automorphisms of the root datum $(M, M^\vee, \mathfrak{R}, \mathfrak{R}^\vee)$ stabilizing the subset $\Delta \subset \mathfrak{R}$. We will say that a group A acts on $\mathbf{G}$ *by pinned automorphisms* if it acts via a group homomorphism

$$A \to \mathrm{Aut}(\mathbf{G}, \mathbf{T}, M, \mathfrak{R}, \Delta, X).$$

In this situation, there is an induced action of A on M preserving $\mathfrak{R}$ and $\mathfrak{R}_+$.

3.3. **Root subgroups, Borel subgroup, and unipotent subgroup.** Recall that for any root $\gamma \in \mathfrak{R}$, there is a closed immersion

$$\exp_\gamma : \mathscr{L}ie(\mathbf{G})^\gamma \hookrightarrow \mathbf{G} \tag{3.2}$$

whose image, denoted by $\mathbf{U}_\gamma$, is a closed subgroup scheme of $\mathbf{G}$: see [DG11b, Exp. XXII, Théorème 1.1].

We endow the subset $\mathfrak{R}_+ \subset \mathfrak{R}$ with some arbitrary order. Then one can consider the product morphism

$$\prod_{\alpha \in \mathfrak{R}_+} \mathbf{U}_\alpha \to \mathbf{G},$$

where the product on the left-hand side (to be understood as fiber product over S) is taken with respect to our chosen order on $\mathfrak{R}_+$. This morphism is a closed immersion, and its image $\mathbf{U}$ is a subgroup scheme which does not depend on the choice of order on $\mathfrak{R}_+$; see [DG11b, Exp. XXII, §5.5]. The product morphism

$$\mathbf{T} \ltimes_S \mathbf{U} \to \mathbf{G}$$

is also a closed immersion, and a homomorphism of group schemes; its image will be denoted $\mathbf{B}$. On geometric fibers of $\mathbf{G} \to S$, $\mathbf{B}$ is a Borel subgroup in the usual sense, and $\mathbf{U}$ is its unipotent radical. Similar considerations using the negative roots $-\mathfrak{R}_+$ produce a closed subgroup scheme denoted $\mathbf{U}^-$.

3.4. **Chevalley systems.** Recall that the set $X = (X_\alpha : \alpha \in \Delta)$ is indexed by the simple roots. A *Chevalley system* is a collection $(Y_\alpha : \alpha \in \mathfrak{R})$ parametrized by $\mathfrak{R}$, where each Y_α is again a nowhere vanishing section of $\mathscr{L}ie(\mathbf{G})^\alpha$, and where the entire collection is subject to certain conditions, spelled out in [DG11b, Exp. XXIII, Définition 6.1]. By [DG11b, Exp. XXIII, Proposition 6.2], a Chevalley system exists. More specifically, the proof of this proposition shows that there exist Chevalley systems which satisfy the following additional conditions:

- for any $\alpha \in \Delta$ we have $X_\alpha = Y_\alpha$;
- for any $\alpha \in \mathfrak{R}_+$ the sections Y_α and $Y_{-\alpha}$ are dual to each other with respect to the pairing of [DG11b, Exp. XX, Corollaire 2.6].

When considering Chevalley systems below we will always tacitly assume (following e.g. the conventions in [BT84, §3.2.2]) that these additional conditions are satisfied. In this case, one can use the notation $(X_\alpha : \alpha \in \mathfrak{R})$ for a Chevalley system, and this system is determined by the "positive" subset $(X_\alpha : \alpha \in \mathfrak{R}_+)$.

[1]As noted in §2.2, in case S is connected, the condition that the restriction to $\mathbf{T}$ is induced by an automorphism of M is automatically satisfied.

According to [DG11b, Exp. XXIII, Corollaire 6.5], one consequence of the conditions in the definition is that in a Chevalley system, if α and β are roots such that $\alpha+\beta$ is also a root, then

$$[X_\alpha, X_\beta] = \pm r X_{\alpha+\beta} \qquad \begin{array}{c}\text{where } r \in \{1,2,3\} \text{ is the smallest positive} \\ \text{integer such that } \beta - r\alpha \text{ is not a root.}\end{array} \tag{3.3}$$

Of course, if $\alpha+\beta$ is not a root then $[X_\alpha, X_\beta]=0$.

Example 3.2. Let us give an example where one can write down (the positive part of) a Chevalley system explicitly in terms of a pinning $(X_\alpha : \alpha \in \Delta)$. Assume that $\mathbf{G}$ is of type D_4, and number the simple roots as $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ with $\langle \alpha_2, \alpha_i^\vee \rangle = -1$ for $i \in \{1,3,4\}$. There are eight other positive roots. One can easily check that the following vectors are the positive part of a Chevalley system:

$$\begin{aligned} X_{\alpha_i+\alpha_2} &= [X_{\alpha_i}, X_{\alpha_2}] \quad (i=1,2,3), & X_{\alpha_2+\alpha_3+\alpha_4} &= [[X_{\alpha_3}, X_{\alpha_2}], X_{\alpha_4}], \\ X_{\alpha_1+\alpha_2+\alpha_3} &= [[X_{\alpha_1}, X_{\alpha_2}], X_{\alpha_3}], & X_{\alpha_1+\alpha_2+\alpha_3+\alpha_4} &= [[[X_{\alpha_1}, X_{\alpha_2}], X_{\alpha_3}], X_{\alpha_4}], \\ X_{\alpha_1+\alpha_2+\alpha_4} &= [[X_{\alpha_1}, X_{\alpha_2}], X_{\alpha_4}], & X_{\alpha_1+2\alpha_2+\alpha_3+\alpha_4} &= [[[[X_{\alpha_1}, X_{\alpha_2}], X_{\alpha_3}], X_{\alpha_4}], X_{\alpha_2}]. \end{aligned}$$

Once a Chevalley system $(X_\alpha : \alpha \in \mathfrak{R}_+)$ as above is fixed, for any $\alpha \in \mathfrak{R}_+$ there exists a unique morphism of S-group schemes

$$\varphi_\alpha : \mathrm{SL}_{2,S} \to \mathbf{G} \tag{3.4}$$

which satisfies

$$\varphi_\alpha \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} = \exp_\alpha(aX_\alpha), \qquad \varphi_\alpha \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} = \exp_{-\alpha}(aX_{-\alpha})$$

for any $a \in \mathbf{G}_{\mathrm{a},S}$, see [DG11b, Exp. XX, Corollaire 2.6]. In this case we automatically have

$$\varphi_\alpha \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} = \alpha^\vee(a)$$

for any $a \in \mathbf{G}_{\mathrm{m},S}$. We set

$$n_\alpha := \varphi_\alpha \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

(This section coincides with the section denoted $w_\alpha(\mathrm{X}_\alpha)$ in [DG11b, Exp. XXIII, Définition 6.1].)

3.5. **Reduction to simply connected quasi-simple groups.** We will say that a pinned reductive group scheme $(\mathbf{G}, \mathbf{T}, M, \mathfrak{R}, \Delta, X)$ is quasi-simple and simply connected if $\mathfrak{R}$ is indecomposable and $\mathfrak{R}^\vee$ generates $M^\vee$ (or, in other words, if $\mathbf{G}_{\overline{s}}$ is quasi-simple and simply connected in the usual sense of semisimple algebraic groups for any geometric point $\overline{s}$ of S). For some constructions below, we will reduce the problem to this case (or to products of such groups) as follows.

The root datum of our group $\mathbf{G}$ with respect to $\mathbf{T}$ is $(M, M^\vee, \mathfrak{R}, \mathfrak{R}^\vee)$. We set $M_{\mathrm{sc}} := \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}\mathfrak{R}^\vee, \mathbb{Z})$, and $M_{\mathrm{sc}}^\vee := \mathbb{Z}\mathfrak{R}^\vee$. We have natural (dual) maps $M_{\mathrm{sc}}^\vee \to M^\vee$ and $M \to M_{\mathrm{sc}}$. The latter morphism is injective on the subset $\mathfrak{R}$, which can therefore also be regarded as a subset of M_{sc}. We then have a morphism of root data

$$(M_{\mathrm{sc}}, M_{\mathrm{sc}}^\vee, \mathfrak{R}, \mathfrak{R}^\vee) \to (M, M^\vee, \mathfrak{R}, \mathfrak{R}^\vee)$$

in the sense of [DG11b, Exp. XXI, Définition 6.1.1]. If we denote by

$$(\mathbf{G}_{\mathrm{sc}}, \mathbf{T}_{\mathrm{sc}}, M_{\mathrm{sc}}, \mathfrak{R}, \Delta, X_{\mathrm{sc}})$$

the pinned reductive group scheme over S with root datum $(M_{\mathrm{sc}}, M_{\mathrm{sc}}^\vee, \mathfrak{R}, \mathfrak{R}^\vee)$, then this morphism corresponds to a morphism of pinned groups

$$\mathbf{G}_{\mathrm{sc}} \to \mathbf{G}.$$

Moreover, for any geometric point $\overline{s}$ of S, this morphism identifies $(\mathbf{G}_{\mathrm{sc}})_{\overline{s}}$ with the simply connected cover of the derived subgroup of $\mathbf{G}_{\overline{s}}$. The roots of $\mathbf{G}$ and $\mathbf{G}_{\mathrm{sc}}$ are the same, and our morphism $\mathbf{G}_{\mathrm{sc}} \to \mathbf{G}$ restricts to an isomorphism on root subgroups with the same label, identifying the given pinnings.

One can make $\mathbf{G}_{\mathrm{sc}}$ more concrete as follows. Consider the decomposition

$$\mathfrak{R} = \bigsqcup_{i \in I} \mathfrak{R}_i$$

of $\mathfrak{R}$ as a direct sum of indecomposable constituents. This determines a direct-sum decomposition

$$M_{\mathrm{sc}} = \bigoplus_{i \in I} M_{\mathrm{sc},i},$$

and hence a product decomposition (where the product is fiber product over S)

$$\mathbf{G}_{\mathrm{sc}} = \prod_{i \in I} \mathbf{G}_{\mathrm{sc},i} \tag{3.5}$$

where $\mathbf{G}_{\mathrm{sc},i}$ is the pinned reductive group scheme over S with root datum

$$(M_{\mathrm{sc},i}, (M_{\mathrm{sc},i})^\vee, \mathfrak{R}_i, \mathfrak{R}_i^\vee).$$

4. Equivalence classes of roots and twisted SL_2-maps

4.1. **An equivalence relation on roots.** In this subsection we consider an arbitrary reduced root system $\mathfrak{R}$ in a real vector space V, and a basis $\Delta \subset \mathfrak{R}$ of $\mathfrak{R}$. We assume we are given a group A and an action of A on V preserving $\mathfrak{R}$ and Δ. We will denote by $\mathfrak{R}_+$ the system of positive roots determined by Δ. (This subset is also preserved by A.) Let us consider the equivalence relation $\sim$ on $\mathfrak{R}_+$ defined as follows:

$$\alpha \sim \beta \quad \text{if} \sum_{\gamma \in A \cdot \alpha} \gamma \text{ and} \sum_{\delta \in A \cdot \beta} \delta \text{ are scalar multiples of one another.} \tag{4.1}$$

(Here we mean scalar multiples in the real vector space V. This relation of course depends on the group A and its action on $\mathfrak{R}$, although it does not appear in the notation.)

First, assume that $\mathfrak{R}$ is indecomposable. In this case this equivalence relation is studied in [Ste68], and this analysis (based on a case-by-case verification) shows that each equivalence class E for $\sim$ is of one of the following two forms.

(i) E is a single A-orbit, and if $\alpha, \beta \in E$ then $\alpha + \beta$ is not a root.
(ii) E is of the form $\{\alpha, a \cdot \alpha, \alpha + a \cdot \alpha\}$ for some $a \in A$ and $\alpha \in \mathfrak{R}$; in this case, $\{\alpha, a \cdot \alpha\}$ is an A-orbit, and $\alpha + a \cdot \alpha$ is fixed by A.

In more detail, if the image of A in the automorphism group of the Dynkin diagram of $\mathfrak{R}$ is a cyclic group, then this statement is [Ste68, Claim (2′) in the proof of Theorem 8.2]. The only case not covered is that in which $\mathfrak{R}$ is of type D_4, and the image of A is the full automorphism group (which is the symmetric group $\mathfrak{S}_3$). In that case, a direct calculation shows that there are six equivalence classes for $\sim$, each of type (i).

As explained in [Ste68], equivalence classes of type (ii) occur if and only if $\mathfrak{R}$ is of type A_{2n} and A acts nontrivially. More explicitly, if we choose a labeling $\alpha_1, \dots, \alpha_{2n}$ of the simple roots such that $\langle \alpha_i, \alpha_j^\vee \rangle = -1$ if $|j - i| = 1$, then the equivalence classes of type (ii) are the sets of the form

$$\{\alpha_i + \cdots + \alpha_n, \quad \alpha_{n+1} + \cdots + \alpha_{2n+1-i}, \quad \alpha_i + \cdots + \alpha_{2n+1-i}\} \tag{4.2}$$

with $i \in \{1, \dots, n\}$.

Now we consider the general case. Write

$$\mathfrak{R} = \bigsqcup_{i \in I} \mathfrak{R}_i \tag{4.3}$$

for the decomposition of $\mathfrak{R}$ as a direct sum of its indecomposable constituents. Given $a \in A$ and $i \in I$ there exists a unique $\sigma(a)(i) \in I$ such that $a(\mathfrak{R}_i) = \mathfrak{R}_{\sigma(a)(i)}$, and this operation defines a group homomorphism

$$\sigma : A \to \mathfrak{S}_I. \tag{4.4}$$

For any $i \in I$, we denote by $V_i \subset V$ the subspace spanned by $\mathfrak{R}_i$, by $A_i = \{a \in A \mid \sigma(a)(i) = i\}$ the stabilizer of the component $\mathfrak{R}_i$, and by $\sim_i$ the equivalence relation defined as above for the action of A_i on the root system $\mathfrak{R}_i$.

Lemma 4.1. *If $\alpha, \beta \in \mathfrak{R}_+$, we have $\alpha \sim \beta$ if and only if there exist $a \in A$ and $i \in I$ such that $a \cdot \alpha$ and β both belong to $\mathfrak{R}_i$, and moreover $(a \cdot \alpha) \sim_i \beta$.*

Proof. Let us write $i \leftrightarrow j$ if i and j belong to the same orbit of the action of A on I via σ. In this case, fix an element $a_{i,j} \in A$ such that $a_{i,j}(\mathfrak{R}_i) = \mathfrak{R}_j$. Then if $\alpha \in \mathfrak{R}_i$, the projection of $\sum_{a \in A} a \cdot \alpha$ to V_j along the decomposition $V = \bigoplus_k V_k$ is $a_{i,j} \cdot (\sum_{a \in A_i} a \cdot \alpha)$ if $i \leftrightarrow j$, and zero otherwise. The claim easily follows. □

This lemma and the analysis of the indecomposable case above show that, in general, each equivalence class E for $\sim$ is of one of the following two forms.

(i′) E is a single A-orbit, and if $\alpha, \beta \in A$ then $\alpha + \beta$ is not a root.

(ii′) $\mathfrak{R}$ contains an indecomposable constituent $\mathfrak{R}'$ of type A_{2n} whose stabilizer A' acts nontrivially on this constituent, and

$$E = \bigsqcup_{a \in A/A'} a \cdot E'$$

where E' is a subset of the form (4.2) in the given indecomposable constituent of $\mathfrak{R}$.

We will say that a positive root β is *special* if it belongs to an equivalence class E of type (ii′) and is a sum of two roots which belong to E. In other words, if E is an equivalence class of type (ii′) and E' is as above, then E' is of the form $\{\alpha, a \cdot \alpha, \alpha + a \cdot \alpha\}$ for some $\alpha \in \mathfrak{R}'$ and $a \in A'$; the special roots in E are those of the form $b \cdot (\alpha + a \cdot \alpha)$ for some $b \in A$. In this situation, E is the union of exactly two A-orbits, one consisting of special roots, and the other of nonspecial roots.

Remark 4.2. Note that each equivalence class E for $\sim$ is "closed under positive combinations" in the following sense: if $\alpha, \beta \in E$ are distinct and $i, j \in \mathbb{Z}_{\geq 1}$ are such that $i\alpha + j\beta \in \mathfrak{R}$, then in fact $i\alpha + j\beta \in E$. If E is of type (ii′), then this claim is clear from the explicit description above. If E is of type (i′), we use the fact that if α, β are positive roots such that $\alpha + \beta \notin \mathfrak{R}$, then there are no $i, j \in \mathbb{Z}_{\geq 1}$ and such that $i\alpha + j\beta \in \mathfrak{R}$. (In fact, since neither $\alpha + \beta$ nor $\alpha - \beta$ are roots we have $\langle \beta, \alpha^\vee \rangle$=0 by [Bou02, Chap. VI, Corollary to Theorem 1]. If we assume for a contradiction that there exists a root of the form $i\alpha + j\beta$ with $i, j \geq 1$, and choose these coefficients such that $i+j$ is minimal, then since $\langle i\alpha + j\beta, \alpha^\vee \rangle = 2i > 0$, by [Bou02, Chap. VI, Corollary to Theorem 1] $(i-1)\alpha + j\beta$ is a root. By minimality we must have $i = 1$, so that $j\beta$ is a root. Since $\mathfrak{R}$ is reduced this forces $j = 1$, which provides a contradiction since $\alpha + \beta$ is not a root.)

In §5.3 we will use the following fact.

Lemma 4.3. *Let $(\alpha_1, \dots, \alpha_r)$ be representatives for the equivalence classes for $\sim$, and let $(\bar{\alpha}_1, \dots, \bar{\alpha}_r)$ be their images in the coinvariants V_A. Then each $\bar{\alpha}_i$ is nonzero, and the lines $(\mathbb{R} \cdot \bar{\alpha}_1, \dots, \mathbb{R} \cdot \bar{\alpha}_r)$ in V_E are pairwise distinct.*

Proof. The proof can be easily reduced to the case $\mathfrak{R}$ is irreducible. In this case, the claim can be checked case-by-case by inspecting the description of roots in each type (see e.g. [Bou02]), and the various possibilities for the action of A. □

4.2. **Chevalley–Steinberg systems.** From now on in this section we fix a pinned reductive group scheme (3.1) over S as in §3.1 (without any special assumption on S), endowed with an action of a group A by pinned automorphisms (see §3.2). Below we will need to consider Chevalley systems which afford some compatibility property with our given action of A. (Such systems are called *Chevalley–Steinberg systems* in [Lan96, Definition 4.3] or [Lou19, Définition 2.1.4].) The statement of this property involves the equivalence relation $\sim$ on $\mathfrak{R}_+$ from §4.1 for the given action of A on $\mathfrak{R}$, seen as a root system in the subspace it generates in $\mathbb{R} \otimes_{\mathbb{Z}} M$.

Proposition 4.4. *There exists a Chevalley system* $(X_\alpha : \alpha \in \mathfrak{R})$ *such that*

$$a \cdot X_\beta = X_{a \cdot \beta} \tag{4.5}$$

for all $a \in A$ *and all nonspecial* $\beta \in \mathfrak{R}_+$.

Proof. First, assume that $\mathfrak{R}$ is indecomposable. In this case, the statement is essentially proved in [Lan96, Proposition 4.4]. Let us give a brief outline of the argument, for the reader's convenience. The basic observation is that making sign changes in a Chevalley system on vectors associated with nonsimple positive roots (and the corresponding changes for the opposite roots) produces a new Chevalley system.

We can of course replace A by its image in $\mathrm{Aut}(\mathbf{G}, \mathbf{T}, M, \mathfrak{R}, \Delta, X)$. Since $\mathbf{G}$ is quasi-simple, the group $\mathrm{Aut}(\mathbf{G}, \mathbf{T}, M, \mathfrak{R}, \Delta, X)$ is the automorphism group of a connected Dynkin diagram, and therefore has order 1, 2, or 6 (the last possibility occurring only in type D_4, where the group is $\mathfrak{S}_3$). The subgroup A thus is cyclic of order 1, 2, 3, or $A \cong \mathfrak{S}_3$. One can then make sign changes on some elements X_α ($\alpha \in \mathfrak{R}_+$) to ensure that the desired condition holds, by considering the various orbits $A \cdot \beta$ of nonspecial roots as follows (see [Lan96, Proposition 4.4] for details).

- If A is cyclic (i.e., of order 1, 2, or 3) and the stabilizer in A of β has odd order (either 1 or 3), then one can ensure that $a \cdot X_\gamma = X_{a \cdot \gamma}$ for all $a \in A$ and $\gamma \in A \cdot \beta$.
- Assume now that A has order 2 and the stabilizer of β is all of A (so that $A \cdot \beta = \{\beta\}$). Denote by a the unique nontrivial element in A. Since β is not special, it is not of the form $\gamma + a \cdot \gamma$ for some $\gamma \in \mathfrak{R}_+$. Then a brief calculation using root system combinatorics shows that $a \cdot X_\beta = X_\beta$ for any $a \in A$.
- When A has order 6, the argument in [Lan96, Proposition 4.4] does not seem to be quite complete. In this case, $\mathbf{G}$ has type D_4, and all equivalence classes for $\sim$ are of type (i). Applying a to each of the equations in Example 3.2 one sees that this Chevalley system satisfies
$$a \cdot X_\beta = X_{a \cdot \beta} \qquad \text{for all } a \in A.$$
(In fact, it is enough to check this for a belonging to some set of generators of the group $A \cong \mathfrak{S}_3$.)

Now, let us explain how to treat the general case. Using the construction of §3.5 one can assume that $\mathbf{G}$ is a product of quasi-simple simply connected groups. (In fact, if $\mathbf{G}_{\mathrm{sc}}$ is as in §3.5, the root subgroups and the notion of Chevalley system coincide for $\mathbf{G}_{\mathrm{sc}}$ and $\mathbf{G}$.) In this case, associated with the decomposition (4.3) is the decomposition

$$\mathbf{G} = \prod_{i \in I} \mathbf{G}_i.$$

Recall also the action of A on I given by σ, see (4.4), and choose a subset $J \subset I$ of representatives for the A-orbits. For any $j \in J$, we also denote by $A_j \subset A$ the stabilizer of j, and choose representatives $a_1^j, \dots, a_{r(j)}^j$ for the quotient A/A_j. We then have a bijection

$$\bigsqcup_{j \in J} \mathfrak{R}_j \times \{1, \dots, r(j)\} \xrightarrow{\sim} \mathfrak{R}$$

given by $(\alpha, k) \mapsto a_k^j \cdot \alpha$ if $\alpha \in \mathfrak{R}_j$, which restricts to a bijection between nonspecial roots on each side. We choose a Chevalley system as in the proposition for each $\mathbf{G}_j$ ($j \in J$), and then set

$$X_{a_k^j \cdot \alpha} = a_k^j \cdot X_\alpha$$

if $\alpha \in \mathfrak{R}_j$. It is easily checked that this procedure produces a Chevalley system with the required property. □

Remark 4.5. In Proposition 4.4 we have specified only what happens for nonspecial roots. In fact, as soon as special roots exist, there does not exist any Chevalley system which satisfies (4.5) for all $a \in A$ and $\alpha \in \mathfrak{R}$. More precisely, this equality sometimes fails by a sign when β is special. (It is instructive to work out this failure in the example of the action of $\mathbb{Z}/2\mathbb{Z}$ on $\mathrm{SL}_{3,\mathbb{Z}}$ considered in §2.3.2.). It is possible to specify these signs explicitly, but this require some choices. Since this will not be necessary below, we omit the details.

4.3. **Twisted SL_2-maps.** From now on we fix a Chevalley system $(X_\alpha : \alpha \in \mathfrak{R})$ as in Proposition 4.4. Our goal in the rest of this section is to explain the construction of analogues of the maps (3.4) for the group $\mathbf{G}^A$. More specifically, these maps will be attached to equivalence classes for $\sim$. In case E is of type (i′), it will be given by a group scheme morphism

$$\imath_E : \mathrm{SL}_{2,S} \to \mathbf{G}^A.$$

In case E is of type (ii′), it will be given by a group scheme morphism

$$\imath_E : (\mathrm{SL}_{3,S})^{\mathbb{Z}/2\mathbb{Z}} \to \mathbf{G}^A,$$

where the action on the left-hand side is that considered in §2.3.2. For orbits of type (i′) the construction is uniform, but depends on the choice of Chevalley system. For orbits of type (ii′) the construction is more ad hoc.

4.3.1. *Equivalence classes of type (i′).* We start with the easier case when E is an equivalence class of type (i′). (In particular, this class consists of nonspecial roots.) For any $\alpha \in E$ we have a morphism φ_α as in (3.4). Moreover, if $a \in A$ and $\alpha \in E$, since $a \cdot X_\alpha = X_{a \cdot \alpha}$, by uniqueness we have

$$a \circ \varphi_\alpha = \varphi_{a \cdot \alpha}$$

(where by abuse we denote by a the action of a on $\mathbf{G}$). By the comments in Remark 4.2, if $\alpha, \alpha' \in E$ are distinct then no positive combination of α and α' is a root, which implies that $\mathbf{U}_\alpha$ and $\mathbf{U}_{\alpha'}$ commute by the commutation relations [DG11b, Exp. XXII, Corollaire 5.5.2]. As a consequence, the map

$$\imath'_E := \prod_{\alpha \in E} \varphi_\alpha : \prod_{\alpha \in E} \mathrm{SL}_{2,S} \to \mathbf{G}$$

is a morphism of group schemes, which is moreover A-equivariant where A acts on the left-hand side by permuting the factors (according to the action on E). Passing to A-fixed points and using the fact that

$$\left(\prod_{\alpha \in E} \mathrm{SL}_{2,S} \right)^A = \mathrm{SL}_{2,S},$$

we obtain the desired morphism $\imath_E$.

An important property of this morphism, which will be used below, is the following. The considerations above show that we have a natural closed immersion of group schemes

$$\prod_{\alpha \in E} \mathbf{U}_\alpha \to \mathbf{G}.$$

(Because the $\mathbf{U}_\alpha$'s appearing here commute with one another, we do not need to specify an order on E.) The image of this morphism is stable under the action of A, and will be denoted $\mathbf{U}_E$. If we denote by $\mathrm{U}_{2,S}$ the subgroup of $\mathrm{SL}_{2,S}$ of upper triangular unipotent matrices, then $\imath_E$ restricts to an isomorphism

$$\mathrm{U}_{2,S} \xrightarrow{\sim} (\mathbf{U}_E)^A. \tag{4.6}$$

Moreover, we have

$$\imath_E \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} = \prod_{\alpha \in E} \alpha^\vee(a). \tag{4.7}$$

4.3.2. *Equivalence classes of type (ii′).* Now, let us fix an equivalence class E of type (ii′). In this case, we use the considerations of §3.5 to reduce the construction to the case where $\mathbf{G}$ is a product of quasi-simple simply connected groups. Namely, consider the group $\mathbf{G}_{\mathrm{sc}}$ constructed in §3.5. The action of A on $\mathbf{G}$ induces an action on its root datum, which in turn provides an action on the root datum of $\mathbf{G}_{\mathrm{sc}}$, and finally an action on $\mathbf{G}_{\mathrm{sc}}$ by pinned group automorphisms. For this action, the morphism $\mathbf{G}_{\mathrm{sc}} \to \mathbf{G}$ is A-equivariant, and hence restricts to a group scheme morphism $(\mathbf{G}_{\mathrm{sc}})^A \to \mathbf{G}^A$. The roots associated with these two group schemes coincide, as does the equivalence relations on $\mathfrak{R}_+$ determined by the actions of A (see §4.1) and the notions of orbits of type (i′) or (ii′). It therefore suffices to construct our morphism for the group $\mathbf{G}_{\mathrm{sc}}$.

Recall next the decomposition (3.5) and the action of A on I determined by (4.4). If we decompose I into its orbits for this action:

$$I = \bigsqcup_{j \in J} I_j$$

and set $\mathbf{G}_{\mathrm{sc}}^j = \prod_{i \in I_j} \mathbf{G}_{\mathrm{sc},i}$, then each $\mathbf{G}_{\mathrm{sc}}^j$ is stable under the action of A, and moreover by Lemma 2.1(2) we have

$$(\mathbf{G}_{\mathrm{sc}})^A = \prod_{j \in J} (\mathbf{G}_{\mathrm{sc}}^j)^A.$$

We can (and will) therefore assume that the action of A on I is transitive.

In this case, all constituents $\mathfrak{R}_i$ have the same type. Since we are interested in equivalence classes of type (ii′), it suffices to consider the case when this type is A_{2n} for some $n \geq 1$. We can also assume that the stabilizer in A of each component acts nontrivially on this component. For any $i \in I$, the intersection $\Delta \cap \mathfrak{R}_i$ is a basis of $\mathfrak{R}_i$. Let us fix an identification of $\mathfrak{R}_i$ and its basis Δ_i with the root system of SL_{2n+1} and its basis from Example 3.1. (There exist exactly two such identifications; we choose one of them.) This determines an identification of pinned reductive group schemes

$$\mathbf{G}_{\mathrm{sc},i} = \mathrm{SL}_{2n+1,S}$$

(where the pinning on the right-hand side is as in Example 3.1). Taking these identifications together we obtain an identification

$$\mathbf{G}_{\mathrm{sc}} = \left(\mathrm{SL}_{2n+1,S}\right)^{\times_S I}.$$

The group of pinned automorphisms of $(\mathrm{SL}_{2n+1,S})^{\times_S I}$ is $(\mathbb{Z}/2\mathbb{Z})^{\times I} \rtimes \mathfrak{S}_I$; our given action of A is therefore determined by a "lift" of (4.4) to a group homomorphism $\tilde{\sigma} : A \to (\mathbb{Z}/2\mathbb{Z})^{\times I} \rtimes \mathfrak{S}_I$.

There exists $k \in \{1, \dots, n\}$ such that our given equivalence class E of type (ii′) is the union of the subsets

$$\{\alpha_k + \cdots + \alpha_n, \quad \alpha_{n+1} + \cdots + \alpha_{2n+1-k}, \quad \alpha_k + \cdots + \alpha_{2n+1-k}\}$$

in the root system of each copy of $\mathrm{SL}_{2n+1,S}$. One can then consider the morphism

$$(f_{k,n})^{\times_S I} : (\mathrm{SL}_{3,S})^{\times_S I} \to (\mathrm{SL}_{2n+1,S})^{\times_S I},$$

see (2.5). This morphism is equivariant with respect to the actions of $(\mathbb{Z}/2\mathbb{Z})^{\times I} \rtimes \mathfrak{S}_I$ on each side, so it induces a morphism

$$\big((\mathrm{SL}_{3,S})^{\times_S I}\big)^{(\mathbb{Z}/2\mathbb{Z})^{\times I} \rtimes \mathfrak{S}_I} \to \big((\mathrm{SL}_{2n+1,S})^{\times_S I}\big)^{(\mathbb{Z}/2\mathbb{Z})^{\times I} \rtimes \mathfrak{S}_I}$$

between fixed points. Here the left-hand side identifies with $(\mathrm{SL}_{3,S})^{\mathbb{Z}/2\mathbb{Z}}$, and our given morphism $\tilde{\sigma}$ determines a morphism

$$\big((\mathrm{SL}_{2n+1,S})^{\times_S I}\big)^{(\mathbb{Z}/2\mathbb{Z})^{\times I} \rtimes \mathfrak{S}_I} \to (\mathbf{G}_{\mathrm{sc}})^A.$$

This construction therefore provides the desired morphism

$$\imath_E : (\mathrm{SL}_{3,S})^{\mathbb{Z}/2\mathbb{Z}} \to (\mathbf{G}_{\mathrm{sc}})^A.$$

Remark 4.6. In the construction above we have chosen an identification of each $\mathfrak{R}_i$ with the root system of type A_{2n}. Changing these identifications amounts to composing the identification $\mathbf{G}_{\mathrm{sc}} = \big(\mathrm{SL}_{2n+1,S}\big)^{\times_S I}$ with the action of a certain element of $(\mathbb{Z}/2\mathbb{Z})^{\times I}$. This change will lead to a different morphism $(\mathrm{SL}_{3,S})^{\times_S I} \to \mathbf{G}_{\mathrm{sc}}$, but it will not affect its restriction $(\mathrm{SL}_{3,S})^{\mathbb{Z}/2\mathbb{Z}} \to (\mathbf{G}_{\mathrm{sc}})^A$. In other words, the morphism $\imath_E$ does not depend on these choices.

The morphism we have constructed here again has a property similar to that explained at the end of §4.3.1. Namely, choosing any order on E we can consider the product morphism

$$\prod_{\alpha \in E} \mathbf{U}_\alpha \to \mathbf{G}.$$

This morphism is a closed immersion, and its image is a subgroup scheme which does not depend on the choice of order; it will be denoted $\mathbf{U}_E$. In particular, this image is stable under the action of A. Recall the subgroup scheme $\mathrm{U}_{3,S}$ of $\mathrm{SL}_{3,S}$ considered in §2.3.2. Then the morphism $\imath_E$ restricts to an isomorphism

$$(\mathrm{U}_{3,S})^{\mathbb{Z}/2\mathbb{Z}} \xrightarrow{\sim} (\mathbf{U}_E)^A. \tag{4.8}$$

To check this claim, one can e.g. use the following fact.

Lemma 4.7. *Let $\mathbf{H}$ be a pinned reductive group scheme over S which is a quasi-simple simply connected group, and assume given a finite set I and an action of a group A on $\mathbf{H}^{\times_S I}$ by pinned automorphisms. As in (4.4) this determines an action of A on I. If this action is transitive, then, for any $i \in I$, projection onto the component parametrized by i induces an isomorphism of group schemes*

$$\big(\mathbf{H}^{\times_S I}\big)^A \xrightarrow{\sim} \mathbf{H}^{A_i}$$

where A_i is the stabilizer of i in A.

Proof. The inverse isomorphism is constructed as follows. By assumption the action of A on I induces a bijection $A/A_i \xrightarrow{\sim} I$. Choose, for any $j \in I$, an element $a_j \in A$ such that $\sigma(a_j)(i) = j$. Then the action of a_j on $\mathbf{H}^{\times_S I}$ identifies the copy of $\mathbf{H}$ indexed by i with that indexed by j. It is easily checked that the assignment

$$h \mapsto \prod_j (a_j \cdot h)$$

induces a morphism $\mathbf{H}^{A_i} \to \big(\mathbf{H}^{\times_S I}\big)^A$ which is inverse to the morphism of the statement. □

Finally, we have a counterpart of (4.7):

$$\imath_E \begin{pmatrix} a & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & a^{-1} \end{pmatrix} = \prod_{\substack{\alpha \in E \\ \alpha \text{ special}}} \alpha^\vee(a). \tag{4.9}$$

5. Flatness and smoothness

5.1. **Statement.** We continue with our pinned reductive group scheme (3.1) over S, and our group A which acts on $\mathbf{G}$ by pinned automorphisms. Let M_A denote the coinvariants for the induced action of A on M (see §2.2). Recall also that A acts on $\mathfrak{R}$, and permutes its indecomposable components.

The following statement is the main result of this paper.

Theorem 5.1. (1) *The group scheme $\mathbf{G}^A$ is flat over S.*

(2) *The group scheme $\mathbf{G}^A$ has geometrically connected fibers over S iff either M_A is torsion-free or S has exactly one residual characteristic $\ell > 0$ and the torsion subgroup of M_A is an ℓ-group.*

(3) *The group scheme $\mathbf{G}^A$ is smooth over S iff the following conditions hold:*
- *the order of the torsion subgroup of M_A is coprime to all residual characteristics of S;*
- *if $\mathfrak{R}$ has an indecomposable component of type A_{2n} for some $n \geq 1$ whose stabilizer in A acts nontrivially on this component, then 2 is not a residual characteristic of S.*

(4) *If S is the spectrum of a mixed characteristic DVR and M_A is torsion-free, then $\mathbf{G}^A$ is a quasi-reductive S-group scheme in the sense of Prasad–Yu [PY06].*

5.2. **Fixed points in the big cell.** The main step in our proof of Theorem 5.1 will be the study of the fixed points of A on the "big cell" in $\mathbf{G}$. Recall the subgroups $\mathbf{U}$, $\mathbf{U}^-$ introduced in §3.3. For any $\alpha \in \mathfrak{R}$ and $a \in A$, the action of a on $\mathbf{G}$ induces an isomorphism

$$\mathbf{U}_\alpha \xrightarrow{\sim} \mathbf{U}_{a \cdot \alpha}.$$

By the independence of $\mathbf{U}$ on the choice of order on $\mathfrak{R}_+$, this implies that this subgroup is stable under the action of A. Similar comments apply to $\mathbf{U}^-$.

By [DG11b, Exp. XXII, Proposition 4.1.2], the multiplication morphism

$$\mathbf{U}^- \times_S \mathbf{T} \times_S \mathbf{U} \to \mathbf{G}$$

is an open immersion. Its image (called the *big cell*) is denoted $\mathbf{C} \subset \mathbf{G}$. This open subscheme is stable under the action of A, and using Lemma 2.1(2) we see that multiplication induces an isomorphism

$$(\mathbf{U}^-)^A \times_S \mathbf{T}^A \times_S \mathbf{U}^A \xrightarrow{\sim} \mathbf{C}^A. \tag{5.1}$$

Moreover, by Lemma 2.1(1) the morphism $\mathbf{C}^A \to \mathbf{G}^A$ induced by the open immersion $\mathbf{C} \to \mathbf{G}$ is itself an open immersion.

Lemma 5.2. (1) *The scheme $\mathbf{T}^A$ is flat over S. It is smooth over S iff the order of the torsion subgroup of M_A is prime to all residual characteristics of S.*

(2) *The scheme $\mathbf{U}^A$ is flat over S. It is smooth over S iff it satisfies the following condition: if $\mathfrak{R}$ has an indecomposable component of type A_{2n} for some $n \geq 1$ whose stabilizer in A acts nontrivially on this component, then 2 is invertible on S.*

(3) *The scheme $\mathbf{C}^A$ is flat over S. It is smooth over S iff the conditions in Theorem 5.1(3) hold.*

Proof. (1) This follows from the discussion in §2.2.

(2) Consider the equivalence relation $\sim$ on the set of positive roots $\mathfrak{R}_+$ that was defined in §4.1 (for the given action of A on $\mathfrak{R}$). In §4.3, we have associated to each equivalence class E for $\sim$ a subgroup scheme $\mathbf{U}_E \subset \mathbf{G}$. After choosing a numbering $E_1, \dots, E_n$ of these equivalence classes, the product morphism induces an isomorphism

$$\mathbf{U}_{E_1} \times_S \cdots \times_S \mathbf{U}_{E_n} \xrightarrow{\sim} \mathbf{U}.$$

Using Lemma 2.1(1) we deduce that multiplication induces an isomorphism

$$(\mathbf{U}_{E_1})^A \times_S \cdots \times_S (\mathbf{U}_{E_n})^A \xrightarrow{\sim} \mathbf{U}^A. \tag{5.2}$$

Here, the factors are described in (4.6) or (4.8).

This shows that $\mathbf{U}^A$ is a product of factors which are either isomorphic to $\mathbf{G}_{\mathrm{a},S}$ or to $(\mathrm{U}_{3,S})^{\mathbb{Z}/2\mathbb{Z}}$; moreover, this second case occurs iff $\mathfrak{R}$ has an indecomposable component of type A_{2n} for some $n \geq 1$ whose stabilizer in A acts nontrivially on this factor. Each of these factors is flat over S (see §2.3.2 for the second case) so $\mathbf{U}^A$ is flat. If 2 is not a residual characteristic of S or if no factor $(\mathrm{U}_{3,S})^{\mathbb{Z}/2\mathbb{Z}}$ occurs then all of these factors are smooth over S (see again §2.3.2), so that $\mathbf{U}^A$ is also smooth over S. On the other hand, if 2 is a residual characteristic of S and a factor $(\mathrm{U}_{3,S})^{\mathbb{Z}/2\mathbb{Z}}$ occurs, then as in §2.3.2 one sees that there exists $s \in S$ such that $\mathrm{Spec}(\kappa(s)) \times_S \mathbf{U}^A$ is not reduced, so that $\mathbf{U}^A$ is not smooth over S.

(3) The conclusion of (2) of course also applies to $(\mathbf{U}^-)^A$. Using (5.1) we deduce that $\mathbf{C}^A$ is flat over S. If the conditions of Theorem 5.1(3) hold, then by (1) and (2) each of $\mathbf{T}^A$, $\mathbf{U}^A$ and $(\mathbf{U}^-)^A$ is smooth over S, so that $\mathbf{C}^A$ is smooth over S by (5.1). Conversely, if one of these conditions fails, then as in (2) one sees that there exists $s \in S$ such that $\mathrm{Spec}(\kappa(s)) \times_S \mathbf{C}^A$ is not reduced, so that $\mathbf{C}^A$ is not smooth over S. $\square$

Example 5.3. If $\mathbf{G}$ is semisimple and either simply connected or of adjoint type, then M_A is free since M admits a basis permuted by A. (In the first case one can take the basis of fundamental weights, and in the second case the basis of simple roots.) Hence in these cases $\mathbf{T}^A$ is a torus.

5.3. **The case of algebraically closed fields.** In this subsection we assume that $S = \mathrm{Spec}(\Bbbk)$ is the spectrum of an algebraically closed field $\Bbbk$. Below we prove in particular in this case that the reduced neutral component $(\mathbf{G}^A)^\circ_{\mathrm{red}}$ is a split reductive group. As explained in the introduction this case was already treated by Adler–Lansky and Haines, but for the reader's convenience we give a self-contained argument. We simultaneously establish a number of facts about the structure theory of this reductive group, listed in Proposition 5.4 below, which will be useful for the further study in Section 6.

In this statement, we denote by $\mathbf{N}$ the normalizer of $\mathbf{T}$ in $\mathbf{G}$, and by $\mathbf{W}$ the Weyl group of $(\mathbf{G}, \mathbf{T})$, i.e. the quotient $\mathbf{N}/\mathbf{T}$. (As explained in [DG70b, Exp. XII, §2], $\mathbf{N}$ is a smooth subgroup scheme of $\mathbf{G}$, and $\mathbf{W}$ is a finite constant group scheme, see also [Con14, Proposition 5.1.6]. For simplicity, we will not distinguish $\mathbf{W}$ from $\mathbf{W}(\Bbbk)$.) Since A stabilizes $\mathbf{T}$, it also stabilizes $\mathbf{N}$, and this action induces an action on $\mathbf{W}$.

Proposition 5.4. *The following assertions hold:*

(1) $(\mathbf{G}^A)^\circ_{\mathrm{red}}$ *is a connected reductive group;*
(2) $(\mathbf{T}^A)^\circ_{\mathrm{red}} \subset (\mathbf{G}^A)^\circ_{\mathrm{red}}$ *is a maximal torus, and* $(\mathbf{B}^A)^\circ_{\mathrm{red}}$ *is a Borel subgroup;*
(3) *the positive roots of* $((\mathbf{G}^A)^\circ_{\mathrm{red}}, (\mathbf{T}^A)^\circ_{\mathrm{red}})$ *determined by* $(\mathbf{B}^A)^\circ_{\mathrm{red}}$ *are in natural bijection with the equivalence classes for the equivalence relation* $\sim$ *on* $\mathfrak{R}_+$ *(see (4.1));*
(4) *for any equivalence class* E*, seen as a positive root for* $((\mathbf{G}^A)^\circ_{\mathrm{red}}, (\mathbf{T}^A)^\circ_{\mathrm{red}})$*, the corresponding root subgroup of* $(\mathbf{G}^A)^\circ_{\mathrm{red}}$ *is* $((\mathbf{U}_E)^A)_{\mathrm{red}}$*;*
(5) *the natural maps provide bijections*

$$\mathbf{N}^A(\Bbbk)/\mathbf{T}^A(\Bbbk) \cong \mathbf{W}^A \cong \mathbf{B}^A(\Bbbk)\backslash\mathbf{G}^A(\Bbbk)/\mathbf{B}^A(\Bbbk);$$

(6) *the open subscheme* $\mathbf{C}^A \subset \mathbf{G}^A$ *is dense;*
(7) *the embedding* $\mathbf{T}^A \to \mathbf{G}^A$ *induces an isomorphism between the groups of connected components of these group schemes;*
(8) *the Weyl group of* $(\mathbf{G}^A)^\circ_{\mathrm{red}}$ *with respect to* $(\mathbf{T}^A)^\circ_{\mathrm{red}}$ *identifies canonically with* $\mathbf{W}^A$*.*

Proof. Let $\mathbf{K}$ be the unipotent radical of $(\mathbf{G}^A)^\circ_{\mathrm{red}}$, and set $\mathbf{R} := (\mathbf{G}^A)^\circ_{\mathrm{red}}/\mathbf{K}$ which is a connected reductive group over $\Bbbk$. Let $q : (\mathbf{G}^A)^\circ_{\mathrm{red}} \to \mathbf{R}$ be the quotient map.

Step 1. The restriction of q *to a map* $(\mathbf{T}^A)^\circ_{\mathrm{red}} \to \mathbf{R}$ *is a closed immersion. In particular,* $\mathbf{R}$ *contains a torus of dimension* $\dim(\mathbf{T}^A)$. This follows from the observation that the subgroup $(\mathbf{T}^A)^\circ_{\mathrm{red}} = \mathrm{D}_\Bbbk(M_A/(M_A)_{\mathrm{tor}})$ is a torus. It therefore has trivial intersection with the unipotent group $\mathbf{K}$.

Step 2. For each equivalence class E*, there is a map*

$$\varphi_E : \mathrm{SL}_{2,\Bbbk} \to \mathbf{G}^A$$

whose restriction to the subgroup $\mathrm{U}_{2,\Bbbk} \subset \mathrm{SL}_{2,\Bbbk}$ *of upper triangular unipotent matrices induces an isomorphism*

$$\mathrm{U}_{2,\Bbbk} \xrightarrow{\sim} (\mathbf{U}_E)^A_{\mathrm{red}}.$$

If E is of type (i′), set $\varphi_E = \imath_E$. We have seen in (4.6) that this map behaves as desired on $\mathrm{U}_{2,\Bbbk}$. (In this case, the scheme $(\mathbf{U}_E)^A$ is already reduced.) If E is of type (ii′), consider the sequences of maps

$$\mathrm{U}_{2,\Bbbk} \hookrightarrow \mathrm{SL}_{2,\Bbbk} \twoheadrightarrow \mathrm{PGL}_{2,\Bbbk} \xrightarrow{(2.3)} (\mathrm{SL}_{3,\Bbbk})^{\mathbb{Z}/2\mathbb{Z}} \xrightarrow{\imath_E} \mathbf{G}^A \qquad \text{if } \mathrm{char}(\Bbbk) \neq 2;$$
$$\mathrm{U}_{2,\Bbbk} \hookrightarrow \mathrm{SL}_{2,\Bbbk} \xrightarrow{(2.4)} (\mathrm{SL}_{3,\Bbbk})^{\mathbb{Z}/2\mathbb{Z}} \xrightarrow{\imath_E} \mathbf{G}^A \qquad \text{if } \mathrm{char}(\Bbbk) = 2.$$

We have seen in (4.8) that $\imath_E$ restricts to an isomorphism $(\mathrm{U}_{3,\Bbbk})^{\mathbb{Z}/2\mathbb{Z}} \xrightarrow{\sim} (\mathbf{U}_E)^A$, so the claim follows from the observation that either (2.3) or (2.4) (depending on the characteristic of $\Bbbk$) induces an isomorphism $\mathrm{U}_{2,\Bbbk} \xrightarrow{\sim} (\mathrm{U}_{3,\Bbbk})^{\mathbb{Z}/2\mathbb{Z}}_{\mathrm{red}}$. (Recall that the scheme $(\mathrm{U}_{3,\Bbbk})^{\mathbb{Z}/2\mathbb{Z}}$ is reduced if and only if $\mathrm{char}(\Bbbk) \neq 2$.)

Step 3. The map $\gamma_E^\vee : \mathbb{G}_{\mathrm{m},\Bbbk} \to \mathbf{G}^A$ *given by*

$$\gamma_E^\vee(a) = \varphi_E \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$$

is a nontrivial cocharacter of $(\mathbf{T}^A)^\circ_{\mathrm{red}}$. The fact that $\gamma_E^\vee$ factors through $(\mathbf{T}^A)^\circ_{\mathrm{red}}$ follows from the fact that $\mathbb{G}_{\mathrm{m},\Bbbk}$ is reduced and connected. To check that $\gamma_E^\vee$ is nonzero, we simply remark that in $M^\vee$, by (4.7) and (4.9) (combined with (2.3) and (2.4) in the latter case), we have

$$\gamma_E^\vee = \begin{cases} \sum_{\alpha \in E} \alpha^\vee & \text{if } E \text{ is of type (i}'\text{)}, \\ 2\sum_{\alpha \in E,\ \alpha \text{ special}} \alpha^\vee & \text{if } E \text{ is of type (ii}'\text{) and } \mathrm{char}(\Bbbk) \neq 2, \\ \sum_{\alpha \in E,\ \alpha \text{ special}} \alpha^\vee & \text{if } E \text{ is of type (ii}'\text{) and } \mathrm{char}(\Bbbk) = 2. \end{cases}$$

For future reference, we rewrite this formula as follows, using the observation that in type (ii′), the sum of all nonspecial roots is equal to the sum of all special roots:

$$\gamma_E^\vee = \begin{cases} \sum_{\alpha \in E} \alpha^\vee & \text{if } E \text{ is of type (i}'\text{)}, \\ & \quad \text{or if } E \text{ is of type (ii}'\text{) and } \mathrm{char}(\Bbbk) \neq 2, \\ \sum_{\alpha \in E,\ \alpha \text{ special}} \alpha^\vee & \text{if } E \text{ is of type (ii}'\text{) and } \mathrm{char}(\Bbbk) = 2. \end{cases} \tag{5.3}$$

Step 4. For each equivalence class E*, let* $\bar{\alpha}_E$ *be the weight by which the torus* $(\mathbf{T}^A)^\circ_{\mathrm{red}}$ *acts on* $\mathscr{L}ie((\mathbf{U}_E)^A_{\mathrm{red}})$*. Then each* $\bar{\alpha}_E$ *is nonzero.* Since $\mathscr{L}ie((\mathbf{U}_E)^A_{\mathrm{red}})$ is a subspace of $\mathscr{L}ie(\mathbf{U}_E)$, it is enough to check that $(\mathbf{T}^A)^\circ_{\mathrm{red}}$ has no nonzero fixed vectors in $\mathscr{L}ie(\mathbf{U}_E)$. Consider the cocharacter

$\gamma_E^\vee$ from Step 3. The claim follows from the fact that $\langle \alpha, \gamma_E^\vee \rangle > 0$ for all $\alpha \in E$. More precisely, a straightforward computation (by reducing to the quasi-simple case) shows that

$$\langle \alpha, \gamma_E^\vee \rangle = 2 \quad \text{for all } \alpha \in E \qquad\qquad \text{if } E \text{ is of type (i}'\text{)},$$

$$\langle \alpha, \gamma_E^\vee \rangle = \begin{cases} 1 & \text{if } \alpha \in E \text{ is nonspecial and } \mathrm{char}(\Bbbk) = 2, \\ 2 & \text{if } \alpha \in E \text{ is nonspecial and } \mathrm{char}(\Bbbk) \neq 2, \\ 2 & \text{if } \alpha \in E \text{ is special and } \mathrm{char}(\Bbbk) = 2, \\ 4 & \text{if } \alpha \in E \text{ is special and } \mathrm{char}(\Bbbk) \neq 2, \end{cases} \qquad \text{and } E \text{ is of type (ii}'\text{)}.$$

Step 5. For each equivalence class E for $\sim$, the morphism $(\mathbf{U}_E)^A_{\mathrm{red}} \to \mathbf{R}$ obtained by restricting q is finite, with kernel the r-th infinitesimal neighborhood of the unit for some $r \geq 1$. The kernel of the morphism under consideration is a subgroup scheme of $(\mathbf{U}_E)^A_{\mathrm{red}} \cong \mathbb{G}_{\mathrm{a},\Bbbk}$, which is stable under the action of $(\mathbf{T}^A)^\circ_{\mathrm{red}}$ by conjugation; if it is not an infinitesimal neighborhood of the unit then it is all of $(\mathbf{U}_E)^A_{\mathrm{red}}$. By Step 2, it follows that the composition

$$\mathrm{U}_{2,\Bbbk} \hookrightarrow \mathrm{SL}_{2,\Bbbk} \xrightarrow{\varphi_E} (\mathbf{T}^A)^\circ_{\mathrm{red}} \subset (\mathbf{G}^A)^\circ_{\mathrm{red}} \to \mathbf{R}$$

is trivial. The kernel of $\mathrm{SL}_{2,\Bbbk} \to \mathbf{R}$ is therefore a normal subgroup scheme of $\mathrm{SL}_{2,\Bbbk}$ that contains $\mathrm{U}_{2,\Bbbk}$. The only such subgroup is all of $\mathrm{SL}_{2,\Bbbk}$: so the map $\mathrm{SL}_{2,\Bbbk} \to \mathbf{R}$ is trivial. But Steps 1 and 3 together imply that this map is nontrivial on $\{\left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix}\right)\} \subset \mathrm{SL}_{2,\Bbbk}$, a contradiction.

Step 6. If E_1 and E_2 are distinct equivalence classes, then $\bar{\alpha}_{E_1}$ and $\bar{\alpha}_{E_2}$ are linearly independent. Consider the morphism $\mathbf{G}_{\mathrm{sc}} \to \mathbf{G}$ as in §3.5, and the maximal torus $\mathbf{T}_{\mathrm{sc}}$ of $\mathbf{G}_{\mathrm{sc}}$. As explained in Example 5.3, $(\mathbf{T}_{\mathrm{sc}})^A$ is reduced and connected. The morphism $\mathbf{G}_{\mathrm{sc}} \to \mathbf{G}$ induces a morphism $(\mathbf{T}_{\mathrm{sc}})^A \to (\mathbf{T}^A)^\circ_{\mathrm{red}}$, hence a morphism relating characters of these tori. Explicitly, we have

$$(\mathbf{T}_{\mathrm{sc}})^A = \mathrm{D}_\Bbbk((M_{\mathrm{sc}})_A), \quad (\mathbf{T}^A)^\circ_{\mathrm{red}} = \mathrm{D}_\Bbbk(M_A/(M_A)_{\mathrm{tor}})$$

where M_{sc} is as in §3.5, i.e. M_{sc} is the lattice of weights of the root system $\mathfrak{R}$, and this morphism is given by the obvious morphism

$$M_A/(M_A)_{\mathrm{tor}} \to (M_{\mathrm{sc}})_A.$$

The desired claim therefore follows from Lemma 4.3, using the embedding $(M_{\mathrm{sc}})_A \to \mathbb{R} \otimes_{\mathbb{Z}} (M_{\mathrm{sc}})_A = (\mathbb{R} \otimes_{\mathbb{Z}} M_{\mathrm{sc}})_A$.

Step 7. Proof of parts (1), (2), (3), *and* (4). All four of these assertions will follow from the fact that $q : (\mathbf{G}^A)^\circ_{\mathrm{red}} \to \mathbf{R}$ is an isomorphism. To prove this fact, it is enough to show the equality

$$\dim((\mathbf{G}^A)^\circ_{\mathrm{red}}) = \dim(\mathbf{R}).$$

It is obvious that $\dim((\mathbf{G}^A)^\circ_{\mathrm{red}}) \geq \dim(\mathbf{R})$; we need only prove the opposite inequality. Let N be the number of equivalence classes for $\sim$. The analysis of $\mathbf{C}^A$ in §5.2 shows that $\dim((\mathbf{G}^A)^\circ_{\mathrm{red}}) = \dim((\mathbf{T}^A)^\circ_{\mathrm{red}}) + 2N$. On the other hand, using Steps 1, 5, and 6 and considering the action of $(\mathbf{T}^A)^\circ_{\mathrm{red}}$ on the Lie algebra of $\mathbf{R}$ we see that $\dim(\mathbf{R}) \geq \dim((\mathbf{T}^A)^\circ_{\mathrm{red}}) + 2N$, so we are done.

Step 8. The natural morphism

$$\mathbf{N}^A(\Bbbk) \to \mathbf{W}^A \tag{5.4}$$

is surjective. By [Hée91, Corollaire 3.5], $\mathbf{W}^A$ admits a system of Coxeter generators in bijection with orbits of A in Δ. For any such orbit the construction of §4.3 provides a twisted SL_2-map with domain either $\mathrm{SL}_{2,\Bbbk}$ or $(\mathrm{SL}_{3,\Bbbk})^{\mathbb{Z}/2\mathbb{Z}}$. In the first, resp. second, case, the image of the matrix $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$, resp. $\left(\begin{smallmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{smallmatrix}\right)$, provides a representative of the corresponding reflection in $\mathbf{W}^A$.

Step 9. Study of the Bruhat decomposition. Recall the Bruhat decomposition

$$\mathbf{G} = \bigsqcup_{w \in \mathbf{W}} \mathbf{U} w \mathbf{B} = \bigsqcup_{w \in \mathbf{W}} \mathbf{B} w \mathbf{B},$$

where each $\mathbf{U}w\mathbf{B} = \mathbf{B}w\mathbf{B}$ is a locally closed subscheme of $\mathbf{G}$. For any $a \in A$, the action of a induces an isomorphism

$$\mathbf{U}w\mathbf{B} \xrightarrow{\sim} \mathbf{U}(a \cdot w)\mathbf{B} = \mathbf{B}(a \cdot w)\mathbf{B};$$

it follows that $\mathbf{G}^A \cap (\mathbf{U}w\mathbf{B}) = \mathbf{G}^A \cap (\mathbf{B}w\mathbf{B})$ is empty unless $w \in \mathbf{W}^A$, so that we have a decomposition $\mathbf{G}^A = \bigsqcup_{w \in \mathbf{W}^A} (\mathbf{U}w\mathbf{B})^A$. By surjectivity of (5.4) each $w \in \mathbf{W}^A$ admits a lift $\dot{w} \in \mathbf{N}^A(\Bbbk)$. Using such a lift and the usual description of Bruhat cells (see e.g. [Jan03, Equation (1) in §II.13.2]) we deduce that for any $w \in \mathbf{W}^A$ we have

$$(\mathbf{U}w\mathbf{B})^A = \mathbf{U}^A \dot{w} \mathbf{B}^A = \mathbf{B}^A \dot{w} \mathbf{B}^A.$$

We conclude that there is a decomposition

$$\mathbf{G}^A = \bigsqcup_{w \in \mathbf{W}^A} \mathbf{U}^A \dot{w} \mathbf{B}^A = \bigsqcup_{w \in \mathbf{W}^A} \mathbf{B}^A \dot{w} \mathbf{B}^A. \tag{5.5}$$

Let us now note that for $w \in \mathbf{W}^A$ and E an equivalence class for $\sim$, we either have $w(E) \subset \mathfrak{R}_+$ or $w(E) \subset -\mathfrak{R}_+$. In the latter case, we write $w(E) < 0$. Using (5.2), we obtain

$$(\mathbf{U}w\mathbf{B})^A = \left(\prod_{\substack{E \text{ equiv. class for } \sim \\ w^{-1}(E) < 0}} (\mathbf{U}_E)^A \right) \dot{w} \mathbf{B}^A. \tag{5.6}$$

It then follows that

$$\dim(\mathbf{U}^A \dot{w} \mathbf{B}^A) = \#\{E \in (\mathfrak{R}_+/\sim) \mid w^{-1}(E) < 0\} + \dim(\mathbf{B}^A).$$

Step 10. Proof of (5). The first bijection follows from Step 8 and the observation that the kernel of (5.4) is $\mathbf{T}(\Bbbk) \cap \mathbf{N}^A(\Bbbk) = \mathbf{T}^A(\Bbbk)$; the second bijection is a restatement of (5.5).

Step 11. Proof of (6). Let $w_0 \in \mathbf{W}$ be the longest element. Since the action of A preserves lengths, we have $w_0 \in \mathbf{W}^A$. The subscheme $\mathbf{B}^A \dot{w}_0 \mathbf{B}^A$ is the unique term of maximal dimension in (5.5). Since all irreducible components of $\mathbf{G}^A$ have the same dimension (because it is a group scheme), we deduce that $\mathbf{B}^A \dot{w}_0 \mathbf{B}^A$ is dense in $\mathbf{G}^A$. Finally, comparing (5.6) (for $\dot{w}_0$) with (5.1), we obtain an identification $\mathbf{B}^A \dot{w}_0 \mathbf{B}^A \cong \mathbf{C}^A$.

Step 12. Proof of (7). By Step 11, the embedding $\mathbf{C}^A \to \mathbf{G}^A$ induces a bijection between sets of connected components. In view of (5.1), and since $(\mathbf{U}^A)_{\mathrm{red}}$ and $((\mathbf{U}^-)^A)_{\mathrm{red}}$ are isomorphic to affine spaces, the same property holds for the embedding

$$\mathbf{T}^A \to \mathbf{C}^A,$$

which proves (7).

Step 13. Proof of (8). Thanks to Step 12, Example 5.3 shows that $\mathbf{G}^A$ is connected if $\mathbf{G}$ is semisimple and simply connected. The study of the morphism (5.4) above therefore shows that this surjection admits a (set theoretic) section which takes values in $\mathbf{N}^A(\Bbbk) \cap (\mathbf{G}^A)^\circ(\Bbbk)$. From now on, we assume that the lifts $\dot{w}$ (for $w \in \mathbf{W}^A$) are chosen in $(\mathbf{G}^A)^\circ_{\mathrm{red}}(\Bbbk) = (\mathbf{G}^A)^\circ(\Bbbk)$. Let us also fix some representatives $t_1, \ldots, t_r \in \mathbf{T}^A(\Bbbk)$ for the connected components of $\mathbf{T}^A$. Then we have

$$\mathbf{N}^A = \bigsqcup_{\substack{w \in \mathbf{W}^A \\ i \in \{1, \ldots, r\}}} \dot{w} t_i (\mathbf{T}^A)^\circ,$$

and hence

$$(\mathbf{N}^A) \cap (\mathbf{G}^A)^\circ_{\mathrm{red}} = \bigsqcup_{w \in \mathbf{W}^A} \dot{w} (\mathbf{T}^A)^\circ_{\mathrm{red}}.$$

On the other hand, there exists a natural closed immersion

$$(\mathbf{N}^A) \cap (\mathbf{G}^A)^\circ_{\mathrm{red}} \to N_{(\mathbf{G}^A)^\circ_{\mathrm{red}}}((\mathbf{T}^A)^\circ_{\mathrm{red}}),$$

and we deduce an embedding

$$\mathbf{W}^A \hookrightarrow N_{(\mathbf{G}^A)^\circ_{\mathrm{red}}}((\mathbf{T}^A)^\circ_{\mathrm{red}})/(\mathbf{T}^A)^\circ_{\mathrm{red}}. \tag{5.7}$$

For any $w \in \mathbf{W}^A$ we similarly have

$$(\mathbf{B}^A \dot{w} \mathbf{B}^A) \cap (\mathbf{G}^A)^\circ_{\mathrm{red}} = (\mathbf{B}^A)^\circ_{\mathrm{red}} \dot{w} (\mathbf{B}^A)^\circ_{\mathrm{red}},$$

hence

$$(\mathbf{G}^A)^\circ_{\mathrm{red}} = \bigsqcup_{w \in \mathbf{W}^A} (\mathbf{B}^A)^\circ_{\mathrm{red}} \dot{w} (\mathbf{B}^A)^\circ_{\mathrm{red}},$$

and finally a bijection

$$\mathbf{W}^A \xrightarrow{\sim} (\mathbf{B}^A)^\circ_{\mathrm{red}}(\Bbbk) \backslash (\mathbf{G}^A)^\circ_{\mathrm{red}}(\Bbbk) / (\mathbf{B}^A)^\circ_{\mathrm{red}}(\Bbbk).$$

Using the Bruhat decomposition for the reductive group $(\mathbf{G}^A)^\circ_{\mathrm{red}}$, we deduce that the morphism (5.7) is an isomorphism, which finally proves (8). ☐

Remark 5.5. Proposition 5.4(3) says that the positive roots for $(\mathbf{G}^A)^\circ_{\mathrm{red}}$ are in bijection with the set of equivalence classes for $\sim$, and in Step 5 of the proof, we introduced the notation $\bar{\alpha}_E$ for the root corresponding to an equivalence class E. This is a character of $(\mathbf{T}^A)^\circ_{\mathrm{red}}$, or, equivalently, an element of $M_A/(M_A)_{\mathrm{tor}}$. It is immediate from the definition of $\mathbf{U}_E$ that $\bar{\alpha}_E$ is the image of some element $\alpha \in E$ under the projection $M \to M_A/(M_A)_{\mathrm{tor}}$ (and this observation was adequate for Step 6 of the proof).

But of which element is $\bar{\alpha}_E$ the image? If E is of type (i′), then it consists of a single A-orbit, and its image in $M_A/(M_A)_{\mathrm{tor}}$ is a singleton. But if E is of type (ii′), its image in $M_A/(M_A)_{\mathrm{tor}}$ consists of two elements: one is the image of the nonspecial roots, and the other is the image of the special roots. The latter is twice the former. We claim that $\bar{\alpha}_E$ is the image in $M_A/(M_A)_{\mathrm{tor}}$ of

$$\begin{cases} \text{any root in } E & \text{if } E \text{ is of type (i}'), \\ \text{any nonspecial root in } E & \text{if } E \text{ is of type (ii}') \text{ and } \mathrm{char}(\Bbbk) \neq 2, \\ \text{any special root in } E & \text{if } E \text{ is of type (ii}') \text{ and } \mathrm{char}(\Bbbk) = 2. \end{cases}$$

To justify the latter two cases, we use (4.8) to reduce the problem to the setting of $\mathbf{G} = \mathrm{SL}_{3,\Bbbk}$. In this setting, the claim follows from (2.2).

With the formula for $\bar{\alpha}_E$ in hand, we read off from Step 5 of the proof that

$$\langle \bar{\alpha}_E, \gamma_E^\vee \rangle = 2$$

for any E. Since $\gamma_E^\vee$ comes from a homomorphism $\mathrm{SL}_{2,\Bbbk} \to \mathbf{G}^A$ as in Step 2, we conclude that $\gamma_E^\vee$ is in fact the coroot for $(\mathbf{G}^A)^\circ_{\mathrm{red}}$ corresponding to $\bar{\alpha}_E$.

Remark 5.6. Let us explain an argument proving the surjectivity of (5.4) which does not rely on the results of [Hée91]. We denote by $\mathfrak{R}^{(A)}$, resp. $\mathfrak{R}_+^{(A)}$, the image of $\mathfrak{R}$, resp. $\mathfrak{R}_+$, in M_A. This subset will be studied more thoroughly in §6.1 below; for now, we note that Steps 5 and 6 in the preceding proof show that the natural map $\mathfrak{R} \to \mathfrak{R}^{(A)}$ induces a bijection $\mathfrak{R}/A \xrightarrow{\sim} \mathfrak{R}^{(A)}$ and that, by parts (3)–(4) of the proposition, the root system $\mathfrak{R}^{(A)\prime}$ of $(\mathbf{G}^A)^\circ_{\mathrm{red}}$ is obtained from $\mathfrak{R}^{(A)}$ by discarding one element from each pair of the form $\{\alpha, 2\alpha\}$. In particular, $\mathfrak{R}_+^{(A)}$ defines a positive system $\mathfrak{R}_+^{(A)\prime}$ in $\mathfrak{R}^{(A)\prime}$. Since A-invariant elements of $\mathbf{W}$ act compatibly on $\mathfrak{R}$ and on M_A, $\mathbf{W}^A$ acts on $\mathfrak{R}^{(A)}$. If $w \in \mathbf{W}^A$ then $w(\mathfrak{R}_+^{(A)\prime})$ is a positive system for $\mathfrak{R}^{(A)\prime}$; hence there exists $n \in N_{(\mathbf{G}^A)^\circ_{\mathrm{red}}}((\mathbf{T}^A)^\circ_{\mathrm{red}})(\Bbbk)$ such that $w(\mathfrak{R}_+^{(A)\prime}) = n(\mathfrak{R}_+^{(A)\prime})$. The argument in the proof of Lemma 6.3(2) below shows that $N_{(\mathbf{G}^A)^\circ_{\mathrm{red}}}((\mathbf{T}^A)^\circ_{\mathrm{red}})(\Bbbk) \subset \mathbf{N}^A(\Bbbk)$. In particular n defines an element w' in $\mathbf{W}^A$ such that $w^{-1}w'$ induces a based automorphism of $\mathfrak{R}$. Hence $w' = w$, which proves the desired claim.

5.4. **Proof of Theorem 5.1.** We can finally explain the proof of Theorem 5.1. For part (1), we will study the multiplication morphism

$$f : \mathbf{C}^A \times_S \mathbf{C}^A \to \mathbf{G}^A,$$

along with its base changes to points or geometric points of S.

First, let $\overline{s} : \mathrm{Spec}(\Bbbk) \to S$ be a geometric point. By Proposition 5.4(6) the open subscheme $\mathbf{C}^A_{\overline{s}} \subset \mathbf{G}^A_{\overline{s}}$ is dense, and hence $f_{\overline{s}}$ is surjective. Since any point of a scheme is the image of a geometric point, it follows that f is surjective.

Next, let $s \in S$, and consider the base change $f_s : \mathbf{C}^A_s \times_{\kappa(s)} \mathbf{C}^A_s \to \mathbf{G}^A_s$. This map factors as the composition

$$\mathbf{C}^A_s \times_{\kappa(s)} \mathbf{C}^A_s \xrightarrow{\text{inclusion}} \mathbf{G}^A_s \times_{\kappa(s)} \mathbf{G}^A_s \xrightarrow[\sim]{(g,h)\mapsto(g,gh)} \mathbf{G}^A_s \times_{\kappa(s)} \mathbf{G}^A_s \xrightarrow{\text{projection to 2nd factor}} \mathbf{G}^A_s.$$

The first map is an open immersion (by Lemma 2.1(1)); the second is an isomorphism; and the third is flat, since $\mathbf{G}^A_s$ is (obviously) flat over $\mathrm{Spec}(\kappa(s))$. We conclude that f_s is flat.

The two preceding paragraphs and Lemma 5.2(3) show that f satisfies the assumptions of the fiberwise criterion for flatness (see [Sta22, Tag 039E]). This criterion implies that $\mathbf{G}^A$ is flat over S, i.e. that (1) holds.

Part (2) follows from Proposition 5.4(7) and the comments following Lemma 2.2.

Regarding part (3), if $\mathbf{G}^A$ is smooth then so is the open subscheme $\mathbf{C}^A$, so that the conditions in the statement must be satisfied by Lemma 5.2(3). Conversely, if these conditions are satisfied then $\mathbf{C}^A$ is smooth. We deduce that for any $s \in S$ the fiber $\mathbf{G}^A_s$ admits a smooth open subscheme containing the unit, and is therefore smooth by [DG70a, II, §5, Théorème 2.1]. Since $\mathbf{G}^A$ is known to be flat over S, by [Sta22, Tag 01V8] this implies that $\mathbf{G}^A$ is smooth, and finishes the proof.

Finally, regarding (4), we have seen that $\mathbf{G}^A$ is flat over S in (1). It is clearly affine, and its generic fiber is smooth by (3) and geometrically connected (and hence connected) by (2). The identity connected component of its geometric special fiber is reductive by Proposition 5.4(1), which finishes the verification of the conditions of the definition in [PY06]. (These conditions include an extra condition on comparison of dimensions, but this condition is automatic when the group scheme is of finite type, which is the case here, as explained in the discussion following [PY06, Theorem 1.2].)

Remark 5.7. Consider the morphism f used in the proof of Theorem 5.1(1). The fiberwise criterion for flatness also implies that f is flat. Since it is also surjective, it is faithfully flat. It is of finite type, hence an fppf cover, and in particular an epimorphism.

Corollary 5.8. *Assume $\mathbf{G}$ and A satisfy the conditions of Theorem 5.1(3). Let $(\mathbf{G}^A)^\circ \subset \mathbf{G}^A$ be the "fiberwise identity component," i.e., the open subgroup scheme characterized by the following property:*

> *For each point $s \in S$, $(\mathbf{G}^A)^\circ_s$ is the connected component of the unit in the smooth group scheme $(\mathbf{G}^A)_s$ over $\kappa(s)$.*

Then $(\mathbf{G}^A)^\circ$ is a split reductive group scheme over S. In particular, if $\mathbf{G}$ is semisimple and either simply connected or of adjoint type, then $\mathbf{G}^A$ is a split reductive group scheme over S.

For the existence of the subgroup scheme $(\mathbf{G}^A)^\circ$, see [DG11a, Exp. $\mathrm{VI_B}$, §3].

Proof. Let N be the order of the torsion subgroup of M_A, multiplied by 2 if $\mathfrak{R}$ has an indecomposable constituent of type A_{2n} with a nontrivial action of its stabilizer in A. The conditions in Theorem 5.1(3) imply that S admits a map to $\mathrm{Spec}(\mathbb{Z}[\frac{1}{N}])$. Then $\mathbf{G}$ (together with the action of A) can be obtained by base change from a pinned reductive group scheme over $\mathrm{Spec}(\mathbb{Z}[\frac{1}{N}])$; by compatibility of the operation $(-)^\circ$ with base change (see [DG11a, Exp. $\mathrm{VI_B}$, Proposition 3.3]), this reduces the proof to the case $S = \mathrm{Spec}(\mathbb{Z}[\frac{1}{N}])$. Now $(\mathbf{G}^A)^\circ$ is smooth, as an open subscheme of the smooth scheme $\mathbf{G}^A$. The open immersion $(\mathbf{G}^A)^\circ \to \mathbf{G}^A$ is quasi-compact because $\mathbf{G}^A$ is noetherian (see [Sta22, Tag 01OX]), so $(\mathbf{G}^A)^\circ$ is quasi-affine over $\mathrm{Spec}(\mathbb{Z}[\frac{1}{N}])$ by [Sta22, Tag 02JR] and [Sta22, Tag 01SN]. In view of [DG11a, Exp. $\mathrm{VI_B}$, Proposition 12.9], it follows that $(\mathbf{G}^A)^\circ$ is affine. Finally, each geometric fiber of $(\mathbf{G}^A)^\circ$ is a connected reductive group, so $(\mathbf{G}^A)^\circ$ is a reductive group scheme. The subgroup $(\mathbf{T}^A)^\circ$ is a split maximal torus of $(\mathbf{G}^A)^\circ$ by Proposition 5.4(2).

For the last assertion, if $\mathbf{G}$ is semisimple and either simply connected or of adjoint type, Example 5.3 and Proposition 5.4(7) imply that every geometric fiber of $\mathbf{G}^A$ is connected, so $(\mathbf{G}^A)^\circ = \mathbf{G}^A$. □

Example 5.9. Let us come back to the example considered in §2.3.1, with the pinning of Example 3.1. (With this choice of pinning, it is easily seen that $\mathbb{Z}/2\mathbb{Z}$ acts by pinned automorphisms.)

(1) First, assume that $n = 1$, i.e. $\mathbf{G} = \mathrm{SL}_{3,\mathbb{Z}}$. By Corollary 5.8,

$$\mathrm{Spec}(\mathbb{Z}[\tfrac{1}{2}]) \times_{\mathrm{Spec}(\mathbb{Z})} (\mathrm{SL}_{3,\mathbb{Z}})^{\mathbb{Z}/2\mathbb{Z}}$$

is a reductive group scheme over $\mathrm{Spec}(\mathbb{Z}[\frac{1}{2}])$. In fact, considering the root data (see §6.1 below) one sees that (2.3) is an isomorphism. On the other hand, $\mathrm{Spec}(\mathbb{F}_2) \times_{\mathrm{Spec}(\mathbb{Z})} (\mathrm{SL}_{3,\mathbb{Z}})^{\mathbb{Z}/2\mathbb{Z}}$ is not reduced, and (2.4) is an isomorphism.

(2) Now, consider the case $n \in \mathbb{Z}_{\geq 2}$. Then $\mathbf{G}^A$ is a flat and geometrically connected group scheme over $\mathbb{Z}$, the restriction

$$\mathrm{Spec}(\mathbb{Z}[\tfrac{1}{2}]) \times_{\mathrm{Spec}(\mathbb{Z})} (\mathrm{SL}_{2n+1,\mathbb{Z}})^{\mathbb{Z}/2\mathbb{Z}}$$

is isomorphic to $\mathrm{SO}_{2n+1,\mathbb{Z}[\frac{1}{2}]}$, but $\mathrm{Spec}(\mathbb{F}_2) \times_{\mathrm{Spec}(\mathbb{Z})} (\mathrm{SL}_{2n+1,\mathbb{Z}})^{\mathbb{Z}/2\mathbb{Z}}$ is nonreduced. The associated reduced group scheme is simple and simply connected of type C_n, i.e. isomorphic to $\mathrm{Sp}_{2n,\mathbb{F}_2}$. In particular, the base change of $\mathbf{G}$ to $\mathbb{Z}_2$ is a quasi-reductive $\mathbb{Z}_2$-group scheme in the sense of Prasad–Yu [PY06] which is nonreductive.

5.5. **The case of general fields.** For completeness, we explain in this subsection how to generalize the results of §5.3 to more general base fields. We therefore assume that $S = \mathrm{Spec}(\Bbbk)$ for some field $\Bbbk$, which we do *not* assume to be algebraically closed. We choose an algebraic closure of $\Bbbk$, which we denote by $\overline{\Bbbk}$. For any scheme X over $\Bbbk$, we set $X_{\overline{\Bbbk}} := \overline{\Bbbk} \otimes_{\Bbbk} X$. With this notation, by (2.1) we have

$$(\mathbf{G}^A)_{\overline{\Bbbk}} = (\mathbf{G}_{\overline{\Bbbk}})^A;$$

this group scheme will be denoted $\mathbf{G}^A_{\overline{\Bbbk}}$.

Proposition 5.10. (1) *The reduced subscheme $(\mathbf{G}^A)_{\mathrm{red}}$ is geometrically reduced; as a consequence it is a subgroup scheme of $\mathbf{G}^A$, and we have $(\mathbf{G}^A)_{\mathrm{red},\overline{\Bbbk}} = (\mathbf{G}^A_{\overline{\Bbbk}})_{\mathrm{red}}$.*

(2) *We have $((\mathbf{G}^A)^\circ_{\mathrm{red}})_{\overline{\Bbbk}} = (\mathbf{G}^A_{\overline{\Bbbk}})^\circ_{\mathrm{red}}$, and $(\mathbf{G}^A)^\circ_{\mathrm{red}}$ is a split reductive group over $\Bbbk$.*

Proof. (1) Consider once again the open subscheme $\mathbf{C}^A \subset \mathbf{G}^A$, and the decompositions (5.1) and (5.2). For any equivalence class E in $\mathfrak{R}_+$ we have the subgroup $((\mathbf{U}_E)^A)_{\mathrm{red}}$ which is isomorphic to $\mathbb{G}_{\mathrm{a},\Bbbk}$ (as in Step 2 of the proof of Proposition 5.4); in particular $(\mathbf{U}^A)_{\mathrm{red}}$ is an affine space over $\Bbbk$. Similarly $((\mathbf{U}^-)^A)_{\mathrm{red}}$ is an affine space over $\Bbbk$. We deduce that $(\mathbf{C}^A)_{\mathrm{red}}$ is the product of $(\mathbf{T}^A)_{\mathrm{red}}$ and an affine space; in particular it is geometrically reduced.

Recall now the morphism

$$f : \mathbf{C}^A \times \mathbf{C}^A \to \mathbf{G}^A$$

from the proof of Theorem 5.1. Since f is faithfully flat it induces an injective morphism $\mathscr{O}(\mathbf{G}^A) \hookrightarrow \mathscr{O}(\mathbf{C}^A \times \mathbf{C}^A)$, hence an embedding

$$\mathscr{O}((\mathbf{G}^A)_{\mathrm{red}}) \hookrightarrow \mathscr{O}((\mathbf{C}^A \times \mathbf{C}^A)_{\mathrm{red}}).$$

Here, since $(\mathbf{C}^A)_{\mathrm{red}}$ is geometrically reduced we have $(\mathbf{C}^A \times \mathbf{C}^A)_{\mathrm{red}} = (\mathbf{C}^A)_{\mathrm{red}} \times (\mathbf{C}^A)_{\mathrm{red}}$, and this scheme is also geometrically reduced. Hence $(\mathbf{G}^A)_{\mathrm{red}}$ is geometrically reduced. The other statements in (1) are immediate consequences.

(2) Since the formation of the neutral component commutes with field extensions (as follows from [Sta22, Tag 04KV]) we have

$$((\mathbf{G}^A)^\circ_{\mathrm{red}})_{\overline{\Bbbk}} = ((\mathbf{G}^A)_{\mathrm{red},\overline{\Bbbk}})^\circ,$$

and by (1) the right-hand side coincides with $(\mathbf{G}^A_{\overline{\Bbbk}})^\circ_{\mathrm{red}}$, which proves the desired equality. Since the latter group is connected and reductive by Proposition 5.4(1), we deduce that $(\mathbf{G}^A)^\circ_{\mathrm{red}}$ is a reductive group over $\Bbbk$. The maximal torus $(\mathbf{T}^A)^\circ_{\mathrm{red}} = \mathrm{D}_{\Bbbk}(M_A/(M_A)_{\mathrm{tor}})$ is split, hence $(\mathbf{G}^A)^\circ_{\mathrm{red}}$ is split. $\square$

Once Proposition 5.10 is established, the other structural properties of Proposition 5.4 follow for general base fields. In particular, here again the connected components of $\mathbf{G}^A$ are in bijection with those of $\mathbf{T}^A$; some of these connected components might therefore not be geometrically connected.

6. Complements

6.1. **Root data for fixed points.** In this subsection we discuss the notion of roots and coroots for the fixed point group schemes $\mathbf{G}^A$. We will give a description that only depends on the following data: a root datum

$$\Phi = (M, \mathfrak{R}, M^\vee, \mathfrak{R}^\vee),$$

a basis $\Delta \subset \mathfrak{R}$, and an action of a group A on Φ preserving Δ. In this setting we will denote by W_Φ the Weyl group of Φ, and by $\mathfrak{R}^{(A)}$, resp. $\mathfrak{R}_+^{(A)}$, resp. $\Delta^{(A)}$, the image of $\mathfrak{R}$, resp. $\mathfrak{R}_+$, resp. Δ, along the quotient map

$$M \to M_A.$$

Considering a pinned reductive group scheme with root datum $(M, \mathfrak{R}, M^\vee, \mathfrak{R}^\vee)$ and basis Δ over an algebraically closed field, Steps 5–6 in the proof of Proposition 5.4 (and the fact that $\mathbf{K}$ is trivial) show that the natural map

$$\mathfrak{R} \to \mathfrak{R}^{(A)}$$

induces a bijection $\mathfrak{R}/A \xrightarrow{\sim} \mathfrak{R}^{(A)}$. In particular, we have

$$\mathfrak{R}^{(A)} = \mathfrak{R}_+^{(A)} \sqcup -\mathfrak{R}_+^{(A)}.$$

Recall the equivalence relation $\sim$ considered in §4.1. If an equivalence class E is of type (ii′), write it as $E = E' \sqcup E''$ with E' being the set of nonspecial roots, and E'' the set of special roots. (Both E' and E'' are A-orbits.) Now, we denote by:

- $\mathfrak{R}^{(A)}_{\mathrm{nd,nm},+} \subset \mathfrak{R}_+^{(A)}$ the subset consisting of restrictions of equivalence classes E of type (i′);
- $\mathfrak{R}^{(A)}_{\mathrm{m},+} \subset \mathfrak{R}_+^{(A)}$ the subset consisting of restrictions of A-orbits of the form E' where $E = E' \sqcup E''$ is an equivalence class of type (ii′);
- $\mathfrak{R}^{(A)}_{\mathrm{d},+} \subset \mathfrak{R}_+^{(A)}$ the subset consisting of restrictions of A-orbits of the form E'' where $E = E' \sqcup E''$ is an equivalence class of type (ii′).

We also set

$$\mathfrak{R}^{(A)}_{\mathrm{nd,nm}} = \mathfrak{R}^{(A)}_{\mathrm{nd,nm},+} \sqcup -\mathfrak{R}^{(A)}_{\mathrm{nd,nm},+}, \quad \mathfrak{R}^{(A)}_{\mathrm{m}} = \mathfrak{R}^{(A)}_{\mathrm{m},+} \sqcup -\mathfrak{R}^{(A)}_{\mathrm{m},+}, \quad \mathfrak{R}^{(A)}_{\mathrm{d}} = \mathfrak{R}^{(A)}_{\mathrm{d},+} \sqcup -\mathfrak{R}^{(A)}_{\mathrm{d},+}.$$

Then we have a partition

$$\mathfrak{R}^{(A)} = \mathfrak{R}^{(A)}_{\mathrm{nd,nm}} \sqcup \mathfrak{R}^{(A)}_{\mathrm{m}} \sqcup \mathfrak{R}^{(A)}_{\mathrm{d}},$$

and the assignment $\gamma \mapsto 2\gamma$ induces a bijection $\mathfrak{R}^{(A)}_{\mathrm{m}} \xrightarrow{\sim} \mathfrak{R}^{(A)}_{\mathrm{d}}$. (Here, "m" stands for "multipliable," "d" for "divisible," and "n" for "non".) We also set

$$\mathfrak{R}^{(A)}_1 = \mathfrak{R}^{(A)}_{\mathrm{nd,nm}} \sqcup \mathfrak{R}^{(A)}_{\mathrm{m}}, \quad \mathfrak{R}^{(A)}_2 = \mathfrak{R}^{(A)}_{\mathrm{nd,nm}} \sqcup \mathfrak{R}^{(A)}_{\mathrm{d}}.$$

Finally, we set

$$\Delta^{(A)}_2 = (\Delta^{(A)} \cap \mathfrak{R}^{(A)}_{\mathrm{nd,nm}}) \sqcup \{2\alpha : \alpha \in \Delta^{(A)} \cap \mathfrak{R}^{(A)}_{\mathrm{m}}\}.$$

It turns out that $\mathfrak{R}^{(A)}$ "extends" to a root datum. For this, one has to define the set of coroots $\mathfrak{R}^{(A),\vee}$ via the following formula. Let $\gamma \in \mathfrak{R}^{(A)}_+$, and let E be the unique equivalence class containing the A-orbit in $\mathfrak{R}_+$ corresponding to γ. Then we set

$$\gamma^\vee = \begin{cases} \sum_{\alpha \in E} \alpha^\vee & \text{if } \gamma \in \mathfrak{R}^{(A)}_{\mathrm{nd,nm}} \sqcup \mathfrak{R}^{(A)}_{\mathrm{m}}; \\ \sum_{\alpha \in E''} \alpha^\vee & \text{if } \gamma \in \mathfrak{R}^{(A)}_{\mathrm{d}}. \end{cases} \tag{6.1}$$

If $\gamma \in -\mathfrak{R}^{(A)}_+$, we set $\gamma^\vee := -(-\gamma)^\vee$. In both cases, we regard these coroots as elements in $(M^\vee)^A$.

If $\mathbf{T}$ is a torus with character lattice M (over a connected scheme S), and if we consider the action on A induced by our given action on M, the set of cocharacters $\mathbf{G}_{\mathrm{m}} \to \mathbf{T}^A$ identifies naturally with $(M^\vee)^A$, where $M^\vee$ is identified with the lattice of cocharacters $\mathbf{G}_{\mathrm{m}} \to \mathbf{T}$. Hence, the formulas above determine cocharacters of $\mathbf{T}^A$. This contrasts with the character lattice M_A of $\mathbf{T}^A$ which may admit a non-trivial torsion subgroup $(M_A)_{\mathrm{tor}}$. The corresponding quotient $M_A/(M_A)_{\mathrm{tor}}$ is the weight lattice of the maximal subtorus scheme of $\mathbf{T}^A$. Steps 5–6 in the proof of Proposition 5.4 show that the quotient morphism $M_A \to M_A/(M_A)_{\mathrm{tor}}$ is injective on $\mathfrak{R}^{(A)}$, so that this set can (and will) also be regarded as a subset in $M_A/(M_A)_{\mathrm{tor}}$. We will denote by $\mathfrak{R}^{(A),\vee}$, resp. $\mathfrak{R}^{(A),\vee}_1$, resp. $\mathfrak{R}^{(A),\vee}_2$, the image of $\mathfrak{R}^{(A)}$, resp. $\mathfrak{R}^{(A)}_1$, resp. $\mathfrak{R}^{(A)}_2$, under the assignment $\gamma \mapsto \gamma^\vee$.

The following proposition is essentially proved in [Hai15, Hai18]; see also [AL19].

Proposition 6.1. (1) *The quadruple $(M_A/(M_A)_{\mathrm{tor}}, \mathfrak{R}^{(A)}_1, (M^\vee)^A, \mathfrak{R}^{(A),\vee}_1)$ is a reduced root datum, with basis $\Delta^{(A)}$ and Weyl group $(W_\Phi)^A$.*

(2) *The quadruple $(M_A/(M_A)_{\mathrm{tor}}, \mathfrak{R}^{(A)}_2, (M^\vee)^A, \mathfrak{R}^{(A),\vee}_2)$ is a reduced root datum, with basis $\Delta^{(A)}_2$ and Weyl group $(W_\Phi)^A$.*

(3) *The quadruple $(M_A/(M_A)_{\mathrm{tor}}, \mathfrak{R}^{(A)}, (M^\vee)^A, \mathfrak{R}^{(A),\vee})$ is a (not necessarily reduced) root datum, with basis $\Delta^{(A)}$ and Weyl group $(W_\Phi)^A$.*

In the rest of this subsection, we briefly explain how this statement can be recovered from the analysis in §5.3. We first consider the first two parts. Assume that S is the spectrum of an algebraically closed field $\Bbbk$, and consider the pinned reductive group scheme $\mathbf{G}$ over S with root datum Φ. By Proposition 5.4 and Remark 5.5, $(\mathbf{G}^A)^\circ_{\mathrm{red}}$ is a connected reductive algebraic group, with maximal torus $(\mathbf{T}^A)^\circ_{\mathrm{red}}$ (whose lattice of characters identifies with $M_A/(M_A)_{\mathrm{tor}}$), and its root system is $\mathfrak{R}^{(A)}_1$ if $\mathrm{char}(\Bbbk) \neq 2$, and $\mathfrak{R}^{(A)}_2$ if $\mathrm{char}(\Bbbk) = 2$. Remark 5.5 also describes its coroots: they are the cocharacters constructed in Step 3 of the proof of Proposition 5.4, in (5.3). Note that that formula agrees with (6.1).

In each case it is easily seen that $\Delta^{(A)}$ is a basis, and it follows from Proposition 5.4(8) that the corresponding Weyl group is $(W_\Phi)^A$.

This justifies the first two cases in Proposition 6.1. The third case follows, using the standard observation that a union of root systems of types B_n and C_n produces a nonreduced root system of type BC_n.

Remark 6.2. In the setting of Corollary 5.8, from the pinning of $\mathbf{G}$ one can obtain a pinning of $(\mathbf{G}^A)^\circ$, with associated root datum $(M_A/(M_A)_{\mathrm{tor}}, \mathfrak{R}_1^{(A)}, (M^\vee)^A, \mathfrak{R}_1^{(A),\vee})$.

6.2. **Weyl group.** In this subsection we prove some results on the interplay between the maximal torus $\mathbf{T}$, its normalizer $\mathbf{N}$, the Weyl group $\mathbf{W}$, and fixed points.

Let us consider once again a pinned reductive group scheme (3.1) over S, and our group A which acts on $\mathbf{G}$ by pinned automorphisms. As in §5.3 (but now over an arbitrary base), we denote by $\mathbf{N}$ the normalizer of $\mathbf{T}$ in $\mathbf{G}$. By [Con14, Proposition 2.1.2], $\mathbf{N}$ is a smooth subgroup scheme of $\mathbf{G}$. Denote by λ the sum of the positive coroots in $\mathfrak{R}^\vee$; then λ defines a cocharacter $\mathbf{G}_{\mathrm{m},S} \to \mathbf{T}$, which is A-invariant and hence takes values in $\mathbf{T}^A$. As explained e.g. in [Con14, Theorem 5.1.13], $\mathbf{T}$ coincides with the centralizer of λ; in particular, $\mathbf{T}$ is its own centralizer in $\mathbf{G}$. The Weyl group $\mathbf{W}$ is the quotient sheaf $\mathbf{N}/\mathbf{T}$ for the fppf topology. By [Con14, Proposition 5.1.6], $\mathbf{W}$ is representable by the constant group scheme $(W_\Phi)_S$ over S, where W_Φ is the Weyl group of the root datum $\Phi = (M, \mathfrak{R}, M^\vee, \mathfrak{R}^\vee)$.

Lemma 6.3. *The following properties hold:*

(1) *the centralizer of* $\mathbf{T}^A$ *in* $\mathbf{G}$ *is representable by* $\mathbf{T}$*; in particular, the centralizer of* $\mathbf{T}^A$ *in* $\mathbf{G}^A$ *is representable by the closed subgroup scheme* $\mathbf{T}^A$*;*
(2) *the normalizer of* $\mathbf{T}^A$ *in* $\mathbf{G}$ *is contained in* $\mathbf{N}$*; in particular, the normalizer of* $\mathbf{T}^A$ *in* $\mathbf{G}^A$ *is representable by the closed subgroup scheme* $\mathbf{N}^A$*;*
(3) *the quotient sheaf* $\mathbf{N}^A/\mathbf{T}^A$ *is representable by the constant S-group scheme* $((W_\Phi)^A)_S$.

Proof. (1) As explained above, the cocharacter λ takes values in $\mathbf{T}^A$; its centralizer in $\mathbf{G}$ (i.e. $\mathbf{T}$) therefore contains the centralizer of $\mathbf{T}^A$. We deduce that $\mathrm{Z}_{\mathbf{G}}(\mathbf{T}^A) = \mathbf{T}$, and then that $\mathrm{Z}_{\mathbf{G}^A}(\mathbf{T}^A) = \mathbf{T}^A$.

(2) The normalizer $\mathrm{N}_{\mathbf{G}}(\mathbf{T}^A)$ of $\mathbf{T}^A$ in $\mathbf{G}$ must preserve the centralizer $\mathrm{Z}_{\mathbf{G}}(\mathbf{T}^A) = \mathbf{T}$. We deduce an inclusion $\mathrm{N}_{\mathbf{G}}(\mathbf{T}^A) \subset \mathbf{N}$. Intersecting with $\mathbf{G}^A$, it follows that $\mathrm{N}_{\mathbf{G}^A}(\mathbf{T}^A) \subset \mathbf{N}^A$. The opposite inclusion is clear.

(3) The quotient sheaf $\mathbf{N}^A/\mathbf{T}^A$ injects into $\mathbf{W}^A = ((W_\Phi)^A)_S$ by definition. The proof of surjectivity is similarly to the proof of surjectivity of (5.4): either by [Hée91, Corollaire 3.5] or because $\mathbf{W}^A$ is the Weyl group of the root datum $(M_A/(M_A)_{\mathrm{tor}}, \mathfrak{R}^{(A)}, (M^\vee)^A, \mathfrak{R}^{(A),\vee})$ (see Lemma 6.1), this group is generated by the simple reflections associated with equivalence classes in $\mathfrak{R}_+$ which intersect Δ. If E is such an equivalence class of type (i′), resp. (ii′), then the corresponding reflection is the image of $\imath_E \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$, resp. $\imath_E \left(\begin{smallmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{smallmatrix}\right)$. □

6.3. **Parabolic and Levi subgroups.** Let us consider once again a pinned reductive group scheme (3.1) over S, and our group A which acts on $\mathbf{G}$ by pinned automorphisms. Recall that given an S-scheme X which is affine over S and endowed with an action of a diagonalizable group scheme $\mathrm{D}_S(N)$, following [May22], for any submonoid $Q \subset N$ we have a closed subscheme $X^Q \subset X$ called the attractor scheme associated with Q. (This construction generalizes the classical notion of attractor for an action of $\mathbf{G}_\mathrm{m}$ used e.g. in [CGP15, Con14].) In this subsection we will consider this construction in the case of the action of $\mathbf{T} = \mathrm{D}_S(M)$, resp. $\mathbf{T}^A = \mathrm{D}_S(M_A)$, on $X = \mathbf{G}$, resp. $X = \mathbf{G}^A$.

Recall the root datum
$$(M_A/(M_A)_{\mathrm{tor}}, \mathfrak{R}^{(A)}, (M^\vee)^A, \mathfrak{R}^{(A),\vee})$$
considered in Proposition 6.1(3), and its basis $\Delta^{(A)}$. We have a canonical identification $\Delta/A \xrightarrow{\sim} \Delta^{(A)}$, and therefore a canonical bijection between subsets of $\Delta^{(A)}$ and A-stable subsets of Δ.

Consider an A-stable subset $\Gamma \subset \Delta$, and denote by $\Gamma^{(A)} \subset \Delta^{(A)}$ its image in $\Delta^{(A)}$. To Γ we can associate a parabolic subgroup $\mathbf{P}$ of $\mathbf{G}$, whose Lie algebra is the sum of $\mathscr{L}ie(\mathbf{B})$ and the root subspaces in $\mathscr{L}ie(\mathbf{G})$ associated with the roots which belong to $\mathbb{Z}\Gamma = \sum_{\gamma \in \Gamma} \mathbb{Z} \cdot \gamma$. Following [Con14, Example 5.2.2], this subgroup scheme can be realized as the attractor subscheme associated with the conjugation action of $\mathbf{G}_{\mathrm{m},S}$ on $\mathbf{G}$ via a cocharacter $\mu \in M^\vee$ which satisfies

$$\begin{cases} \langle \mu, \alpha^\vee \rangle > 0 & \text{if } \alpha \in \Delta \smallsetminus \Gamma; \\ \langle \mu, \alpha^\vee \rangle = 0 & \text{if } \alpha \in \Gamma. \end{cases}$$

We also have the Levi factor $\mathbf{M}$ of $\mathbf{P}$ containing $\mathbf{T}$, which can be realized as the fixed point subscheme associated with the same action of $\mathbf{G}_{\mathrm{m},S}$, see [Con14, Proposition 5.4.5 and its proof], and its unipotent radical $\mathbf{U}_\mathbf{P}$, see [Con14, Corollary 5.2.5]. The theory of [May22] allows us to make this construction more canonical, by suppressing the choice of the cocharacter μ. Namely, denote by N_Γ the submonoid of M generated by $\Delta \sqcup (-\Gamma)$. The following claim is an immediate generalization of [May22, Proposition 7.3].

Lemma 6.4. *We have*

$$\mathbf{P} = \mathbf{G}^{N_\Gamma}, \quad \mathbf{M} = \mathbf{G}^{\mathbb{Z}\Gamma}.$$

Proof. The proof is the same as that of [May22, Proposition 7.3]. Namely, choose $\mu \in M^\vee$ as above, which we now consider as a morphism $M \to \mathbb{Z}$. The induced morphism $\mathbf{G}_{\mathrm{m},S} = \mathrm{D}_S(\mathbb{Z}) \to \mathrm{D}_S(M) = \mathbf{T}$ is precisely μ seen as a cocharacter. By [May22, Lemma 5.6] we have $\mathbf{G}^{\mu^{-1}(\mathbb{Z}_{\geq 0})} = \mathbf{G}^{\mathbb{Z}_{\geq 0}}$ where on the left-hand side we regard $\mathbf{G}$ with the action of $\mathbf{T}$, and on the right-hand side we regard $\mathbf{G}$ with the action of $\mathbf{G}_{\mathrm{m},S}$ via μ. As explained above the right-hand side is known to coincide with $\mathbf{P}$. Using the fact that $N_\Gamma \subset \mu^{-1}(\mathbb{Z}_{\geq 0})$, in view of [May22, Remark 3.5] we deduce that $\mathbf{G}^{N_\Gamma} \subset \mathbf{P}$. On the other hand we have $\mathbf{P} = \mathbf{M} \ltimes \mathbf{U}_\mathbf{P}$. It is easily seen that both $\mathbf{M}$ and $\mathbf{U}_\mathbf{P}$ are contained in $\mathbf{G}^{N_\Gamma}$, which implies that $\mathbf{P} \subset \mathbf{G}^{N_\Gamma}$ and finishes the proof of the first claim. The proof of the second one is similar. $\square$

Now, let us consider the submonoid $N_\Gamma^{(A)}$ of M_A generated by $\Delta^{(A)} \sqcup (-\Gamma^{(A)})$ and the subgroup $\mathbb{Z}\Gamma^{(A)}$ generated by $\Gamma^{(A)}$.

Proposition 6.5. *The natural morphisms*

$$\mathbf{P}^A \to (\mathbf{G}^A)^{N_\Gamma^{(A)}} \quad \text{and} \quad \mathbf{M}^A \to (\mathbf{G}^A)^{\mathbb{Z}\Gamma^{(A)}}$$

are isomorphisms.

Proof. The proof is similar to that of Lemma 6.4; we explain the first case, and leave the easy modifications to treat the second case to the reader. The cocharacter μ considered above can be chosen to be A-invariant. (In fact one can reduce the situation to that of a semisimple and simply connected group, and then average any cocharacter satisfying our conditions along the action of A.) Then μ takes values in $\mathbf{T}^A$, or in other words the associated morphism $M \to \mathbb{Z}$ (still denoted μ) factors through a morphism $\overline{\mu} : M_A \to \mathbb{Z}$. Since the action of $\mathbf{G}_{\mathrm{m},S}$ via μ commutes with the action of A we have

$$\mathbf{P}^A = (\mathbf{G}^{\mathbb{Z}_{\geq 0}})^A = (\mathbf{G}^A)^{\mathbb{Z}_{\geq 0}}.$$

By [May22, Lemma 5.6] we have $(\mathbf{G}^A)^{\mathbb{Z}_{\geq 0}} = (\mathbf{G}^A)^{\overline{\mu}^{-1}(\mathbb{Z}_{\geq 0})}$. Since $N_\Gamma^{(A)} \subset \overline{\mu}^{-1}(\mathbb{Z}_{\geq 0})$, we deduce that

$$(\mathbf{G}^A)^{N_\Gamma^{(A)}} \subset \mathbf{P}^A.$$

On the other hand, if T is an S-scheme and $x : T \to \mathbf{P}^A$ is a T-point of $\mathbf{P}^A$, then since $\mathbf{P} = \mathbf{G}^{N_\Gamma}$ we have a canonical morphism

$$x' : \mathrm{A}_S(N_\Gamma) \to \mathbf{G}$$

where the left-hand side is the S-scheme associated with the monoid N_Γ following [May22, §3]. This morphism is easily seen to be A-invariant; it therefore restricts to a morphism $x'' : (\mathrm{A}_S(N_\Gamma))^A \to \mathbf{G}^A$. We also have a canonical morphism $N_\Gamma \to N_\Gamma^{(A)}$, which induces a morphism $\mathrm{A}_S(N_\Gamma^{(A)}) \to \mathrm{A}_S(N_\Gamma)$. This morphism is A-equivariant with respect to the trivial action on its domain, so it factors through a morphism $\mathrm{A}_S(N_\Gamma^{(A)}) \to (\mathrm{A}_S(N_\Gamma))^A$. Composing with x'' we obtain a morphism

$$\mathrm{A}_S(N_\Gamma^{(A)}) \to \mathbf{G}^A.$$

This construction shows that $\mathbf{P}^A \subset (\mathbf{G}^A)^{N_\Gamma^{(A)}}$, which finishes the proof. □

In particular, given $\gamma \in \Delta^{(A)}$, we can consider the parabolic subgroup and its Levi factor $\mathbf{M}_\gamma \subset \mathbf{P}_\gamma \subset \mathbf{G}$ associated with the A-orbit in Δ corresponding to γ, and the corresponding fixed points $(\mathbf{M}_\gamma)^A \subset \mathbf{G}$. The following claim will be required in [ALRR23].

Corollary 6.6. *The unique subgroup scheme of $\mathbf{G}^A$ that contains $(\mathbf{M}_\gamma)^A$ for every $\gamma \in \Delta^{(A)}$ is $\mathbf{G}^A$ itself.*

Proof. Let $\mathbf{H} \subset \mathbf{G}^A$ be a subgroup scheme that contains all the subgroups $(\mathbf{M}_\gamma)^A$ for $\gamma \in \Delta^{(A)}$. Then $\mathbf{H}$ contains $\mathbf{T}^A$, and also the elements in $\mathbf{N}^A$ corresponding to the simple reflections in the Weyl group $\mathbf{W}^A$ constructed in the proof of Lemma 6.3(3). Since these elements generate $\mathbf{W}^A$, we deduce that $\mathbf{H}$ contains $\mathbf{N}^A$. Since any root in $\mathfrak{R}_1^{(A)}$ is $\mathbf{W}^A$-conjugate to a root in $\Delta^{(A)}$ (see §6.1), we then deduce that $\mathbf{H}$ contains each subgroup $(\mathbf{U}_E)^A$ with E an equivalence class in $\mathfrak{R}_+$, and therefore that it contains $\mathbf{C}^A$ (see (5.1) and (5.2)). Finally, since the multiplication morphism $\mathbf{C}^A \times \mathbf{C}^A \to \mathbf{G}^A$ is an epimorphism (see Remark 5.7), this implies that $\mathbf{H} = \mathbf{G}^A$. □

6.4. **Center and isogenies.** Let us consider once again a pinned reductive group scheme (3.1) over S, and our group A which acts on $\mathbf{G}$ by pinned automorphisms. Recall the definition of the center of a group scheme; see [Con14, Definition 2.2.1]. (By definition, this center is a sheaf of groups, which is not necessarily representable.) In particular let $\mathbf{Z} \subset \mathbf{G}$ denote the center of $\mathbf{G}$. By [Con14, Theorem 3.3.4], $\mathbf{Z}$ is representable by a diagonalizable group scheme; more explicitly we have

$$\mathbf{Z} = \bigcap_{\alpha \in \Delta} \ker(\alpha) = \mathrm{D}_S(M/\mathbb{Z}\mathfrak{R}).$$

By functoriality, $\mathbf{Z}$ is preserved by the A-action, so we may form its fixed-point scheme $\mathbf{Z}^A$, which is again a diagonalizable group scheme: more explicitly, by Lemma 2.2 we have

$$\mathbf{Z}^A = \mathrm{D}_S((M/\mathbb{Z}\mathfrak{R})_A). \tag{6.2}$$

Lemma 6.7. *The closed subgroup scheme $\mathbf{Z}^A$ represents the center of $\mathbf{G}^A$.*

Proof. Denote by $\mathbf{Z}'$ the center of $\mathbf{G}^A$. Since $\mathbf{Z}$ is the center of $\mathbf{G}$, we have $\mathbf{Z}^A \subset \mathbf{Z}'$. On the other hand, we know by Lemma 6.3 that $\mathbf{T}^A$ is its own centralizer, so $\mathbf{Z}'$ is contained in $\mathbf{T}^A$. Considering the $\mathbf{T}^A$-action on each $(\mathbf{U}_E)^A$, we obtain that

$$\mathbf{Z}' \subset \bigcap_{\alpha \in \Delta^{(A)}} \ker(\alpha),$$

which in view of (6.2) shows that $\mathbf{Z}' \subset \mathbf{Z}^A$, and hence that $\mathbf{Z}' = \mathbf{Z}^A$. □

Recall the group scheme $\mathbf{G}_{\mathrm{sc}}$ considered in §3.5, and denote by $\mathbf{Z}_{\mathrm{sc}}$ its center. The natural morphism $f : \mathbf{G}_{\mathrm{sc}} \to \mathbf{G}$ restricts to a morphism $\mathbf{Z}_{\mathrm{sc}} \to \mathbf{Z}$. It is a standard fact that the natural morphism

$$\mathbf{G}_{\mathrm{sc}} \times_S^{\mathbf{Z}_{\mathrm{sc}}} \mathbf{Z} \to \mathbf{G}$$

is an isomorphism, where the left-hand side is the (fppf) quotient of $\mathbf{G}_{\mathrm{sc}} \times_S \mathbf{Z}$ by the action of $\mathbf{Z}_{\mathrm{sc}}$ defined by $z \cdot (g, h) = (gz^{-1}, f(z)h)$. In other words, we have an exact sequence of fppf sheaves of groups

$$1 \to \mathbf{Z}_{\mathrm{sc}} \to \mathbf{G}_{\mathrm{sc}} \times_S \mathbf{Z} \to \mathbf{G} \to 1 \tag{6.3}$$

where the first morphism is the natural antidiagonal embedding. The next lemma provides a version of this result for the group $\mathbf{G}^A$, which will be used in [ALRR23].

Proposition 6.8. *The natural map*

$$(\mathbf{G}_{\mathrm{sc}})^A \times_S^{(\mathbf{Z}_{\mathrm{sc}})^A} \mathbf{Z}^A \to \mathbf{G}^A$$

is an isomorphism, where the left-hand side is the quotient of $(\mathbf{G}_{\mathrm{sc}})^A \times_S \mathbf{Z}^A$ by the action of $(\mathbf{Z}_{\mathrm{sc}})^A$ induced by the action of $\mathbf{Z}_{\mathrm{sc}}$ on $\mathbf{G}_{\mathrm{sc}} \times_S \mathbf{Z}$ considered above.

Proof. The given map is clearly a monomorphism of group objects in fppf sheaves, as taking A-fixed points is a left exact functor. To conclude the proof, it therefore suffices to check surjectivity of the morphism $\mathbf{G}_{\mathrm{sc}}^A \times_S \mathbf{Z}^A \to \mathbf{G}^A$ in the fppf topology. Since the multiplication map $\mathbf{C}^A \times_S \mathbf{C}^A \to \mathbf{G}^A$ is an epimorphism of fppf sheaves (see Remark 5.7), we are reduced to showing surjectivity of $\mathbf{C}_{\mathrm{sc}}^A \times_S \mathbf{Z}^A \to \mathbf{C}^A$ where $\mathbf{C}_{\mathrm{sc}}$ is the big cell of $\mathbf{G}_{\mathrm{sc}}$. But passing to simply connected covers does not affect root groups so, due to the decomposition (5.1), it is enough to show the surjectivity of the morphism

$$\mathbf{T}_{\mathrm{sc}}^A \times_S \mathbf{Z}^A \to \mathbf{T}^A.$$

In order to see this, we remark that $\mathbf{T}/\mathbf{Z} = \mathbf{T}_{\mathrm{adj}} := \mathrm{D}_S(\mathbb{Z}\mathfrak{R})$. By left exactness of fixed points, we can embed $\mathbf{T}^A/\mathbf{Z}^A$ into $(\mathbf{T}_{\mathrm{adj}})^A$, reducing the problem to proving that the morphism $(\mathbf{T}_{\mathrm{sc}})^A \to (\mathbf{T}_{\mathrm{adj}})^A$ is surjective. In fact, in view of [DG70b, Exp. VIII, §3] or [Oes14, §5.3], to prove this claim it suffices to prove that the natural morphism $(\mathbb{Z}\mathfrak{R})_A \to (M_{\mathrm{sc}})_A$ is injective, which follows from the fact that these $\mathbb{Z}$-modules are free of finite rank (because $\mathbb{Z}\mathfrak{R}$ and M_{sc} both have a basis permuted by A), and that the given morphism becomes an isomorphism after tensor product with $\mathbb{Q}$. □

6.5. **Further study of the case of** SL_3**.** In this subsection we prove a technical statement regarding the group scheme $(\mathrm{SL}_{3,\mathbb{Z}_2})^{\mathbb{Z}/2\mathbb{Z}}$ for the action considered in §2.3.2 (see Proposition 6.9) that will be required in the companion paper [ALRR23].

If we set

$$n := \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

then n is a $\mathbb{Z}$-point of $(\mathrm{SL}_{3,\mathbb{Z}})^{\mathbb{Z}/2\mathbb{Z}}$. Let us denote by $\mathrm{C}_{3,\mathbb{Z}}$ the big cell constructed from the pinning of Example 3.1. We also let $\mathrm{U}_{3,\mathbb{Z}}$ be as in §2.3.2, $\mathrm{U}_{3,\mathbb{Z}}^-$ be the similar group of lower triangular matrices, and $\mathrm{T}_{3,\mathbb{Z}}$ be the maximal torus of Example 3.1. With this notation, $(\mathrm{SL}_{3,\mathbb{Z}})^{\mathbb{Z}/2\mathbb{Z}}$ has an affine open covering with two open subsets given explicitly by

$$(\mathrm{C}_{3,\mathbb{Z}})^{\mathbb{Z}/2\mathbb{Z}} \cong (\mathrm{U}_{3,\mathbb{Z}}^-)^{\mathbb{Z}/2\mathbb{Z}} \times_{\mathrm{Spec}(\mathbb{Z})} (\mathrm{T}_{3,\mathbb{Z}})^{\mathbb{Z}/2\mathbb{Z}} \times_{\mathrm{Spec}(\mathbb{Z})} (\mathrm{U}_{3,\mathbb{Z}})^{\mathbb{Z}/2\mathbb{Z}}$$

and

$$n \cdot (\mathrm{C}_{3,\mathbb{Z}})^{\mathbb{Z}/2\mathbb{Z}} \cong (\mathrm{U}_{3,\mathbb{Z}}^-)^{\mathbb{Z}/2\mathbb{Z}} \times_{\mathrm{Spec}(\mathbb{Z})} (\mathrm{T}_{3,\mathbb{Z}})^{\mathbb{Z}/2\mathbb{Z}} \times_{\mathrm{Spec}(\mathbb{Z})} (\mathrm{U}_{3,\mathbb{Z}})^{\mathbb{Z}/2\mathbb{Z}}.$$

(To see that these two subschemes cover $(\mathrm{SL}_{3,\mathbb{Z}})^{\mathbb{Z}/2\mathbb{Z}}$, it suffices to check that they contain all points over $\mathrm{Spec}(\mathbb{Z}[\frac{1}{2}])$ and over $\mathrm{Spec}(\mathbb{F}_2)$, and this follows from the analysis in §5.3.) As explained in §2.3.2, we have

$$(\mathrm{U}_{3,\mathbb{Z}}^-)^{\mathbb{Z}/2\mathbb{Z}} \cong (\mathrm{U}_{3,\mathbb{Z}})^{\mathbb{Z}/2\mathbb{Z}} \cong \mathrm{Spec}(\mathbb{Z}[x, y]/(x^2 - 2y)),$$

and the considerations in §2.2 show that we have an isomorphism

$$\mathbb{G}_{\mathrm{m},\mathbb{Z}} \xrightarrow{\sim} (\mathrm{T}_{3,\mathbb{Z}})^{\mathbb{Z}/2\mathbb{Z}} \tag{6.4}$$

given explicitly by

$$a \mapsto \mathrm{diag}(a, 1, a^{-1}).$$

For any commutative ring A, we denote by $\mathrm{SL}_{3,A}$, $\mathrm{C}_{3,A}$, etc., the schemes obtained by base change to $\mathrm{Spec}(A)$. We will be particularly interested in the cases where A is $\mathbb{Z}_2$ or $\mathbb{F}_2$. Our goal is to prove the following statement.

Proposition 6.9. *Let $\mathbf{H}$ be a flat affine group scheme over $\mathbb{Z}_2$, and let*

$$\pi : \mathbf{H} \to (\mathrm{SL}_{3,\mathbb{Z}_2})^{\mathbb{Z}/2\mathbb{Z}}$$

be a morphism of group schemes such that $\pi_{|\mathrm{Spec}(\mathbb{F}_2)}$ is surjective (at the level of topological spaces). Then the schematic image of $\pi_{|\mathrm{Spec}(\mathbb{F}_2)}$ is $(\mathrm{SL}_{3,\mathbb{F}_2})^{\mathbb{Z}/2\mathbb{Z}}$.

We start with two preliminary lemmas.

Lemma 6.10. *The only closed subgroup schemes of $(\mathrm{SL}_{3,\mathbb{F}_2})^{\mathbb{Z}/2\mathbb{Z}}$ containing the subgroup $(\mathrm{SL}_{3,\mathbb{F}_2})^{\mathbb{Z}/2\mathbb{Z}}_{\mathrm{red}}$ are $(\mathrm{SL}_{3,\mathbb{F}_2})^{\mathbb{Z}/2\mathbb{Z}}_{\mathrm{red}}$ and $(\mathrm{SL}_{3,\mathbb{F}_2})^{\mathbb{Z}/2\mathbb{Z}}$.*

Proof. Let $\mathbf{K}$ be a closed subgroup scheme of $(\mathrm{SL}_{3,\mathbb{F}_2})^{\mathbb{Z}/2\mathbb{Z}}$ containing $(\mathrm{SL}_{3,\mathbb{F}_2})^{\mathbb{Z}/2\mathbb{Z}}_{\mathrm{red}}$. The closed immersion $(\mathrm{T}_{3,\mathbb{F}_2})^{\mathbb{Z}/2\mathbb{Z}} \hookrightarrow (\mathrm{SL}_{3,\mathbb{F}_2})^{\mathbb{Z}/2\mathbb{Z}}$ factors through $(\mathrm{SL}_{3,\mathbb{F}_2})^{\mathbb{Z}/2\mathbb{Z}}_{\mathrm{red}}$, and hence through $\mathbf{K}$, and $(\mathrm{T}_{3,\mathbb{F}_2})^{\mathbb{Z}/2\mathbb{Z}}$ identifies with $\mathbb{G}_{\mathrm{m},\mathbb{F}_2}$, see (6.4). We will consider the attractor, resp. repeller, scheme associated with the conjugation action of this subgroup on $\mathrm{SL}_{3,\mathbb{F}_2}$ and $(\mathrm{SL}_{3,\mathbb{F}_2})^{\mathbb{Z}/2\mathbb{Z}}_{\mathrm{red}}$, in the sense of [CGP15, §2.1]. The attractor, resp. repeller, for the action on $\mathrm{SL}_{3,\mathbb{F}_2}$ is the standard positive, resp. negative, Borel subgroup, see e.g. [Con14, Theorem 5.1.13 and its proof]. Hence the corresponding attractor, resp. repeller, for the action on $(\mathrm{SL}_{3,\mathbb{F}_2})^{\mathbb{Z}/2\mathbb{Z}}$ identifies with

$$(\mathrm{T}_{3,\mathbb{F}_2})^{\mathbb{Z}/2\mathbb{Z}} \times_{\mathrm{Spec}(\mathbb{F}_2)} (\mathrm{U}_{3,\mathbb{F}_2})^{\mathbb{Z}/2\mathbb{Z}}, \quad \text{resp.} \quad (\mathrm{U}^-_{3,\mathbb{F}_2})^{\mathbb{Z}/2\mathbb{Z}} \times_{\mathrm{Spec}(\mathbb{F}_2)} (\mathrm{T}_{3,\mathbb{F}_2})^{\mathbb{Z}/2\mathbb{Z}}.$$

In view of [CGP15, Proposition 2.1.8(3)], we deduce that

$$\mathbf{K} \cap (\mathrm{C}_{3,\mathbb{F}_2})^{\mathbb{Z}/2\mathbb{Z}} = \big(\mathbf{K} \cap (\mathrm{U}^-_{3,\mathbb{F}_2})^{\mathbb{Z}/2\mathbb{Z}}\big) \times_{\mathrm{Spec}(\mathbb{F}_2)} (\mathrm{T}_{3,\mathbb{F}_2})^{\mathbb{Z}/2\mathbb{Z}} \times_{\mathrm{Spec}(\mathbb{F}_2)} \big(\mathbf{K} \cap (\mathrm{U}_{3,\mathbb{F}_2})^{\mathbb{Z}/2\mathbb{Z}}\big).$$

Now, we observe that the matrix description for $(\mathrm{U}_{3,\mathbb{F}_2})^{\mathbb{Z}/2\mathbb{Z}}$ provides a short exact sequence of group schemes

$$1 \to (\mathrm{U}_{3,\mathbb{F}_2})^{\mathbb{Z}/2\mathbb{Z}}_{\mathrm{red}} \to (\mathrm{U}_{3,\mathbb{F}_2})^{\mathbb{Z}/2\mathbb{Z}} \to \alpha_{2,\mathbb{F}_2} \to 1.$$

Since $\mathbf{K}$ contains $(\mathrm{U}_{3,\mathbb{F}_2})^{\mathbb{Z}/2\mathbb{Z}}_{\mathrm{red}}$, it follows that $\mathbf{K} \cap (\mathrm{U}_{3,\mathbb{F}_2})^{\mathbb{Z}/2\mathbb{Z}}$ is either $(\mathrm{U}_{3,\mathbb{F}_2})^{\mathbb{Z}/2\mathbb{Z}}$ or $(\mathrm{U}_{3,\mathbb{F}_2})^{\mathbb{Z}/2\mathbb{Z}}_{\mathrm{red}}$. Since $\mathbf{K}(\mathbb{F}_2)$ contains the image of the element n considered above, we have

$$\mathbf{K} \cap (\mathrm{U}^-_{3,\mathbb{F}_2})^{\mathbb{Z}/2\mathbb{Z}} = \begin{cases} (\mathrm{U}^-_{3,\mathbb{F}_2})^{\mathbb{Z}/2\mathbb{Z}} & \text{if } \mathbf{K} \cap (\mathrm{U}_{3,\mathbb{F}_2})^{\mathbb{Z}/2\mathbb{Z}} = (\mathrm{U}_{3,\mathbb{F}_2})^{\mathbb{Z}/2\mathbb{Z}}; \\ (\mathrm{U}^-_{3,\mathbb{F}_2})^{\mathbb{Z}/2\mathbb{Z}}_{\mathrm{red}} & \text{if } \mathbf{K} \cap (\mathrm{U}_{3,\mathbb{F}_2})^{\mathbb{Z}/2\mathbb{Z}} = (\mathrm{U}_{3,\mathbb{F}_2})^{\mathbb{Z}/2\mathbb{Z}}_{\mathrm{red}}. \end{cases}$$

As a conclusion, we have either

$$\mathbf{K} \cap (\mathrm{C}_{3,\mathbb{F}_2})^{\mathbb{Z}/2\mathbb{Z}} = (\mathrm{C}_{3,\mathbb{F}_2})^{\mathbb{Z}/2\mathbb{Z}} \quad \text{or} \quad \mathbf{K} \cap (\mathrm{C}_{3,\mathbb{F}_2})^{\mathbb{Z}/2\mathbb{Z}} = (\mathrm{C}_{3,\mathbb{F}_2})^{\mathbb{Z}/2\mathbb{Z}}_{\mathrm{red}}.$$

Using once again the fact that $\mathbf{K}(\mathbb{F}_2)$ contains the image of n, and the description of the open cover of $(\mathrm{SL}_{3,\mathbb{Z}})^{\mathbb{Z}/2\mathbb{Z}}$ considered above, in the first case we deduce that $\mathbf{K} = (\mathrm{SL}_{3,\mathbb{F}_2})^{\mathbb{Z}/2\mathbb{Z}}$, and in the second case that $\mathbf{K} = (\mathrm{SL}_{3,\mathbb{F}_2})^{\mathbb{Z}/2\mathbb{Z}}_{\mathrm{red}}$. □

The second lemma uses the notion of dilatation (or affine blow-up) from [MRR20]. We will apply this construction to the scheme $(\mathrm{SL}_{3,\mathbb{Z}_2})^{\mathbb{Z}/2\mathbb{Z}}$, the principal subscheme $(\mathrm{SL}_{3,\mathbb{F}_2})^{\mathbb{Z}/2\mathbb{Z}}$, and either the closed subscheme $(\mathrm{SL}_{3,\mathbb{F}_2})^{\mathbb{Z}/2\mathbb{Z}}_{\mathrm{red}} \subset (\mathrm{SL}_{3,\mathbb{F}_2})^{\mathbb{Z}/2\mathbb{Z}}$ or the closed subscheme $(\mathrm{T}_{3,\mathbb{F}_2})^{\mathbb{Z}/2\mathbb{Z}} \subset (\mathrm{SL}_{3,\mathbb{F}_2})^{\mathbb{Z}/2\mathbb{Z}}$. The construction of [MRR20] provides two affine schemes

$$\mathrm{Bl}^{(\mathrm{SL}_{3,\mathbb{F}_2})^{\mathbb{Z}/2\mathbb{Z}}}_{(\mathrm{SL}_{3,\mathbb{F}_2})^{\mathbb{Z}/2\mathbb{Z}}_{\mathrm{red}}}\Big((\mathrm{SL}_{3,\mathbb{Z}_2})^{\mathbb{Z}/2\mathbb{Z}}\Big) \quad \text{and} \quad \mathrm{Bl}^{(\mathrm{SL}_{3,\mathbb{F}_2})^{\mathbb{Z}/2\mathbb{Z}}}_{(\mathrm{T}_{3,\mathbb{F}_2})^{\mathbb{Z}/2\mathbb{Z}}}\Big((\mathrm{SL}_{3,\mathbb{Z}_2})^{\mathbb{Z}/2\mathbb{Z}}\Big)$$

endowed with canonical morphisms to $(\mathrm{SL}_{3,\mathbb{Z}_2})^{\mathbb{Z}/2\mathbb{Z}}$. Moreover, by the universal property of dilatations (see [MRR20, Proposition 2.6]) there exists a canonical morphism

$$\mathrm{Bl}^{(\mathrm{SL}_{3,\mathbb{F}_2})^{\mathbb{Z}/2\mathbb{Z}}}_{(\mathrm{T}_{3,\mathbb{F}_2})^{\mathbb{Z}/2\mathbb{Z}}}\Big((\mathrm{SL}_{3,\mathbb{Z}_2})^{\mathbb{Z}/2\mathbb{Z}}\Big) \to \mathrm{Bl}^{(\mathrm{SL}_{3,\mathbb{F}_2})^{\mathbb{Z}/2\mathbb{Z}}}_{(\mathrm{SL}_{3,\mathbb{F}_2})^{\mathbb{Z}/2\mathbb{Z}}_{\mathrm{red}}}\Big((\mathrm{SL}_{3,\mathbb{Z}_2})^{\mathbb{Z}/2\mathbb{Z}}\Big) \tag{6.5}$$

over $(\mathrm{SL}_{3,\mathbb{Z}_2})^{\mathbb{Z}/2\mathbb{Z}}$.

Lemma 6.11. *The morphism (6.5) restricts to an isomorphism over* $(\mathrm{C}_{3,\mathbb{Z}_2})^{\mathbb{Z}/2\mathbb{Z}}$.

Proof. By compatibility of dilatations with flat base change (see [MRR20, Lemma 2.7]), we have

$$\mathrm{Bl}^{(\mathrm{SL}_{3,\mathbb{F}_2})^{\mathbb{Z}/2\mathbb{Z}}}_{(\mathrm{SL}_{3,\mathbb{F}_2})^{\mathbb{Z}/2\mathbb{Z}}_{\mathrm{red}}}\Big((\mathrm{SL}_{3,\mathbb{Z}_2})^{\mathbb{Z}/2\mathbb{Z}}\Big) \times_{(\mathrm{SL}_{3,\mathbb{Z}_2})^{\mathbb{Z}/2\mathbb{Z}}} (\mathrm{C}_{3,\mathbb{Z}_2})^{\mathbb{Z}/2\mathbb{Z}} \cong \mathrm{Bl}^{(\mathrm{C}_{3,\mathbb{F}_2})^{\mathbb{Z}/2\mathbb{Z}}}_{(\mathrm{C}_{3,\mathbb{F}_2})^{\mathbb{Z}/2\mathbb{Z}}_{\mathrm{red}}}\Big((\mathrm{C}_{3,\mathbb{Z}_2})^{\mathbb{Z}/2\mathbb{Z}}\Big),$$

with

$$\mathrm{Bl}^{(\mathrm{C}_{3,\mathbb{F}_2})^{\mathbb{Z}/2\mathbb{Z}}}_{(\mathrm{C}_{3,\mathbb{F}_2})^{\mathbb{Z}/2\mathbb{Z}}_{\mathrm{red}}}\Big((\mathrm{C}_{3,\mathbb{Z}_2})^{\mathbb{Z}/2\mathbb{Z}}\Big) = \mathrm{Bl}^{\mathrm{Spec}(\mathbb{F}_2[x,y,z^{\pm1},x',y']/(x^2,(x')^2))}_{\mathrm{Spec}(\mathbb{F}_2[y,z^{\pm1},y'])}\Big(\mathrm{Spec}(\mathbb{Z}_2[x,y,z^{\pm1},x',y']/(x^2-2y,(x')^2-2y'))\Big).$$

Now one checks that the right-hand side identifies with

$$\mathrm{Spec}(\mathbb{Z}_2[u,y,z^{\pm1},u',y']/(2u^2-y,2(u')^2-y')),$$

with the morphism to $\mathrm{Spec}(\mathbb{Z}_2[x,y,z^{\pm1},x',y']/(x^2-2y,(x')^2-2y'))$ corresponding to the ring homomorphism

$$\mathbb{Z}_2[x,y,z^{\pm1},x',y']/(x^2-2y,(x')^2-2y') \to \mathbb{Z}_2[u,y,z^{\pm1},u',y']/(2u^2-y,2(u')^2-y')$$

defined by $x \mapsto 2u$ and $x' \mapsto 2u'$. (In fact the given scheme satisfies the universal property of [MRR20, Proposition 2.6].) This description shows that the restriction of the canonical morphism

$$\mathrm{Bl}^{(\mathrm{C}_{3,\mathbb{F}_2})^{\mathbb{Z}/2\mathbb{Z}}}_{(\mathrm{C}_{3,\mathbb{F}_2})^{\mathbb{Z}/2\mathbb{Z}}_{\mathrm{red}}}\Big((\mathrm{C}_{3,\mathbb{Z}_2})^{\mathbb{Z}/2\mathbb{Z}}\Big) \to (\mathrm{C}_{3,\mathbb{Z}_2})^{\mathbb{Z}/2\mathbb{Z}}$$

to $\mathrm{Spec}(\mathbb{F}_2)$ factors through $\mathrm{Spec}(\mathbb{F}_2[z^{\pm1}])$ since x, x', y, y' get sent to 0; by the universal property of dilatations (and again compatibility of dilatations with flat base change) we deduce a canonical morphism

$$\mathrm{Bl}^{(\mathrm{SL}_{3,\mathbb{F}_2})^{\mathbb{Z}/2\mathbb{Z}}}_{(\mathrm{SL}_{3,\mathbb{F}_2})^{\mathbb{Z}/2\mathbb{Z}}_{\mathrm{red}}}\Big((\mathrm{SL}_{3,\mathbb{Z}_2})^{\mathbb{Z}/2\mathbb{Z}}\Big) \times_{(\mathrm{SL}_{3,\mathbb{Z}_2})^{\mathbb{Z}/2\mathbb{Z}}} (\mathrm{C}_{3,\mathbb{Z}_2})^{\mathbb{Z}/2\mathbb{Z}} \to \mathrm{Bl}^{(\mathrm{SL}_{3,\mathbb{F}_2})^{\mathbb{Z}/2\mathbb{Z}}}_{(\mathrm{T}_{3,\mathbb{F}_2})^{\mathbb{Z}/2\mathbb{Z}}}\Big((\mathrm{SL}_{3,\mathbb{Z}_2})^{\mathbb{Z}/2\mathbb{Z}}\Big) \times_{(\mathrm{SL}_{3,\mathbb{Z}_2})^{\mathbb{Z}/2\mathbb{Z}}} (\mathrm{C}_{3,\mathbb{Z}_2})^{\mathbb{Z}/2\mathbb{Z}}.$$

We claim that this morphism is an inverse to the restriction of (6.5). Indeed, both are flat affine schemes over $\mathrm{Spec}(\mathbb{Z}_2)$ with generic fiber equal to $(\mathrm{SL}_{3,\mathbb{Q}_2})^{\mathbb{Z}/2\mathbb{Z}}$, so to prove the claim it is enough to see that their global sections inside those of $(\mathrm{SL}_{3,\mathbb{Q}_2})^{\mathbb{Z}/2\mathbb{Z}}$ coincide. But each containment is implied by the existence of a map of spectra in the opposite direction. □

Proof of Proposition 6.9. Let π be as in the statement, and assume that the schematic image of $\pi_{|\mathrm{Spec}(\mathbb{F}_2)}$ is not $(\mathrm{SL}_{3,\mathbb{F}_2})^{\mathbb{Z}/2\mathbb{Z}}$. By Lemma 6.10, we conclude that the schematic image of this morphism is $(\mathrm{SL}_{3,\mathbb{F}_2})^{\mathbb{Z}/2\mathbb{Z}}_{\mathrm{red}}$. By the universal property of dilatations (see [MRR20, Proposition 2.6]), we then have a unique factorization

$$\pi : \mathbf{H} \to \mathrm{Bl}^{(\mathrm{SL}_{3,\mathbb{F}_2})^{\mathbb{Z}/2\mathbb{Z}}}_{(\mathrm{SL}_{3,\mathbb{F}_2})^{\mathbb{Z}/2\mathbb{Z}}_{\mathrm{red}}}\Big((\mathrm{SL}_{3,\mathbb{Z}_2})^{\mathbb{Z}/2\mathbb{Z}}\Big) \to (\mathrm{SL}_{3,\mathbb{Z}_2})^{\mathbb{Z}/2\mathbb{Z}}.$$

Using Lemma 6.11, we deduce that the restriction of $\pi_{|\mathrm{Spec}(\mathbb{F}_2)}$ to $(\mathrm{C}_{3,\mathbb{F}_2})^{\mathbb{Z}/2\mathbb{Z}}$ factors through $(\mathrm{T}_{3,\mathbb{F}_2})^{\mathbb{Z}/2\mathbb{Z}}$, yielding a contradiction. □

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GAITSGORY'S CENTRAL FUNCTOR AND THE ARKHIPOV–BEZRUKAVNIKOV EQUIVALENCE IN MIXED CHARACTERISTIC

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Abstract. We show that the nearby cycles functor for the p-adic Hecke stack at parahoric level is perverse t-exact, by developing a theory of Wakimoto filtrations at Iwahori level, and that it lifts to the $\mathbb{E}_1$-center. We apply these tools to construct the Arkhipov–Bezrukavnikov functor for p-adic affine flag varieties at Iwahori level, and prove that it is an equivalence for all classical groups and also exceptional groups of type E_6 and E_7.

Contents

1. Introduction

Let F be a p-adic field with ring of integers O and residue field k, and let G be a connected reductive F-group with a parahoric O-model $\mathcal{G}$. The first goal of this paper is to define a p-adic analogue of the Gaitsgory central functor [Gai01] sending perverse sheaves on the Hecke stack $\mathrm{Hk}_{G,C}$ over the completed algebraic closure C of F to central perverse sheaves on the Hecke stack $\mathrm{Hk}_{\mathcal{G},\bar{k}}$. The second goal of the paper is, when $\mathcal{G} = \mathcal{I}$ is Iwahori, to also define a p-adic analogue of the Arkhipov–Bezrukavnikov functor [AB09] relating coherent sheaves on the dual Springer variety $\hat{\mathcal{N}}_{\mathrm{Spr}}$ to constructible étale sheaves on $\mathrm{Hk}_{\mathcal{I},\bar{k}}$. During the introduction, we will assume for simplicity that G is split and that the coefficients of our sheaves equal $\bar{\mathbb{Q}}_\ell$, as these hypothesis hold for most of the paper. We begin by recalling some of the representation-theoretic aspects at the level of Grothendieck groups.

1.1. **Hecke algebras.** We assume G is a pinned split connected reductive F-group, i.e. equipped with a choice of Borel subgroup B, a maximal split torus $T \subset B \subset G$, and pinning isomorphisms for the root groups attached to positive simple roots with respect to B. The corresponding

Iwahori–Weyl group $W = N(F)/T(O)$, where N is the normalizer of $T \subset G$, admits a length function ℓ which makes (W, ℓ) into a quasi-Coxeter group.

Let $\mathbb{H}$ denote the affine Hecke algebra. Recall that the Iwahori-Matsumoto presentation, see [IM65], defines $\mathbb{H}$ as the $\mathbb{Z}[q^{\pm 1/2}]$-algebra generated by a basis $\{T_w | w \in W\}$ modulo relations $T_w T_{w'} = T_{ww'}$ if $\ell(w) + \ell(w') = \ell(ww')$, and $(T_s + q^{-1/2})(T_s - q^{1/2}) = 0$ for all length 1 elements s. Let $I \subset G(F)$ be the Iwahori subgroup associated with B. Then, the Iwahori-Hecke algebra $\mathcal{H} := C_c(I\backslash G(F)/I)$ is the space of compactly supported smooth functions on $I\backslash G(F)/I$. Fixing a Haar measure on $G(F)$ so that I has measure 1, convolution of functions equips $\mathcal{H}$ with the structure of a unital associative algebra. The affine Hecke algebra specializes to the Iwahori-Hecke algebra via the isomorphism

$$\mathbb{H} \otimes_{\mathbb{Z}[q^{\pm 1}]} \bar{\mathbb{Q}}_\ell \simeq \mathcal{H},$$

where $\bar{\mathbb{Q}}_\ell$ is regarded as a $\mathbb{Z}[q^{\pm 1/2}]$-algebra by mapping q to the cardinality of k (and thus choosing a square root of this integer in $\bar{\mathbb{Q}}_\ell$). Let $\mathbb{H}_{\mathrm{f}} \subset \mathbb{H}$ be the finite Hecke algebra associated with the finite Weyl group and the set of simple reflections. The *antispherical module* is defined as

$$\mathbb{M}^{\mathrm{as}} := \mathbb{Z}[q^{\pm 1/2}]^{\mathrm{sgn}} \otimes_{\mathbb{H}_{\mathrm{f}}} \mathbb{H},$$

where $T_w \in \mathbb{H}_{\mathrm{f}}$ acts by multiplication by $(-1)^{\ell(w)} q^{1/2}$ on $\mathbb{Z}[q^{\pm 1/2}]^{\mathrm{sgn}}$. We also define the anti-spherical module for $\mathcal{H}$ to be $\mathcal{M}^{\mathrm{as}} := \mathbb{M} \otimes_{\mathbb{Z}[q^{\pm 1/2}]} \bar{\mathbb{Q}}_\ell$.

According to Grothendieck's sheaf-function dictionary, the space of functions on the set of $\mathbb{F}_q$-points of a scheme has the category of complexes of coherent or constructible sheaves as its geometric counterpart. Let $\hat{G}$ denote the Langlands dual group of G over $\bar{\mathbb{Q}}_\ell$, and $\hat{U} \subset \hat{B} \subset \hat{G}$ be a Borel subgroup and its unipotent radical. Recall the Springer resolution of the nilpotent cone $\hat{\mathcal{N}} \subset \mathrm{Lie}(\hat{G})$

$$p_{\mathrm{Spr}} : \hat{\mathcal{N}}_{\mathrm{Spr}} = \hat{G} \times^{\hat{B}} \mathrm{Lie}\, \hat{U} \to \hat{\mathcal{N}}.$$

The Steinberg variety is defined as $\hat{\mathrm{St}} := \hat{\mathcal{N}}_{\mathrm{Spr}} \times_{\hat{\mathcal{N}}} \hat{\mathcal{N}}_{\mathrm{Spr}}$. Kazhdan-Lusztig [KL87] showed that the affine Hecke algebra is isomorphic to the Grothendieck group $K_0([\hat{G} \times \mathbb{G}_m \backslash \hat{\mathrm{St}}])$, where the latter has an algebra structure induced by convolution. In particular, the antispherical module $\mathbb{M}^{\mathrm{as}}$ is identified with $K_0([\hat{G} \times \mathbb{G}_m \backslash \hat{\mathcal{N}}_{\mathrm{Spr}}])$. If we forget the $\mathbb{G}_m$-equivariance, then we recover both the Iwahori-Hecke algebra $\mathcal{H}$ and its anti-spherical module $\mathcal{M}^{\mathrm{as}}$.

On the other hand, it follows from the work of Iwahori–Matsumoto [IM65] that the Iwahori-Hecke algebra coincides with $K_0(\mathcal{P}(\mathrm{Hk}_{\mathcal{I}}))$ where $\mathcal{P}$ denotes the category of perverse sheaves on the Hecke stack $\mathrm{Hk}_{\mathcal{I}} = L^+\mathcal{I}\backslash LG/L^+\mathcal{I}$. The natural action of $\mathcal{H}$ on $\mathcal{M}^{\mathrm{as}}$ induces a surjective map $\mathcal{H} \to \mathcal{M}^{\mathrm{as}}$ with kernel generated by the Kazhdan-Lusztig basis elements indexed by the $w \in W$, which are not minimal in their left W_{fin}-coset. This leads us to consider the *antispherical category* of perverse sheaves $\mathcal{P}_{\mathrm{as}}(\mathrm{Hk}_{\mathcal{I}})$ given as the Serre quotient by the IC-sheaves index by those w. Another approach to realize $\mathcal{M}^{\mathrm{as}}$ is as the I-invariants of the compact induction to $G(F)$ of a generic character χ of the unipotent radical I_u^{op} of the opposite Iwahori subgroup. This is the so-called Iwahori–Whittaker model and its categorification plays an important role later in our arguments.

We have observed that there is an abundance of spaces and sheaves that seem related to affine Hecke algebras and their anti-spherical modules. In the next sections, we will explain how to upgrade these isomorphisms to equivalences of stable ∞-categories. A guiding principle for this is the fact that there are certain objects which serve as building blocks for the various categories and we must track down where they get sent to. This is motivated by Bernstein's construction of translation elements θ_ν in the affine Hecke algebra whose trace along the finite Weyl orbits are central, and also of an isomorphism between the spherical Hecke algebra and the center of the Iwahori-Hecke algebra.

1.2. **The central functor.** The first goal of this paper is to fully develop the central functor $\mathcal{Z}$ for p-adic groups, in analogy with Gaitsgory's central functor from [Gai01] in the function field case. The correct geometric setup for this construction is naturally the world of diamonds or more generally v-sheaves on perfectoids of characteristic p. Indeed, for any parahoric O-model $\mathcal{G}$, we have a Hecke stack $\mathrm{Hk}_{\mathcal{G}}$ defined over $\mathrm{Spd}(O)$ and we can define a nearby cycles functor $R\Psi$ by pull-push along the geometric fibers over a complete algebraic closure C of F and the residue field k. Indeed, this was already partially exploited in [AGLR22] to define certain complexes of sheaves $\mathcal{Z}(V) = R\Psi(\mathrm{Sat}(V))$, where Sat denotes the geometric Satake equivalence of Fargues–Scholze [FS21]. It was proved in [AGLR22] that these complexes are algebraic and constructible, that they carry certain centrality isomorphisms, and finally that they are supported at the V-admissible locus $\mathcal{A}_{\mathcal{G},V}$.

Two very important properties of the functor $\mathcal{Z}$ remained however elusive in [AGLR22], namely verifying that the centrality isomorphisms of $\mathcal{Z}$ satisfy various expected compatibilities that make it into a central functor, and that it lands in the category of perverse sheaves. Our most important results in this direction can be resumed as follows:

Theorem 1.1 (Theorems 4.15 and 4.17, Corollaries 4.20 and 4.21)**.** *The functor $\mathcal{Z}\colon \mathrm{Rep}(\hat{G}) \to \mathcal{D}_{\mathrm{ula}}(\mathrm{Hk}_{\mathcal{G},k})$ lifts to an $\mathbb{E}_2$-monoidal functor towards the $\mathbb{E}_1$-center of the right side. Moreover, each $\mathcal{Z}(V)$ is perverse, and has unipotent monodromy. If $\mathcal{G} = \mathcal{I}$ is Iwahori, then $\mathcal{Z}(V)$ is convolution-exact and admits a Wakimoto filtration whose associated graded equals $\mathcal{J}(V_{|\hat{T}})$.*

Let us explain a bit of the notions and ingredients that go into the above theorem. Our treatment of the centrality of $\mathcal{Z}$ is to our best knowledge the only one that uses the machinery of stable ∞-categories, which entails additional higher coherent homotopies. An important technical tool is the notion of an abstract six-functor formalism in the sense of Mann [Man22a, Man22b], which allows us to work at the level of the category of correspondences. Once we are there, we perform the usual fusion trick of looking at the disjoint locus of $(\mathrm{Spd}(O_C))^2$ and conclude the desired monoidality via full faithfulness of pullback away from the diagonal for those sheaves which are perverse over $(\mathrm{Spd}C)^2$. To obtain this full faithfulness, we apply a certain calculation of nearby cycles of kimberlites from [GL24].

Trying to prove perversity of $\mathcal{Z}(V)$ was the genesis of this project. Contrary to the function field case, we cannot rely on Artin vanishing to provide us with this crucial fact. Instead, we first consider a Iwahori $\mathcal{I}$ and look at the Wakimoto functor $\mathcal{J}\colon \mathrm{Rep}(\hat{T}) \to \mathcal{P}(\mathrm{Hk}_{\mathcal{I}})$ following [AB09], but defined instead at the level of complexes as in [AR]. The centrality of $\mathcal{Z}(V)$ implies that it lies in the full subcategory generated by the essential image of $\mathcal{J}$ under extensions. Each graded piece can then be recovered by invoking geometric Satake and a certain orthogonality with respect to the constant terms $\mathrm{CT}_{B^{\mathrm{op}}}$, which proves perversity and the existence of a Wakimoto filtration all at once. This differs considerably from the strategy in [AB09], which exploits both perversity and convolution-exactness (known in equicharacteristic by Artin vanishing). Perversity in the general parahoric case can be deduced from the Iwahori one, based on a suggestion of Achar, which we learned from Cass–Scholbach–van den Hoven [CvdHS24], who adapted our argument for Iwahori models to their setting. From the Wakimoto filtration, we can also deduce the convolution-exactness and unipotency of the monodromy operator induced by the Galois group (note that $\mathrm{Sat}(V)$ descends to $\mathrm{Spd}(F)$ with trivial inertia action). In the meantime, the Wakimoto filtration has been decisively used in [GL24] to give a new proof of unibranchness (i.e., topological normality) of local models in complete generality.

1.3. **The AB functor and Iwahori–Whittaker sheaves.** The next part of our paper carries out the construction of the various functors from [AB09] and, except for treating ∞-categorical questions carefully, does not significantly diverge from it. Recall that we work here with the

Springer stack $[\hat{G}\backslash\hat{\mathcal{N}}_{\mathrm{Spr}}]$ that resolves the corresponding nilpotent stack of $\hat{G}$. We have the following result:

Theorem 1.2 (Proposition 5.7)**.** *There is a monoidal functor* $\mathcal{F}\colon \mathrm{Perf}([\hat{G}\backslash\hat{\mathcal{N}}_{\mathrm{Spr}}]) \to \mathcal{D}_{\mathrm{cons}}(\mathrm{Hk}_{\mathcal{I}})$ *extending* $\mathcal{Z} \times \mathcal{J}$*.*

Let us briefly describe the construction of $\mathcal{F}$ following [AB09]. The main idea consists in defining an analogous functor starting from the quotient stack $[\hat{G} \times \hat{T}\backslash\hat{\mathcal{N}}^{\mathrm{af}}_{\mathrm{Spr}}]$ containing $[\hat{G}\backslash\hat{\mathcal{N}}_{\mathrm{Spr}}]$ as a dense open stack, where $\hat{\mathcal{N}}^{\mathrm{af}}_{\mathrm{Spr}}$ is the affine hull of the canonical $\hat{T}$-torsor over $\hat{\mathcal{N}}_{\mathrm{Spr}}$. The projective objects of the category of coherent sheaves on the enlarged stack can be mapped to $\mathcal{P}(\mathrm{Hk}_{\mathcal{I}})$ by using the functors $\mathcal{Z}$, $\mathcal{J}$, and the nilpotent monodromy endomorphism of the former, after verifying the Plücker relations. After deriving this functor to perfect complexes on the affine stack, we are reduced to showing vanishing on complexes supported at the complement of $[\hat{G}\backslash\hat{\mathcal{N}}_{\mathrm{Spr}}]$.

Next, we study the category $\mathcal{D}_{\mathrm{cons}}(\mathrm{Hk}_{\mathcal{IW}})$ of Iwahori–Whittaker sheaves. This is the stable ∞-category of $(L^+\mathcal{I}^{\mathrm{op}}_{\mathrm{u}}, \mathcal{L})$-twisted equivariant constructible sheaves on $\mathrm{Fl}_{\mathcal{I}}$, where $\mathcal{L}$ is a certain character sheaf obtained via the Artin–Schreier cover. It also carries a perverse t-structure and the category $\mathcal{P}(\mathrm{Hk}_{\mathcal{IW}})$ is a highest weight category in the sense of Beilinson–Ginzburg–Soergel [BGS96], with simple and tilting objects indexed by $\mathbb{X}_*(T)$. We get a perverse t-exact averaging functor $\mathrm{av}_{\mathcal{IW}}\colon \mathcal{D}_{\mathrm{cons}}(\mathrm{Hk}_{\mathcal{I}}) \to \mathcal{D}_{\mathrm{cons}}(\mathrm{Hk}_{\mathcal{IW}})$ given by left convolution against the simple object attached to $0 \in \mathbb{X}_*(T)$ and we denote by $\mathcal{Z}_{\mathcal{IW}}$, resp. $\mathcal{F}_{\mathcal{IW}}$, the composition $\mathrm{av}_{\mathcal{IW}} \circ \mathcal{Z}$, resp. $\mathrm{av}_{\mathcal{IW}} \circ \mathcal{F}$.

Theorem 1.3 (Theorem 7.9)**.** *If G has enough minuscules, then $\mathcal{Z}_{\mathcal{IW}}(V)$ is tilting.*

Notice that in [AB09] there is no assumption on G. Unfortunately, we are missing a crucial ingredient replacing the $\mathbb{G}_{m,k}$-action given by rotating the uniformizer, which is impossible in the p-adic setting. Contrary to what was asserted in an initial version of this paper, the impact of this gap is very mild, and we get the result for all groups with enough minuscules, see Definition 7.8, a class comprised of all classical groups, and also exceptional ones of types E_6 and E_7. This oversight on our side was brought to our attention by the work of Dhillon–Taylor [DT25].

Let us explain how the proof works, so the reader can better grasp the gap above. First of all, the tilting property propagates under convolution and can be verified on adjoint quotients, so we may assume V is either minuscule or quasi-minuscule by a lemma of Ngô–Polo [NP01]. In the minuscule case, all the weights are comprised in a finite Weyl orbit, so one can easily verify the given property. In the quasi-minuscule case, we must still handle the (co)restriction of $\mathcal{Z}_{\mathcal{IW}}(V)$ to the weight 0 Iwahori–Whittaker cell. Here, the vanishing can be achieved by calculating the alternating sum of the Ext groups via an argument on Grothendieck groups and finally bound a $\mathrm{Hom}(\mathcal{Z}_{\mathcal{IW}}(1), \mathcal{Z}_{\mathcal{IW}}(V))$ accordingly. Now, this bound is achieved in [AB09] via the theory of the regular quotient described below together with the fact that the monodromy operator is defined for every sheaf in $\mathcal{P}(\mathrm{Hk}_{\mathcal{I}})$ as it is induced by the $\mathbb{G}_{m,k}$-action given by rotation. For groups with enough minuscules, we can realise every representation up to central isogeny as a direct summand of a tensor product of minuscule representations, so we still get the tilting property.

The last step in proving that $\mathcal{F}_{\mathcal{IW}}$ is an equivalence (now, necessarily assuming enough minuscules) revolves around the regular orbit $\mathcal{O}_{\mathrm{r}} \subset \hat{\mathcal{N}}$ inside the nilpotent cone. The Springer resolution is an isomorphism above this $\hat{G}$-orbit, and hence we should find a category of étale sheaves that plays a similar role. For this, we look at the Serre quotient $\mathcal{P}_0(\mathrm{Hk}_{\mathcal{I}})$ of perverse sheaves on the Hecke stack obtained by modding out IC sheaves with positive-dimensional support.

Theorem 1.4 (Proposition 8.3, Proposition 10.8)**.** *If G has enough minuscules, then there is a symmetric monoidal equivalence $\mathcal{P}_0(\mathrm{Hk}_{\mathcal{I}}) \xrightarrow{\sim} \mathrm{Rep}(Z_{\hat{G}}(n_0))$, where n_0 is a regular nilpotent element.*

Together with the tilting property, this result is key in order to prove the Arkhipov–Bezrukavnikov equivalence, as it induces certain injections of Hom maps. Let us remark that the most delicate point in the above theorem consists in showing that n_0 is regular. For this, we use the theory of weights by descending $\mathcal{Z}(V)$ to a mixed sheaf and calculating its monodromy filtration. In [AB09], one applies Gabber's local weight-monodromy theorem, see [BB93], stating that the weight filtration equals the monodromy filtration, and then calculates the former via the affine Hecke algebra. This is not available for our nearby cycles, unless μ is minuscule, by work of Hansen–Zavyalov [HZ23] combined with the representability theorem in [AGLR22]. Again, we can only easily reproduce this argument for groups with enough minuscules. It would be possible to adapt a different argument due to Bezrukavnikov–Riche–Rider [BRR20], but this would lead us into some detours that seem unnecessary, as we do not have the tilting property for other groups.

Let us finish by stating the second main result of this paper, i.e. the AB equivalence for p-adic groups with enough minuscules:

Theorem 1.5 (Theorem 9.1)**.** *If G has enough minuscules, then the functor $\mathcal{F}_{\mathcal{IW}} \colon \mathrm{Perf}([\hat{G}\backslash\hat{\mathcal{N}}_{\mathrm{Spr}}]) \to \mathcal{D}_{\mathrm{cons}}(\mathrm{Hk}_{\mathcal{IW}})$ is an equivalence.*

We strongly believe that this result must also hold for general split connected reductive groups G. Yun–Zhu have announced in conference talks regarding work of preparation of Hemo–Zhu, see also [Zhu20], a proof of the full Bezrukavnikov equivalence [Bez16] for p-adic groups, that builds on a colimit presentation in terms of double quotients of parahoric jet groups due to Tao–Travkin [TT20]. Recently, Bando [Ban23] also gave a distinct proof of the Bezrukavnikov equivalence for p-adic groups by comparing constructible-étale sheaves in equi- and mixed characteristic via an ingenious geometric construction. However, these previous methods do not yield concrete knowledge about the central functor, whereas our paper places $\mathcal{Z}$ right at the center of it all. We also think that our functor $\mathcal{Z}$ will naturally appear in the picture if one studies étale sheaves on p-adic Hecke stacks, see, e.g., the unibranchness theorem of [GL24], and thus it must play a role in comparing the Zhu [Zhu20] and the Fargues–Scholze [FS21] variants of a categorical p-adic local Langlands correspondence. A natural task for the future will be to explain if and how all of the previous approaches fit together, namely by comparing a priori different central functors.

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1.5. **Notation.** Unfortunately, we will have to use a lot of notations. Thus, let's get over this and define the following objects, which will occur in the whole text.

First, let us discuss scheme-theoretic notations.

- p a prime,
- F a p-adic field with ring of integers O, and (perfect) residue field k.
- $\overline{F}$ an algebraic closure of F and $\Gamma := \mathrm{Gal}(\overline{F}/F)$ the absolute Galois group,
- $\breve{\Gamma} \subseteq \Gamma$ the inertia subgroup and $\Gamma^{\mathrm{un}} := \Gamma/\breve{\Gamma}$ the unramified quotient,
- $\breve{F}$ the completion of the maximal unramified extension of F in $\overline{F}$, $\breve{O} \subseteq \breve{F}$ its ring of integers, and $\overline{k}$ the residue field of $\breve{O}$,
- G a quasi-split reductive group over F,
- $S \subseteq T \subseteq B \subseteq G$ a maximal split torus $S \subseteq G$, T its centralizer (a maximal torus in G as G is quasi-split), and a Borel $B \subseteq G$ containing T,
- $B^- \subseteq G$ is the opposite Borel of B,
- $N := N_G(T)$ denotes the normalizer of T in G,
- $\mathcal{T}$ the connected Néron model of T over O,
- If H/F is a torus, then $\mathbb{X}_\bullet(H)$, resp. $\mathbb{X}^\bullet(H)$ denote the groups of (geometric) cocharacters, resp. characters of H,
- $\mathbb{X}_\bullet := \mathbb{X}_\bullet(T)$, $\mathbb{X}^\bullet := \mathbb{X}_\bullet(T)$,
- $\bar{\mathbb{X}}_\bullet := \mathbb{X}_{\bullet,\breve{\Gamma}}$, where the subscript $\breve{\Gamma}$ denotes the coinvariants.
- $\mathbb{X}_\bullet^+$, $\mathbb{X}^{\bullet,+}$ denote the dominant cocharacters resp. dominant characters of T with respect to B,
- $\mathbb{X}_\bullet(S)^+$, $\mathbb{X}^{\bullet,+}(S)$ denote the dominant cocharacters resp. dominant characters of S with respect to B,

Next, let us introduce combinatorial notations.

- $W := N(\breve{F})/\mathcal{T}(\breve{O})$ the Iwahori-Weyl group of T, also called extended affine Weyl group,
- $\mathcal{A}(G,S)$ the appartment associated with S, identified with $\mathbb{X}_\bullet(S)_\mathbb{R}$ for pinned G,
- $\mathbf{a} \subseteq \mathcal{A}(G,S)$ a fixed alcove,
- $\mathbf{f} \subseteq \mathcal{A}(G,S)$ a facet contained in the closure of $\mathbf{a}$,
- $\mathbb{S} \subseteq W$ the set of reflections at the walls of $\mathbf{a}$, also called the set of simple reflections,
- $W_{\mathrm{af}} \subseteq W$ the affine Weyl group, which is the Coxeter group generated by the simple reflections,
- $\Omega_\mathbf{a}$ the stabilizer of $\mathbf{a}$, which yields an isomorphism
$$W \cong W_{\mathrm{af}} \rtimes \Omega_\mathbf{a}. \tag{1.1}$$
- $\ell\colon W \to \mathbb{N}_{\geq 0}$ the length function on W, i.e., the unique function $\ell(-)\colon W \to \mathbb{N}_{\geq 0}$, which extends the length function $\ell(-)\colon W_{\mathrm{af}} \to \mathbb{N}_{\geq 0}$ on the Coxeter group W_{af}, such that $\ell(\tau) = 0$ for $\tau \in \Omega_\mathbf{a}$.
- $\leq$ is the Bruhat order on W, i.e., $w \leq w'$ for $w = (w_{\mathrm{af}}, \tau), w' = (w'_{\mathrm{af}}, \tau') \in W \cong W_{\mathrm{af}} \rtimes \pi_1(G)_{\breve{\Gamma}}$ if and only if $\tau = \tau'$ and $w_{\mathrm{af}} \leq w'_{\mathrm{af}}$ for the Bruhat order $\leq$ on W_{af} coming from its Coxeter structure,
- $W_{\mathrm{fin}} = N(\breve{F})/T(\breve{F})$ the finite Weyl group, which sits in an exact sequence
$$1 \to \bar{\mathbb{X}}_\bullet \to W \to W_{\mathrm{fin}} \to 1. \tag{1.2}$$
- $t_{\bar{\nu}} \in W$ is the translation element associated with $\bar{\nu} \in \bar{\mathbb{X}}_\bullet$.

- $w_{\bar{\nu}}$ denotes the minimal length element in the coset $W_{\mathrm{fin}} t_{\bar{\nu}}$ for $\bar{\nu} \in \bar{\mathbb{X}}_\bullet$, upon choosing an origin for $\mathcal{A}(G, S)$.

Now let us define notations related to affine flag varieties and perfect geometry.

- $\mathrm{Alg}_k^{\mathrm{perf}}$ the category of perfect k-algebras,
- for a scheme X over k we denote by X^{pf} its perfection,
- for $R \in \mathrm{Alg}_k^{\mathrm{perf}}$ we set $W_O(R) := O \otimes_{W(k)} W(-)$, the ring of O-Witt vectors,
- if X/O is an affine scheme of finite type, then $L^+X \colon (\mathrm{Alg}^{\mathrm{perf}}) \to (\mathrm{Sets}),\ R \mapsto X(W_O(R))$ is the positive loop functor for X,
- if Z/F is an affine scheme of finite type, then $LZ \colon (\mathrm{Alg}^{\mathrm{perf}}) \to (\mathrm{Sets}),\ R \mapsto Z(W_O(R)[1/p])$ is the loop functor for Z,
- $\mathcal{I}/O$ the Iwahori group scheme for G associated with the alcove $\mathbf{a}$,
- $\mathcal{G}/O$ the parahoric model of G associated with the facet $\mathbf{f}$
- the quotient of étale sheaves $\mathrm{Fl}_{\mathcal{G}} := LG/L^+\mathcal{G}$ is the (partial) affine flag variety for $\mathcal{G}$,
- the quotient stack $\mathrm{Hk}_{\mathcal{G}} = [L^+\mathcal{G} \backslash \mathrm{Fl}_{\mathcal{G}}]$ in the étale topology is the Hecke stack for $\mathcal{G}$,
- the quotient stack $\mathrm{Hk}_{(\mathcal{I},\mathcal{G})} = [L^+\mathcal{I} \backslash \mathrm{Fl}_{\mathcal{G}}]$ in the étale topology is the Hecke stack for the pair $(\mathcal{I}, \mathcal{G})$.

Next, let us introduce some cohomological notations.

- $\ell \neq p$ a prime,
- Λ an algebraic extension of $\mathbb{F}_\ell$ or $\mathbb{Q}_\ell$.
- $\mathcal{D}_{\text{ét}}(-) := \mathcal{D}_{\text{ét}}(-, \Lambda)$ denotes the ∞-derived category of "étale sheaves of Λ-modules" on a perfect scheme, or a small v-stack.
- $\mathcal{D}_{\text{ét}}(\mathrm{Hk}_{\mathcal{G}}) := \mathcal{D}_{\text{ét}}(\mathrm{Hk}_{\mathcal{G}}, \Lambda)^{\mathrm{bd}}$ denotes the ∞-category of "étale sheaves of Λ-modules" on $\mathrm{Hk}_{\mathcal{G}}$, whose support is a finite subset of the underlying topological space of $\mathrm{Hk}_{\mathcal{G}}$.

Finally, we collect our notations for the "coherent" side. Note that we consider these objects usually under the assumption that Λ is a field extension of $\mathbb{Q}_\ell$.

- $\hat{G}$ denotes the dual group of G over Λ,
- $\hat{T} \subseteq \hat{G}$ denotes the dual torus to T, and we identify $\mathbb{X}^\bullet(\hat{T}) \cong \mathbb{X}_\bullet(T)$,
- $\hat{G}' := \hat{G} \times \hat{T}$,
- $\hat{T} \subseteq \hat{B} \subseteq \hat{G}$ denotes the Borel subgroup with dominant characters identifying with $\mathbb{X}_\bullet(T)^+$,
- $\hat{U} \subseteq \hat{B}$ is the unipotent radical of $\hat{B}$ with Lie algebra $\mathrm{Lie}(\hat{U})$,
- $\hat{\mathfrak{g}} := \mathrm{Lie}(\hat{G})$ denotes the Lie algebra of $\hat{G}$,
- $\hat{\mathcal{N}} \subseteq \hat{\mathfrak{g}}$ is the nilpotent cone, i.e., the closed subscheme of nilpotent elements,
- $p_{\mathrm{Spr}} \colon \hat{\mathcal{N}}_{\mathrm{Spr}} := \hat{G} \times^{\hat{B}} \mathrm{Lie}(\hat{U}) \to \hat{\mathcal{N}}$ denotes the Springer resolution of the nilpotent cone,
- $\hat{\mathcal{N}}_{\mathrm{Spr}}^{\mathrm{qaf}} := \hat{G} \times^{\hat{U}} \mathrm{Lie}(\hat{U}) \to \hat{\mathcal{N}}_{\mathrm{Spr}}$ denotes the canonical $\hat{T}$-torsor over $\hat{\mathcal{N}}_{\mathrm{Spr}}$,
- $\hat{\mathcal{X}} := \mathrm{Spec}(\mathcal{O}(\hat{G}/\hat{U}))$ is the affine closure of the quasi-affine scheme $\hat{G}/\hat{U}$,

2. Geometry of the affine flag variety

In this section, we want to recall the geometry of the (Witt vector) partial affine flag variety $\mathrm{Fl}_{\mathcal{G}}$, which was first considered as an algebraic space in [Zhu17, Section 1.4]. Its representability by an ind-(perfected projective k-scheme) was then proven in [BS17, Corollary 9.6] via reduction to $G = \mathrm{GL}_n$ and the construction of the determinant line bundle there. Let us note that the base change $\mathrm{Fl}_{\mathcal{G},\bar{k}}$ is the affine flag variety of the parahoric group $\mathcal{G} \otimes_O \breve{O}$. Hence, geometric questions for $\mathrm{Fl}_{\mathcal{G}}$ often reduce to the case $F = \breve{F}$. Our treatment will focus especially on $L^+\mathcal{I}$-equivariant subvarieties of $\mathrm{Fl}_{\mathcal{G}}$.

2.1. **Schubert varieties and convolution.** During the entire paper, we will assume that the group G is *residually split*. In fact, almost all of our arguments with sheaves take place when $F = \breve{F}$, except for a brief appearance of mixed sheaves, for which residual splitness is a lax enough assumption. This simplifies the Galois action on the Iwahori–Weyl group.

Lemma 2.1. *The following conditions are equivalent:*

(1) *The Γ-action on W is trivial.*
(2) *Γ acts trivially on $\bar{\mathbb{X}}_\bullet$.*
(3) *G is residually split, i.e., the reductive quotient $\mathcal{G}_k^{\mathrm{red}}$ of the special fiber of every parahoric O-model of G is split over k.*

Proof. By [KP23, Proposition 7.10.10] the group G is residually split if and only if Γ acts trivially on $\bar{\mathbb{X}}_\bullet$. If Γ acts trivially on W, then as well on $\bar{\mathbb{X}}_\bullet \subseteq W$. Assume now that Γ acts trivially on $\bar{\mathbb{X}}_\bullet$. As W is generated by W_{af} and $\bar{\mathbb{X}}_\bullet$ it suffices to show that Γ acts trivially on W_{af}. But W_{af} embeds Γ-equivariantly into the group of affine transformations on $\bar{\mathbb{X}}_{\bullet,\mathbb{R}}$, and the Γ-action on the latter is trivial. □

The geometry of the Iwahori orbits on the affine flag variety is summarized in the next lemma.

Lemma 2.2. (1) *The map $N(\breve{F}) \to \mathrm{Fl}_{\mathcal{G}},\ n \mapsto n \cdot L^+\mathcal{G}$ induces a bijection*

$$W/W_{\mathbf{f}} = \mathcal{T}(\breve{O})\backslash N(\breve{F})/(N(\breve{F}) \cap \mathcal{G}(\breve{O})) \to \mathrm{Hk}_{(\mathcal{I},\mathcal{G})} \tag{2.1}$$

on underlying topological spaces, i.e, the $L^+\mathcal{I}$-orbits $\mathrm{Fl}_{(\mathcal{I},\mathcal{G}),w} := L^+\mathcal{I} \cdot w \subset \mathrm{Fl}_{\mathcal{G}}$ are indexed by $W/W_{\mathbf{f}}$.

(2) *The $L^+\mathcal{I}$-orbits on $\mathrm{Fl}_{\mathcal{G}}$ form a stratification of $\mathrm{Fl}_{\mathcal{G}}$, i.e., the closure $\mathrm{Fl}_{(\mathcal{I},\mathcal{G}),\leq w}$ of a Schubert cell $\mathrm{Fl}_{(\mathcal{I},\mathcal{G}),w}$ is a union of Schubert cells. More precisely, it is the unique closed perfect subscheme such that*

$$|\mathrm{Fl}_{(\mathcal{I},\mathcal{G}),\leq w}| = \bigcup_{w' \leq w} |\mathrm{Fl}_{(\mathcal{I},\mathcal{G}),w'}| \tag{2.2}$$

for the Bruhat order on W.

Proof. Statement (1) is essentially the Bruhat decomposition, see [BT72, Théorème 6.5] or [KP23, Theorem 7.8.1],

$$W/W_{\mathbf{f}} \cong \mathcal{I}(\breve{O})\backslash G(\breve{F})/\mathcal{G}(\breve{O}) \tag{2.3}$$

(applied over all formally unramified extensions of $\breve{O}$). Since G is residually split, we conclude by étale descent that all points of $\mathrm{Hk}_{(\mathcal{I},\mathcal{G})}$ are k-rational and enumerated by $W/W_{\mathbf{f}}$. Then (2) follows by considering convolution and the Demazure resolutions, cf. [Zhu17, Section 1.4] for details. □

Definition 2.3. The (perfect) projective schemes $\mathrm{Fl}_{(\mathcal{I},\mathcal{G}),\leq w}$ are called Schubert varieties, while their open and dense subschemes $\mathrm{Fl}_{(\mathcal{I},\mathcal{G}),w}$ are called (Iwahori) Schubert cells.

If $n \geq 1$, the contracted product

$$\underbrace{\mathrm{Fl}_{\mathcal{G}} \tilde{\times} \cdots \tilde{\times} \mathrm{Fl}_{\mathcal{G}}}_{n-\text{factors}} := LG \times^{L^+\mathcal{G}} \cdots \times^{L^+\mathcal{G}} LG/L^+\mathcal{G} \tag{2.4}$$

is called the n-fold convolution product of $\mathrm{Fl}_{\mathcal{G}}$. The multiplication morphism

$$m := m_{\mathrm{Fl}_{\mathcal{G}}} \colon \mathrm{Fl}_{\mathcal{G}} \tilde{\times} \cdots \tilde{\times} \mathrm{Fl}_{\mathcal{G}} \to \mathrm{Fl}_{\mathcal{G}},\ \overline{(g_1, \ldots, g_n)} \mapsto g_1 \cdots g_n L^+\mathcal{G} \tag{2.5}$$

has interesting geometric properties. If $X_1, \ldots, X_n \subseteq \mathrm{Fl}_{\mathcal{G}}$ are (locally) closed $L^+\mathcal{G}$-stable subschemes and $Y_1, \ldots, Y_n \subseteq LG$ their preimages, then we set

$$X_1 \tilde{\times} \ldots \tilde{\times} X_n := Y_1 \times^{L^+\mathcal{G}} \ldots \times^{L^+\mathcal{G}} Y_n/L^+\mathcal{G}, \tag{2.6}$$

which is a (locally) closed subscheme of $\mathrm{Fl}_{\mathcal{G}}\tilde{\times}\dots\tilde{\times}\mathrm{Fl}_{\mathcal{G}}$. A similar discussion can be had with the convolution product $\mathrm{Fl}_{\mathcal{I}}\tilde{\times}\dots\tilde{\times}\mathrm{Fl}_{\mathcal{I}}\tilde{\times}\mathrm{Fl}_{\mathcal{G}}$, where the parahoric $\mathcal{G}$ appears only in the last factor and all the other intermediate terms are given by the Iwahori $\mathcal{I}$.

Lemma 2.4. *Let $w_1,\dots,w_n \in W$ and assume that $w_1\cdots w_n$ is right $W_{\mathbf{f}}$-minimal and reduced.*

(1) *The map*
$$\mathrm{Fl}_{\mathcal{I},w_1}\tilde{\times}\cdots\tilde{\times}\mathrm{Fl}_{\mathcal{I},w_n} \to \mathrm{Fl}_{\mathcal{G}} \tag{2.7}$$
has image in $\mathrm{Fl}_{(\mathcal{I},\mathcal{G}),w_1\cdots w_n}$ and induces an isomorphism
$$\mathrm{Fl}_{\mathcal{I},w_1}\tilde{\times}\cdots\tilde{\times}\mathrm{Fl}_{\mathcal{I},w_n} \cong \mathrm{Fl}_{(\mathcal{I},\mathcal{G}),w_1\cdots w_n}. \tag{2.8}$$
(2) *We have*
$$m(\mathrm{Fl}_{\mathcal{I},\le w_1}\tilde{\times}\cdots\tilde{\times}\mathrm{Fl}_{\mathcal{I},\le w_n}) \subseteq \mathrm{Fl}_{(\mathcal{I},\mathcal{G}),\le w_1\cdots w_n} \tag{2.9}$$
and the map
$$\mathrm{Fl}_{\mathcal{I},\le w_1}\tilde{\times}\cdots\tilde{\times}\mathrm{Fl}_{\mathcal{I},\le w_n} \to \mathrm{Fl}_{(\mathcal{I},\mathcal{G}),\le w_1\cdots w_n} \tag{2.10}$$
is (perfectly) proper and birational.

(3) *If $w \in W$ and $\tau \in \Omega_{\mathbf{a}}$, then*
$$\mathrm{Fl}_{(\mathcal{I},\mathcal{G}),w} \to \mathrm{Fl}_{(\mathcal{I},\mathcal{G}),\tau w},\ gL^+\mathcal{G} \mapsto \tau g L^+\mathcal{G} \tag{2.11}$$
is an isomorphism.

Proof. It suffices to check the statements in the case that $F = \breve{F}$. Using induction on n, one reduces the first statement to the case that $n = 2$. Now,
$$\mathcal{I}(\breve{O})w_1\mathcal{I}(\breve{O})\cdot\mathcal{I}(\breve{O})w_2\mathcal{G}(\breve{O}) \cong \mathcal{I}(\breve{O})w_1w_2\mathcal{G}(\breve{O}) \tag{2.12}$$
by the right $W_{\mathbf{f}}$-minimality and reducedness assumptions on w_1w_2: indeed, this follows from the theory of Tits systems, which we can apply by [BT72, Théorème 6.5] and [BT84, Corollaire 6.4.7]. This implies the claim when also applied over all formally unramified extensions of $\breve{O}$. The second statement follows from the first, and the third follows from the first and second as $\mathcal{I}(\breve{O})\tau\mathcal{I}(\breve{O}) = \mathcal{I}(\breve{O})\tau$ for $\tau \in \Omega_{\mathbf{a}}$. □

For example, if $w = \tau s_1\dots s_n$ is a right $W_{\mathbf{f}}$-minimal reduced word, with s_i being simple reflections and τ stabilizing $\mathbf{a}$, then $\mathrm{Fl}_{\mathcal{I},\le\tau}\tilde{\times}\mathrm{Fl}_{\mathcal{I},\le s_1}\tilde{\times}\dots\tilde{\times}\mathrm{Fl}_{\mathcal{I},\le s_n} \to \mathrm{Fl}_{(\mathcal{I},\mathcal{G}),\le w}$ defines the Demazure resolution. Studying Demazure resolutions yields the following important geometric consequences.

Lemma 2.5. *Let $w \in W/W_{\mathbf{f}}$ and denote by $w_{\min} \in W$ its right $W_{\mathbf{f}}$-minimal representative.*

(1) $\mathrm{Fl}_{(\breve{\mathcal{I}},\breve{\mathcal{G}}),w} \cong \mathbb{A}_k^{\ell(w_{\min}),\mathrm{pf}}$*, in particular* $\dim \mathrm{Fl}_{(\mathcal{I},\mathcal{G}),w} = \ell(w_{\min})$.

(2) *If $\ell(w_{\min}) = 0$, i.e., $w_{\min} = \tau \in \Omega_{\mathbf{a}}$, then $\mathrm{Fl}_{(\mathcal{I},\mathcal{G}),w} = \mathrm{Fl}_{(\mathcal{I},\mathcal{G}),\le w}$ is isomorphic to* $\mathrm{Spec}(k)$.

(3) *If $\ell(w_{\min}) = 1$, i.e., $w_{\min} = \tau s$ with s a simple reflection and $\tau \in \Omega_{\mathbf{a}}$, then*
$$\mathrm{Fl}_{(\mathcal{I},\mathcal{G}),\le w} \cong \mathbb{P}_k^{1,\mathrm{pf}}. \tag{2.13}$$

Proof. Using 2.4 one reduces to the case that $w = s$ is a simple reflection. In this case, one checks $\mathrm{Fl}_{\breve{\mathcal{I}},\le s} \cong \mathbb{P}_{\bar{k}}^{1,\mathrm{pf}}$ by hand, cf. [Zhu17, Section 1.4]. The remaining assertions follow. □

Let us note that the morphism
$$(\mathrm{pr}, m)\colon \mathrm{Fl}_{\mathcal{G}}\tilde{\times}\mathrm{Fl}_{\mathcal{G}} \to \mathrm{Fl}_{\mathcal{G}}\times\mathrm{Fl}_{\mathcal{G}},\ \overline{(g_1,g_2)} \mapsto (\overline{g_1}, \overline{g_1g_2}) \tag{2.14}$$
is an isomorphism, i.e., convolution products are secretly just products. Given now $L^+\mathcal{G}$-stable locally closed perfect subschemes $X_1, X_2, Y \subseteq \mathrm{Fl}_{\mathcal{G}}$ such that $m(X_1\tilde{\times}X_2) \subseteq Y$, we can factor $m\colon X_1\tilde{\times}X_2 \to Y$ as
$$X_1\tilde{\times}X_2 \xrightarrow{(\mathrm{pr},m)} X_1 \times Y \xrightarrow{\text{projection}} Y. \tag{2.15}$$

A similar discussion holds if we replace the $L^+\mathcal{G}$-equivariant convolution by the $L^+\mathcal{I}$-equivariant version. For the Iwahori convolutions in $\mathrm{Fl}_{\mathcal{G}}$, we get the following important affineness statement.

Lemma 2.6. *Let $w \in W$ and $X \subset \mathrm{Fl}_{\mathcal{G}}$ be a closed $L^+\mathcal{I}$-stable perfect subscheme. Then the map*

$$m\colon \mathrm{Fl}_{\mathcal{I},w} \tilde{\times} X \to \mathrm{Fl}_{\mathcal{G}} \tag{2.16}$$

is affine. If $\mathcal{G} = \mathcal{I}$ is Iwahori, then the map

$$m\colon X \tilde{\times} \mathrm{Fl}_{\mathcal{I},w} \to \mathrm{Fl}_{\mathcal{I}} \tag{2.17}$$

is also affine. In particular, the inclusion $j_w\colon \mathrm{Fl}_{\mathcal{I},w} \to \mathrm{Fl}_{\mathcal{I}}$ is affine for any $w \in W$.

More generally, the affineness of $j_w\colon \mathrm{Fl}_{(\mathcal{I},\mathcal{G}),w} \to \mathrm{Fl}_{\mathcal{G}}$ for $w \in W/W_{\mathbf{f}}$ follows from Lemma 2.5 because $\mathrm{Fl}_{(\mathcal{I},\mathcal{G}),\leq w} \to \mathrm{Fl}_{\mathcal{G}}$ is a closed immersion and $j_w\colon \mathrm{Fl}_{(\mathcal{I},\mathcal{G}),w} \to \mathrm{Fl}_{(\mathcal{I},\mathcal{G}),\leq w}$ is affine because the target is separated.

Proof. We may assume that $F = \breve{F}$. The proof of [AR, Lemma 4.1.6] applies, and we recall its argument. Using the above remarks, we can write $m\colon \mathrm{Fl}_{\mathcal{I},w} \tilde{\times} X \to \mathrm{Fl}_{\mathcal{G}}$ as the composition

$$\mathrm{Fl}_{\mathcal{I},w} \tilde{\times} X \to \mathrm{Fl}_{\mathcal{I},w} \tilde{\times} \mathrm{Fl}_{\mathcal{G}} \overset{(\mathrm{pr},m)}{\cong} \mathrm{Fl}_{\mathcal{I},w} \times \mathrm{Fl}_{\mathcal{G}} \xrightarrow{\text{projection}} \mathrm{Fl}_{\mathcal{G}} \tag{2.18}$$

of morphisms of ind-perfect schemes. The first morphism is affine as $X \subset \mathrm{Fl}_{\mathcal{G}}$ is a closed immersion, and the second is affine by Lemma 2.5. If $\mathcal{G}$ is Iwahori, then the affineness of $m\colon X \tilde{\times} \mathrm{Fl}_{\mathcal{I},w} \to \mathrm{Fl}_{\mathcal{I}}$ can be checked after passing to the quotient Hecke stacks

$$[L^+\mathcal{I} \backslash X \times^{L^+\mathcal{I}} \mathrm{Fl}_{\mathcal{I},w}] \to \mathrm{Hk}_{\mathcal{I}} \tag{2.19}$$

Forgetting about $L^+\mathcal{I}$-equivariance on the *right*, we reduce to the previous case. □

We need the following result later on, but only at Iwahori level.

Lemma 2.7. *Let $X \subset \mathrm{Fl}_{\mathcal{I}}$ be a locally closed $L^+\mathcal{I}$-stable perfect subscheme. Then there exists a finite subset $S_X \subset W$ such that for any $w \in W$ we have*

$$m(X \tilde{\times} \mathrm{Fl}_{\mathcal{I},w}) \subset \cup_{x \in S_X} \mathrm{Fl}_{\mathcal{I},xw}, \quad m(\mathrm{Fl}_{\mathcal{I},w} \tilde{\times} X) \subset \cup_{x \in S_X} \mathrm{Fl}_{\mathcal{I},wx}.$$

Proof. The proof is via an induction argument on $\ell(w)$ which is similar to the equal characteristic setting, [AR, Lemma 4.4.2]. We sketch the proof here. Of course, it suffices to treat the case $X = \mathrm{Fl}_{\mathcal{I},w}$ for some $w \in W$. Let first $X = \mathrm{Fl}_{\mathcal{I},w}$ for some $w \in W$ with $\ell(w) = 0$. Take $S_X = \{w\}$, then the statement holds by noting that $\ell(xw) = \ell(wx) = \ell(x)$ for any $x \in W$.

Assume now $X = \mathrm{Fl}_{\mathcal{I},w}$ for some $w \in W$ with $\ell(w) > 0$. Write $w = w_1 s_1 = s_2 w_2$, where $\ell(w_1) = \ell(w_2) = \ell(w) - 1$. The induction hypothesis implies that there exist finite subsets $S_{X_1}, S_{X_2} \subset W$ such that

$$m(\mathrm{Fl}_{\mathcal{I},w_1} \tilde{\times} \mathrm{Fl}_{\mathcal{I},w'}) \subset \cup_{x \in S_{X_1}} \mathrm{Fl}_{\mathcal{I},xw'}, \quad m(\mathrm{Fl}_{\mathcal{I},w'} \tilde{\times} \mathrm{Fl}_{\mathcal{I},w_2}) \subset \cup_{x \in S_{X_2}} \mathrm{Fl}_{\mathcal{I},w'x},$$

for any $w' \in W$. Note that for any $w'' \in W$,

$$\mathrm{Fl}_{\mathcal{I},s_1} \tilde{\times} \mathrm{Fl}_{\mathcal{I},w''} \subset \mathrm{Fl}_{\mathcal{I},w''} \sqcup \mathrm{Fl}_{\mathcal{I},s_1 w''}, \quad \mathrm{Fl}_{\mathcal{I},w''} \tilde{\times} \mathrm{Fl}_{\mathcal{I},s_2} \subset \mathrm{Fl}_{\mathcal{I},w''} \sqcup \mathrm{Fl}_{\mathcal{I},w'' s_2},$$

and

$$\mathrm{Fl}_{\mathcal{I},w_1} \tilde{\times} \mathrm{Fl}_{\mathcal{I},s_1} \simeq \mathrm{Fl}_{\mathcal{I},s_2} \tilde{\times} \mathrm{Fl}_{\mathcal{I},w_2} \simeq \mathrm{Fl}_{\mathcal{I},w}$$

by 2.4. We conclude that $S_X := S_{X_1} \cup S_{X_1} s_1 \cup S_{X_2} \cup s_2 S_{X_2}$ is the desired finite subset, thereby concluding the proof. □

2.2. **Constant terms and semi-infinite orbits.** Throughout this section, we assume $\mathcal{G} = \mathcal{I}$ is Iwahori. Let $U \subset B$ be the unipotent radical. Then we get the Iwasawa decompositions ([KP23, Theorem 3.3.3])

$$W = N(\breve{F})/\mathcal{T}(\breve{O}) \simeq U(\breve{F})\backslash G(\breve{F})/\mathcal{I}(\breve{O}), \ w \mapsto U(\breve{F})w\mathcal{I}(\breve{O}). \tag{2.20}$$

Geometrically, this yields the semi-infinite orbits.

Definition 2.8. For $w \in W$ we set $\mathcal{S}_w := LU \cdot w \subset \mathrm{Fl}_{\mathcal{I}}$.

By [AGLR22, Section 5] the $\mathcal{S}_w, w \in W$, are represented by locally closed ind-(perfect schemes) and coincide with the connected components of the attractor $\mathrm{Fl}_{\mathcal{I}}^+$ for a regular action by $\mathbb{G}_m$. More precisely, take a regular coweight $\chi \colon \mathbb{G}_{m,F} \to S$, i.e., such that the centralizer of χ in G is T (for example, the sum of all positive coroots). Then B is the attractor locus

$$G^+ = \{g \in G \mid t \mapsto \chi(t)g\chi(t)^{-1} \text{ extends to } \mathbb{A}^1_F\} \tag{2.21}$$

for the conjugation action of $\mathbb{G}_{m,F}$ on G. More generally, if $B \subseteq P \subseteq G$ is any parabolic subgroup, then there exists a character $\psi \colon \mathbb{G}_{m,F} \to S$ such that $P = G^+$ is the attractor locus for the $\mathbb{G}_{m,F}$-action on G by conjugation. The centralizer M of ψ is then a Levi subgroup of P.

The cocharacter χ extends to a group homomorphism $\mathbb{G}_{m,O} \to \mathcal{S}$ by the universal property of connected Néron models. By conjugation, we deduce a $L^+\mathbb{G}_{m,O}$-action on $\mathrm{Fl}_{\mathcal{I}}$ and we restrict it to $\mathbb{G}_{m,k}$ along the Teichmüller lift map. We get the decomposition

$$\mathrm{Fl}_{\mathcal{I}}^+ = \coprod_{w \in W} \mathcal{S}_w, \text{ where } \mathcal{S}_w := LU \cdot w \subset \mathrm{Fl}_{\mathcal{I}}, \tag{2.22}$$

of the attractor locus, cf. [AGLR22, Section 5]. Similarly, the repeller locus $\mathrm{Fl}_{\mathcal{I}}^-$, i.e., the attractor locus for the inverted $\mathbb{G}_m$-action, decomposes as

$$\mathrm{Fl}_{\mathcal{I}}^- = \coprod_{w \in W} \mathcal{S}_w^-, \text{ where } \mathcal{S}_w^- := LU^- \cdot w \subset \mathrm{Fl}_{\mathcal{I}}, \tag{2.23}$$

where U^- denotes the unipotent radical of the opposite Borel B^- of B. The semi-infinite orbits are relevant for computing constant term functors. Let

$$i^+ \colon \mathrm{Fl}_{\mathcal{I}}^+ \to \mathrm{Fl}_{\mathcal{I}}, \ i^- \colon \mathrm{Fl}_{\mathcal{I}}^- \to \mathrm{Fl}_{\mathcal{I}} \tag{2.24}$$

be the inclusions (=disjoint union of disjoint locally closed immersions). Let $\mathrm{Fl}_{\mathcal{I}}^0 \subseteq \mathrm{Fl}_{\mathcal{I}}$ be the fixed point locus of $\mathbb{G}_m$ and let

$$q^+ \colon \mathrm{Fl}_{\mathcal{I}}^+ \to \mathrm{Fl}_{\mathcal{I}}^0, \ q^- \colon \mathrm{Fl}_{\mathcal{I}}^- \to \mathrm{Fl}_{\mathcal{I}}^0 \tag{2.25}$$

be the natural morphism given by evaluating at $0 \in \mathbb{A}^1_k$ resp. $\infty \in \mathbb{A}^1_k$.

Remark 2.9. The natural inclusion $\mathrm{Fl}_{\mathcal{T}} \to \mathrm{Fl}_{\mathcal{I}}^0$ induced by the $\mathbb{G}_m$-equivariant morphism $\mathcal{T} \to \mathcal{I}$, with $\mathbb{G}_m$ acting trivially on $\mathcal{T}$, is not an isomorphism. Namely, $\mathrm{Fl}_{\mathcal{T}}$ is the perfect constant k-scheme associated with the set $\bar{\mathbb{X}}_\bullet$, while $\mathrm{Fl}_{\mathcal{I}}^0$ is associated with the set W.

Following [AGLR22, Section 6.3] we can now define the constant term functor (associated with B and $\mathcal{I}$).

Definition 2.10. We set

$$\mathrm{CT}_B := Rq_!^+ \circ i^{+,*} \colon \mathcal{D}_{\text{ét}}(\mathrm{Hk}_{\mathcal{I}}) \to \mathcal{D}_{\text{ét}}(L^+\mathcal{T}\backslash \mathrm{Fl}_{\mathcal{I}}^0) \tag{2.26}$$

By Braden's theorem the natural map $Rq_!^+ \circ i^{+,*} \to Rq_*^- \circ Ri^{-,!}$ is an isomorphism, cf. [FS21, Theorem IV.6.5], [AGLR22, Section 6]. This implies excellent formal properties of the constant term functor.

Remark 2.11. Let $A \in \mathcal{D}_{\text{ét}}(\mathrm{Hk}_{\mathcal{I}})$ and $w \in W = \mathrm{Hk}_{\mathcal{I}}(k)$. By proper base change the fiber of $\mathrm{CT}_B(A)$ over w is calculated by $R\Gamma_c(\mathcal{S}_w, A_{|\mathcal{S}_w})$.

To use the formula in 2.11 we establish the following lemma.

Lemma 2.12. *Let $w \in W$ be such that $w(b) - b \in \bar{\mathbb{X}}_\bullet^+$, where b denotes the barycenter of the standard alcove* **a**.

(1) *$\mathrm{Fl}_{\mathcal{I},w} = L^+\mathcal{U} \cdot w$, where $\mathcal{U} \subseteq \mathcal{I}$ denotes the scheme-theoretic closure of U in $\mathcal{I}$.*
(2) *$\mathrm{Fl}_{\mathcal{I},w} = \mathcal{S}_w \cap \mathrm{Fl}_{\mathcal{I},\leq w}$.*

Proof. The first claim is equation (5.11) in the proof of [AGLR22, Lemma 5.3]. The first claim implies $\mathrm{Fl}_{\mathcal{I},w} \subseteq \mathcal{S}_w \cap \mathrm{Fl}_{\mathcal{I},\leq w}$ as $\mathcal{S}_w = LU \cdot w$. Let $x \in \mathrm{Fl}_{\breve{\mathcal{I}},\leq w} \setminus \mathrm{Fl}_{\mathcal{I},w}$. By (perfect) properness of $\mathrm{Fl}_{\mathcal{I},\leq w}$ the orbit map $\gamma\colon \mathbb{G}_{m,k} \to \mathrm{Fl}_{\mathcal{I},\leq w}$, $t \mapsto \chi(t)x$ extends to a $\mathbb{G}_m$-equivariant map $\tilde{\gamma}\colon \mathbb{A}^1_k \to \mathrm{Fl}_{\mathcal{I},\leq w}$. As $\mathrm{Fl}_{\mathcal{I},w}$ is open in $\mathrm{Fl}_{\mathcal{I},\leq w}$ and $W \subseteq \mathrm{Fl}_{\mathcal{I}}$ is exactly the set of $\mathbb{G}_{m,k}$-fixed points, 2.2 implies that $x \in \mathcal{S}_{w'}$ for $w' := \tilde{\gamma}(0) < w$ because $\mathcal{S}_{w'}$ is exactly the subscheme of points contracting to w' under the $\mathbb{G}_m$-action. □

Next we describe the closure relations for the stratification of $\mathrm{Fl}_{\mathcal{I}}$ by the $\mathcal{S}_w, w \in W$. As this is a geometric questions, we may assume $F = \breve{F}$ for this. Then we have to define the dominant cocharacters $\bar{\mathbb{X}}_\bullet^+$ in $\bar{\mathbb{X}}_\bullet$. Recall that $\mathbb{X}_\bullet(S)^+$ denotes the (B-)dominant cocharacters for S. As we assumed $F = \breve{F}$, we get that

$$\mathbb{X}_\bullet(S)_{\mathbb{Q}} \cong \bar{\mathbb{X}}_{\bullet,\mathbb{Q}}, \tag{2.27}$$

and thus we can define $\bar{\mathbb{X}}_\bullet^+$ as the preimage of $\mathbb{X}_\bullet(S)^+_{\mathbb{Q}}$ under the map $\bar{\mathbb{X}}_\bullet \to \bar{\mathbb{X}}_{\bullet,\mathbb{Q}}$. Given $\bar{\mathbb{X}}_\bullet$ we can now define the semi-infinite Bruhat order $\preceq$ on W, which depends on B. Namely, set $w \preceq w'$ if and only if for the Bruhat order $t_{n\bar{\nu}}w \leq t_{n\bar{\nu}}w'$ for all $\bar{\nu} \in \bar{\mathbb{X}}_\bullet^+$ and $n \gg 0$.

Lemma 2.13. *For $w, w' \in W$ we have $\mathcal{S}_w \subseteq \overline{\mathcal{S}_{w'}}$ if and only if $w \preceq w'$.*

Proof. This is [AGLR22, Proposition 5.4], where $\preceq$ is denoted by $\leq^{\frac{\infty}{2}}$. □

If $w = t_{\bar{\mu}}, w' = t_{\bar{\nu}}$, then $w \preceq w'$ if and only if $\bar{\nu} - \bar{\mu}$ lies in $\bar{\mathbb{X}}_\bullet^+$. In particular, on $\bar{\mathbb{X}}_\bullet^+$ the two orders $\leq, \preceq$ agree. We will constantly use the equality $\ell(t_{\bar{\nu}}) = \langle 2\bar{\rho}, \bar{\nu}\rangle$ for $\bar{\nu}$ dominant, see [Zhu14, Lemma 9.1].

3. Cohomology of the affine flag variety

In this section, we want to study cohomology of constructible sheaves on the Hecke stack $\mathrm{Hk}_{\mathcal{I}}$. In particular, we will

(1) introduce Wakimoto-filtered complexes in mixed characteristic, following [AB09], [AR] and [Zhu14],
(2) calculate the constant terms of Wakimoto sheaves,
(3) show that central objects for convolution are Wakimoto-filtered.

In this section we assume that $F = \breve{F}$, and thus in particular that k is algebraically closed.

3.1. **(Co)standard functors.** The considerations that are going to come have an easy shadow on Grothendieck groups. Recall that we always assume our ring of coefficients Λ to be a field.

For $w \in W/W_{\mathbf{f}}$ let $j_w\colon \mathrm{Fl}_{(\mathcal{I},\mathcal{G}),w} \to \mathrm{Fl}_{\mathcal{G}}$ be the locally closed affine immersion. Note that j_w is $L^+\mathcal{I}$-equivariant, and hence descends to a morphism

$$j_w\colon \mathrm{Hk}_{(\mathcal{I},\mathcal{G}),w} \to \mathrm{Hk}_{(\mathcal{I},\mathcal{G})} \tag{3.1}$$

of stacks, where on the left side $\mathrm{Hk}_{(\mathcal{I},\mathcal{G}),w} := [L^+\mathcal{I}\backslash \mathrm{Fl}_{(\mathcal{I},\mathcal{G}),w}]$, and that we will usually denote in the same way. Define the standard object

$$\Delta_{(\mathcal{I},\mathcal{G}),w} := j_{w,!}(\Lambda)[\ell(w_{\min})] \in \mathcal{D}_{\text{ét}}(\mathrm{Hk}_{(\mathcal{I},\mathcal{G})}) \tag{3.2}$$

and the costandard object

$$\nabla_{(\mathcal{I},\mathcal{G}),w} := Rj_{w,*}(\Lambda)[\ell(w_{\min})] \in \mathcal{D}_{\text{ét}}(\text{Hk}_{(\mathcal{I},\mathcal{G})}) \tag{3.3}$$

associated with $w \in W/W_{\mathbf{f}}$.

Let

$$\mathcal{D}_{\text{cons}}(\text{Hk}_{(\mathcal{I},\mathcal{G})}) \subset \mathcal{D}_{\text{ét}}(\text{Hk}_{(\mathcal{I},\mathcal{G})}) \tag{3.4}$$

be the full subcategory consisting of objects with perfect stalks. Let $K_0(\text{Hk}_{(\mathcal{I},\mathcal{G})})$ be the Grothendieck group of $\mathcal{D}_{\text{cons}}(\text{Hk}_{(\mathcal{I},\mathcal{G})})$. Since points in $\text{Fl}_{\mathcal{G}}$ have connected stabilizers under the $L^+\mathcal{I}$-action, $K_0(\text{Hk}_{(\mathcal{I},\mathcal{G})})$ identifies with the Grothendieck group of the category of $L^+\mathcal{I}$-equivariant sheaves on $\text{Fl}_{\mathcal{G}}$. Consequently, $K_0(\text{Hk}_{(\mathcal{I},\mathcal{G})})$ is a free abelian group on the classes of the intersection complexes $\text{IC}_{(\mathcal{I},\mathcal{G}),w} = j_{!*}(\Lambda)$ of $\text{Fl}_{(\mathcal{I},\mathcal{G}),\leq w}$.

Via convolution $K_0(\text{Hk}_{\mathcal{I}})$ is naturally a ring, cf. [AR, Section 5.2], and $K_0(\text{Hk}_{(\mathcal{I},\mathcal{G})})$ is a left $K_0(\text{Hk}_{\mathcal{I}})$-module. In fact, this ring identifies with the integral group ring of W as we recall now.

Lemma 3.1. *The maps*

$$\theta\colon K_0(\text{Hk}_{\mathcal{I}}) \to \mathbb{Z}[W],\ [\mathcal{F}] \mapsto \sum_{w\in W} (-1)^{\ell(w)}\chi(\text{Fl}_{\mathcal{I},w}, j_w^*\mathcal{F})w \tag{3.5}$$

and

$$\vartheta\colon \mathbb{Z}[W] \to K_0(\text{Hk}_{\mathcal{I}}),\ w \mapsto (-1)^{\ell(w)}[\nabla_{\mathcal{I},w}] \tag{3.6}$$

are inverse ring isomorphisms.

Here, $\chi(\text{Fl}_{\mathcal{I},w}, j_w^*\mathcal{F})$ denotes the Euler characteristic. In the equal characteristic case, a proof is given in [AR, Lemma 5.2.1].

Proof. If $s \in W$ and $\ell(s) = 1$, then by Lemma 2.5, we have $\text{Fl}_{\mathcal{I},\leq s} \cong \mathbb{P}_k^{1,\text{pf}}$ and we get distinguished triangles

$$\text{IC}_{\mathcal{I},e} \to \Delta_{\mathcal{I},s} \to \text{IC}_{\mathcal{I},s} \xrightarrow{+1}, \quad \text{IC}_{\mathcal{I},s} \to \nabla_{\mathcal{I},s} \to \text{IC}_{\mathcal{I},e} \xrightarrow{+1}$$

because $\text{IC}_{\mathcal{I},s}$ identifies with the underived pushforward $R^0 j_{s,*}(\Lambda)$. Thus, $\theta([\nabla_{\mathcal{I},s}]) = \theta([\Delta_{\mathcal{I},s}]) = s$ and $\theta(\vartheta(s)) = s$. It then follows from 3.2 below that θ, ϑ are ring homomorphisms and in fact $\theta([\Delta_{\mathcal{I},w}]) = \theta([\nabla_{\mathcal{I},w}]) = w$ for any $w \in W$. Now, ϑ is surjective because the $[\Delta_{\mathcal{I},w}] = [\nabla_{\mathcal{I},w}]$ generate $K_0(\text{Hk}_{\mathcal{I}})$. This finishes the proof. $\square$

We will now study convolutions of standard and costandard sheaves. Before proceeding, we upgrade these objects to actual functors. Recall that $\mathcal{D}_{\text{ét}}(\text{Hk}_{\mathcal{I}})$ has a natural monoidal structure in the sense of [Lur17, Definition 4.1.1.10] as we will see later on, see §4.2. For now, it suffices to construct the underlying bifunctor as follows. First, consider the diagram

$$\begin{tikzcd} & \text{Hk}_{\mathcal{I}} \times \text{Hk}_{\mathcal{I}} \arrow[dl, "\text{pr}_1"'] \arrow[dr, "\text{pr}_2"] & {[L^+\mathcal{I}\backslash LG \times^{L^+\mathcal{I}} LG/L^+\mathcal{I}]} \arrow[l, "p"'] \arrow[r, "m"] & \text{Hk}_{\mathcal{I}} \\ \text{Hk}_{\mathcal{I}} & & \text{Hk}_{\mathcal{I}} & \end{tikzcd} \tag{3.7}$$

of ind-v-stacks on perfect schemes, with pr_1, pr_2 the two projections onto the first and second factors, respectively, p the natural morphism, and m the (quotient by the left $L^+\mathcal{I}$-action of the) convolution morphism discussed in 2.1. Now, for any $\mathcal{F}_1, \mathcal{F}_2 \in \mathcal{D}_{\text{ét}}(\text{Hk}_{\mathcal{I}})$, define

$$\mathcal{F}_1 * \mathcal{F}_2 := Rm_!(p^*(\mathcal{F}_1 \boxtimes^{\mathbb{L}} \mathcal{F}_2)). \tag{3.8}$$

Note that m is (perfectly) proper and thus $Rm_! = Rm_*$. The full subcategory $\mathcal{D}_{\text{cons}}(\text{Hk}_{\mathcal{I}})$ is stable under convolution. A similar diagram can be used to define a convolution product $*$ that realizes $\mathcal{D}_{\text{cons}}(\text{Hk}_{(\mathcal{I},\mathcal{G})})$ as a left module of $\mathcal{D}_{\text{cons}}(\text{Hk}_{\mathcal{I}})$.

Note that the full subcategory of $\mathcal{D}_{\mathrm{cons}}(\mathrm{Hk}_{(\mathcal{I},\mathcal{G})})$ whose objects are supported at the origin identifies with $\mathcal{D}_{\mathrm{cons}}([*/L^+\mathcal{I}])$. Convolution restricts to the usual tensor product on these sheaves which is symmetric monoidal. Furthermore, note that $L^+\mathcal{I}$ is an extension of its reductive quotient, which is naturally isomorphic to the special fiber S_k of the connected Néron model $\mathcal{S}$ of S, by a connected pro-unipotent group. By [FS21, Proposition VI.4.1], we can therefore identify $\mathcal{D}_{\mathrm{cons}}([*/L^+\mathcal{I}])$ with $\mathcal{D}_{\mathrm{cons}}([*/S_k])$ via pullback along $[*/L^+\mathcal{I}] \to [*/S_k]$. This is convenient, because $\mathcal{D}_{\mathrm{cons}}(\mathrm{Hk}_{(\mathcal{I},\mathcal{G}),w})$ also identifies with $\mathcal{D}_{\mathrm{cons}}([*/S_k])$ for any $w \in W/W_{\mathbf{f}}$, since S_k maps isomorphically to the reductive quotient of the stabilizer group $L^+\mathcal{I} \cap wL^+\mathcal{G}w^{-1}$.

Moreover, the above abstract nonsense allows us to regard standard and costandard objects as functors by tensoring. Indeed, we define the *standard* and *costandard* functors:

$$\Delta_{(\mathcal{I},\mathcal{G}),w} \colon \mathcal{D}_{\mathrm{cons}}([*/S_k]) \to \mathcal{D}_{\mathrm{cons}}(\mathrm{Hk}_{(\mathcal{I},\mathcal{G})}),\ M \mapsto M * \Delta_w, \tag{3.9}$$

$$\nabla_{(\mathcal{I},\mathcal{G}),w} \colon \mathcal{D}_{\mathrm{cons}}([*/S_k]) \to \mathcal{D}_{\mathrm{cons}}(\mathrm{Hk}_{(\mathcal{I},\mathcal{G})}),\ M \mapsto M * \nabla_{(\mathcal{I},\mathcal{G}),w}, \tag{3.10}$$

and one checks easily that there are isomorphisms $\Delta_{(\mathcal{I},\mathcal{G}),w}(M) \simeq j_{w!}M[\ell(w)]$ and $\nabla_{(\mathcal{I},\mathcal{G}),w}(M) \simeq Rj_{w*}M[\ell(w)]$ of functors. Since j_w is an affine morphism (Lemma 2.6), both functors are t-exact by [BBDG18, Corollaire 4.1.3] for the natural t-structure on $\mathcal{D}_{\mathrm{cons}}([*/S_k])$ and the perverse t-structure on $\mathcal{D}_{\mathrm{cons}}(\mathrm{Hk}_{(\mathcal{I},\mathcal{G})})$, cf. [AGLR22, Definition 6.8] for the latter.

We start with the following lemma on the convolution of standard and costandard objects.

Lemma 3.2. *For any $w_1 \in W$ and $w_2 \in W/W_{\mathbf{f}}$ such that $\ell(w_1) + \ell(w_{2,\min}) = \ell((w_1w_2)_{\min})$, there exist canonical isomorphisms*

$$\Delta_{\mathcal{I},w_1} * \Delta_{(\mathcal{I},\mathcal{G}),w_2} \simeq \Delta_{(\mathcal{I},\mathcal{G}),w_1w_2}$$
$$\nabla_{\mathcal{I},w_1} * \nabla_{(\mathcal{I},\mathcal{G}),w_2} \simeq \nabla_{(\mathcal{I},\mathcal{G}),w_1w_2},$$

satisfying the obvious associativity constraint.

Proof. In equicharacteristic, this statement can be found in [AB09, Lemma 8(a)] with a proof given in [AR, Lemma 4.1.4 (1),(2)]. The same proof applies here. For any $w_1 \in W$ and $w_2 \in W$ such that w_1w_2 is right $W_{\mathbf{f}}$-minimal and reduced, the convolution morphism $\mathrm{Fl}_{\mathcal{I},w_1}\tilde{\times}\mathrm{Fl}_{(\mathcal{I},\mathcal{G}),w_2} \to \mathrm{Fl}_{(\mathcal{I},\mathcal{G}),w_1w_2}$ is an isomorphism by Lemma 2.4, so the constant complex $m_!\Lambda$ identifies with $\underline{\Lambda}$. This yields the desired isomorphisms after !- or $*$-extension and shifts. Indeed, $Rm_!(j_{w_1,!}\Lambda\tilde{\boxtimes}j_{w_2,!}\Lambda) \cong j_{w_1w_2,!}\Lambda$ and $Rm_*(Rj_{w_1,*}\Lambda\tilde{\boxtimes}Rj_{w_2,*}\Lambda) \cong Rj_{w_1w_2,*}\Lambda$. □

Recall that in a monoidal category, an object is called left-invertible (resp. right-invertible) if multiplication on the left (resp. right) is an equivalence.

Lemma 3.3. *For any $w \in W$, the objects $\nabla_{\mathcal{I},w}$ and $\Delta_{\mathcal{I},w}$ are both left- and right-invertible in the monoidal category $\mathcal{D}_{\mathrm{cons}}(\mathrm{Hk}_{\mathcal{I}})$. More concretely, there exist (non-canonical) isomorphisms*

$$\nabla_{\mathcal{I},w} * \Delta_{\mathcal{I},w^{-1}} \simeq \delta_{\mathcal{I},e} \simeq \Delta_{\mathcal{I},w} * \nabla_{\mathcal{I},w^{-1}}.$$

where $\delta_{\mathcal{I},e} := \Delta_{\mathcal{I},e} = \nabla_{\mathcal{I},e}$ with $e \in W$ is the identity element and $\delta_{\mathcal{I},e}$ is the unit object for convolution.

Proof. It is clear that $\delta_{\mathcal{I},e}$ is the unit for convolution. The existence of the searched for isomorphisms is stated over a Laurent series field in [AB09, Lemma 8(a)] and proved in [AR, Lemma 4.1.4(3)]. Again the same proof applies. The non-canonical isomorphism can be obtained by induction on $\ell(w)$ provided we construct them for all simple reflections s and $\tau \in \Omega_{\mathbf{a}}$. The case for $\tau \in \Omega_{\mathbf{a}}$ is clear as $\mathrm{Fl}_{\mathcal{I},\tau} = \mathrm{Fl}_{\mathcal{I},\leq\tau}$ by Lemma 2.5. Let $s \in \mathbb{S}$. Notice now that the twisted product $\mathrm{Fl}_{\mathcal{I},\leq s}\tilde{\times}\mathrm{Fl}_{\mathcal{I},\leq s}$ identifies with the direct product $\mathrm{Fl}_{\mathcal{I},\leq s} \times \mathrm{Fl}_{\mathcal{I},\leq s}$ via the morphism (pr_1, m), cf. Lemma 2.5. In particular, it is isomorphic to $\mathbb{P}^{1,\mathrm{pf}}_k \times_{\mathrm{Spec}(k)} \mathbb{P}^{1,\mathrm{pf}}_k$. Under this identification $\mathrm{Fl}_{\mathcal{I},e}\tilde{\times}\mathrm{Fl}_{\mathcal{I},\leq s}$ identifies with $\{\infty\} \times_{\mathrm{Spec}(k)} \mathbb{P}^{1,\mathrm{pf}}_k$ and $\mathrm{Fl}_{\mathcal{I},\leq s}\tilde{\times}\{\infty\}$ with the diagonal $\Delta^{\mathrm{pf}}_{\mathbb{P}^1_k}$. Here,

$\infty \in \mathbb{P}_k^{1,\mathrm{pf}} \cong \mathrm{Fl}_{\mathcal{I},\leq s}$ denotes the point eI. From here the argument from [AR, Lemma 4.1.4.(3)] applies (using that perfection does not alter the étale site). ☐

Remark 3.4. As noted in the discussion after [AR, Remark 4.1.5], the isomorphisms in 3.3 depend on various choices, e.g., a decomposition of w into elements of length ≤ 1. For functorial purposes, one can often work with any inverse of $\nabla_{\mathcal{I},w}$, while using $\Delta_{\mathcal{I},w^{-1}}$ for practical computations.

As usual one is interested in understanding what happens on the abelian subcategory $\mathcal{P}(\mathrm{Hk}_{\mathcal{I}})$ arising as the heart of the perverse t-structure. While it is not stable under the monoidal structure of $\mathcal{D}_{\mathrm{cons}}(\mathrm{Hk}_{\mathcal{I}})$, we can still benefit from the semiperversity below. We formulate it also for general parahorics $\mathcal{G}$, because it plays a key role in the perversity of central sheaves.

Lemma 3.5. *For any $w \in W$, left convolution with $\Delta_{\mathcal{I},w}$ (resp. $\nabla_{\mathcal{I},w}$) defines a left (resp. right) exact endofunctor of $\mathcal{D}(\mathrm{Hk}_{(\mathcal{I},\mathcal{G})})$. If $\mathcal{G} = \mathcal{I}$ is Iwahori, the same holds for right convolution. In particular, for any other $v \in W$, we have $\Delta_{\mathcal{I},w} * \nabla_{\mathcal{I},v}, \nabla_{\mathcal{I},w} * \Delta_{\mathcal{I},v} \in \mathcal{P}(\mathrm{Hk}_{\mathcal{I}})$.*

Proof. Note that by definition $\mathcal{D}^{\leq 0}_{\mathrm{cons}}(\mathrm{Hk}_{(\mathcal{I},\mathcal{G})})$ (resp. $\mathcal{D}^{\geq 0}_{\mathrm{cons}}(\mathrm{Hk}_{(\mathcal{I},\mathcal{G})})$) is spanned by the non-negative (resp. non-positive) shifts of the $\Delta_{(\mathcal{I},\mathcal{G}),v}$ (resp. $\nabla_{(\mathcal{I},\mathcal{G}),v}$) for $v \in W/W_{\mathbf{f}}$. By Lemma 2.6, the convolution map $m\colon \mathrm{Fl}_{\mathcal{I},w}\tilde{\times}\mathrm{Fl}_{(\mathcal{I},\mathcal{G}),\leq v} \to \mathrm{Fl}_{\mathcal{I}}$ is affine. Now,

$$\Delta_{\mathcal{I},w} * \nabla_{(\mathcal{I},\mathcal{G}),v} = Rm_!(\Lambda\tilde{\boxtimes}\nabla_v) \tag{3.11}$$

with $\Lambda\tilde{\boxtimes}\nabla_{(\mathcal{I},\mathcal{G}),v}$ perverse by our assumption and Lemma 2.6. Thus $\Delta_{\mathcal{I},w} * \nabla_{(\mathcal{I},\mathcal{G}),v}$ is concentrated in non-negative perverse degrees because !-pushforward of affine morphisms is left exact for the perverse t-structure, cf. [BBDG18, Corollaire 4.1.2]. On the other hand,

$$\nabla_{\mathcal{I},w} * \Delta_{(\mathcal{I},\mathcal{G}),v} = Rm_*(\Delta_w\tilde{\boxtimes}\Lambda) \tag{3.12}$$

is concentrated in non-positive perverse degrees by [BBDG18, Théorème 4.1.1]. If $\mathcal{G} = \mathcal{I}$ is Iwahori, then by symmetry we can run the same arguments for the right convolution. This finishes the proof. ☐

During the remainder of this section, we will no longer need the general parahoric case. So we assume that $\mathcal{G} = \mathcal{I}$ is Iwahori and suppress it from the index of the standard and costandard sheaves.

Lemma 3.6. *For any $w_1, w_2 \in W$, the perverse sheaves $\Delta_{w_1} * \nabla_{w_2}, \nabla_{w_1} * \Delta_{w_2}$ are both supported on $\mathrm{Fl}_{\mathcal{I},\leq w_1 w_2}$, and restrict to $\Lambda[\ell(w_1 w_2)]$ on $\mathrm{Fl}_{\mathcal{I},w_1 w_2}$.*

Proof. The proof is similar to the equal characteristic case, cf. [AR, 4.1.10], and we sketch it here. The Euler characteristic

$$\theta\colon K_0(\mathrm{Hk}_{\mathcal{I}}) \longrightarrow \mathbb{Z}[W], \quad [\mathcal{F}] \longmapsto \sum_{w \in W} (-1)^{\ell(w)} \chi(\mathrm{Fl}_{\mathcal{I},w}, j_w^* \mathcal{F}) w \tag{3.13}$$

defines a ring homomorphism. By the proof of 3.1 we know that $\theta([\Delta_w]) = \theta([\nabla_w]) = w$ for any $w \in W$. Now let $w \in W$ be any element such that $\mathrm{Fl}_{\mathcal{I},w}$ is open in the support of $\Delta_{w_1} * \nabla_{w_2}$. Then the coefficient of w in $\theta([\Delta_{w_1} * \nabla_{w_2}]) \in \mathbb{Z}[W]$ does not vanish. By perversity, see Lemma 3.5, and $\mathcal{I}$-equivariance it is a non-zero multiple of the Euler characteristic of the cohomology of $\Lambda[\ell(w)]$ on $\mathrm{Fl}_{\mathcal{I},w} \cong \mathbb{A}_k^{\ell(w),\mathrm{pf}}$. Now,

$$\theta([\Delta_{w_1} * \nabla_{w_2}]) = \theta([\Delta_{w_1}])\theta([\nabla_{w_2}]) = w_1 w_2, \tag{3.14}$$

and thus $w = w_1 w_2$ and $\Delta_{w_1} * \nabla_{w_2}$ is supported on $\mathrm{Fl}_{\mathcal{I},\leq w_1 w_2}$. By perversity and I-equivariance, we must have that

$$j^*_{w_1 w_2}(\Delta_{w_1} * \nabla_{w_2}) \cong \Lambda^{\oplus m}[\ell(w_1 w_2)] \tag{3.15}$$

for some $m \geq 1$. But the coefficient of $w_1 w_2$ is 1, so $m = 1$ as desired. The statement for $\nabla_{w_1} * \Delta_{w_2}$ follows similarly. □

If $\mathcal{F} \in \mathcal{D}_{\text{ét}}(\text{Hk}_{\mathcal{I}})$, then

$$\text{supp}(\mathcal{F}) = \{w \in W \mid j_w^* \mathcal{F} \neq 0\} \tag{3.16}$$

is the support of $\mathcal{F}$, and

$$\text{cosupp}(\mathcal{F}) = \{w \in W \mid j_w^! \mathcal{F} \neq 0\} \tag{3.17}$$

its cosupport.

We need the following geometric consequence of Lemma 2.7. In equal characteristic this is [AR, Proposition 4.4.4.].

Proposition 3.7. *For any $\mathcal{F} \in \mathcal{D}_{\text{cons}}(\text{Hk}_{\mathcal{I}})$, there exists a finite subset $A_{\mathcal{F}} \subset W$ such that for any $w \in W$,*

(1) $\text{supp}(\Delta_w * \mathcal{F}) \subseteq w \cdot A_{\mathcal{F}}$, $\quad \text{cosupp}(\nabla_w * \mathcal{F}) \subseteq w \cdot A_{\mathcal{F}}$,
(2) $\text{supp}(\mathcal{F} * \Delta_w) \subseteq A_{\mathcal{F}} \cdot w$, $\quad \text{cosupp}(\mathcal{F} * \nabla_w) \subseteq A_{\mathcal{F}} \cdot w$.

Proof. Let $X \subseteq \text{Fl}_{\mathcal{I}}$ be a closed finite union of I-orbits such that $\text{supp}(\mathcal{F}) \subseteq X$. Set $A_{\mathcal{F}} := S_X$ with S_X as in 2.7, i.e.,

$$m(X \tilde{\times} \text{Fl}_{\mathcal{I},w}) \subseteq A_{\mathcal{F}} \cdot w \tag{3.18}$$

and

$$m(\text{Fl}_{\mathcal{I},w} \tilde{\times} X) \subseteq w \cdot A_{\mathcal{F}} \tag{3.19}$$

for all $w \in W$. Now the proper base change theorem implies that $\text{supp}(\mathcal{F} * \Delta_w) \subseteq A_{\mathcal{F}} \cdot w$ and $\text{supp}(\Delta_w * \mathcal{F}) \subseteq w \cdot A_{\mathcal{F}}$ for any $w \in W$. Because we assumed that X is closed, we can also use that Rm_* commutes with !-restrictions (by the adjoint version of the proper base change theorem) to see that $\text{cosupp}(\mathcal{F} * \nabla_w) \subseteq A_{\mathcal{F}} \cdot w$ and $\text{cosupp}(\nabla_w * \mathcal{F}) \subseteq w \cdot A_{\mathcal{F}}$ for any $w \in W$. This finishes the proof. □

Regarding the products $\Delta_{w_1} * \Delta_{w_2}$, $\nabla_{w_1} * \nabla_{w_2}$ for $w_1, w_2 \in W$ we note the following.

Lemma 3.8. *Let $w_1, w_2 \in W$, then*

(1) *$\Delta_{w_1} * \Delta_{w_2}$ lies in the smallest full subcategory of $\mathcal{D}_{\text{cons}}(\text{Hk}_{\mathcal{I}})$, which is closed under extensions, and contains $\Delta_w[n]$ for $w \in W$ and $n \in \mathbb{Z}_{\leq 0}$.*
(2) *$\nabla_{w_1} * \nabla_{w_2}$ lies in the smallest full subcategory of $\mathcal{D}_{\text{cons}}(\text{Hk}_{\mathcal{I}})$, which is closed under extensions, and contains $\nabla_w[n]$ for $w \in W$ and $n \in \mathbb{Z}_{\geq 0}$.*

Proof. Given the results of this section the argument of [AR, Lemma 6.5.8] applies. □

Let $\Omega_{\mathbf{a}} \subset W$ be the stabilizer of the fundamental alcove $\mathbf{a}$, i.e., the subset of length 0 elements.

Lemma 3.9. *Given $w \in W$, let $\tau \in \Omega_{\mathbf{a}}$ be the unique element contained in $\text{Fl}_{\mathcal{I},\leq w}$. Then, the sheaf IC_τ appears with multiplicity 1 in the Jordan–Hölder series of ∇_w and equals its top. Dually, IC_τ appears with multiplicity 1 inside Δ_w as its socle.*

Proof. This follows from the same proof of [BBM04, Lemma 2.1]. The idea is to argue by induction on the length of w. Besides the combinatorics of Coxeter groups, one only has to know that $\text{Fl}_{\mathcal{I}} \to \text{Fl}_{\mathcal{J}_s}$ is a $\mathbb{P}^{1,\text{pf}}_k$-bundle locally for the étale topology that actually splits over Schubert cells (use root groups to see this latter property). Here, $\mathcal{I} \to \mathcal{J}_s$ is the minimal parahoric fixing the wall of the fundamental alcove $\mathbf{a}$ fixed by s. □

3.2. **Wakimoto sheaves.** Let $w_1, w_2 \in W$. In general,

$$\Delta_{w_1} * \Delta_{w_2} \ncong \Delta_{w_1 w_2},\ \nabla_{w_1} * \nabla_{w_2} \ncong \nabla_{w_1 w_2} \tag{3.20}$$

unless $\ell(w_1) + \ell(w_2) = \ell(w_1 w_2)$, cf. 3.2. In this subsection, we want to remedy this fact by introducing objects

$$\mathcal{J}_{\bar{\nu}} \in \mathcal{D}_{\text{ét}}(\text{Hk}_{\mathcal{I}}) \tag{3.21}$$

for $\bar{\nu} \in \bar{\mathbb{X}}_\bullet$ (recall the embedding $\bar{\mathbb{X}}_\bullet \to W,\ \bar{\nu} \mapsto t_{\bar{\nu}}$) such that

$$\mathcal{J}_{\bar{\nu}_1} * \mathcal{J}_{\bar{\nu}_2} \cong \mathcal{J}_{\bar{\nu}_1 + \bar{\nu}_2} \tag{3.22}$$

for all $\bar{\nu}_1, \bar{\nu}_2 \in \bar{\mathbb{X}}_\bullet$, and

$$\mathcal{J}_{\bar{\nu}} \cong \Delta_{t_{\bar{\nu}}} \tag{3.23}$$

if $\bar{\nu} \in -\bar{\mathbb{X}}_\bullet^+$. Note that by Lemma 3.3 this already forces

$$\mathcal{J}_{\bar{\nu}} \cong \nabla_{t_{\bar{\nu}}} \tag{3.24}$$

if $\bar{\nu} \in \bar{\mathbb{X}}_\bullet^+$. In fact, we must have

$$\mathcal{J}_{\bar{\nu}} \cong \Delta_{t_{\bar{\nu}_2}} * \nabla_{\bar{\nu}_1} \tag{3.25}$$

if we write $\bar{\nu} = \bar{\nu}_1 - \bar{\nu}_2$ with $\bar{\nu}_1, \bar{\nu}_2 \in \bar{\mathbb{X}}_\bullet^+$ (which is always possible). Note that $\ell(t_{\bar{\nu}_1}) + \ell(t_{\bar{\nu}_2}) = \ell(t_{\bar{\nu}_1} t_{\bar{\nu}})$ if $\bar{\nu}_1, \bar{\nu}_2 \in \bar{\mathbb{X}}_\bullet^+$. Hence, Lemma 3.2 implies that the above formula for $\mathcal{J}_{\bar{\nu}}$ is independent (up to isomorphism) of $\bar{\nu}_1, \bar{\nu}_2$. To get a more canonical construction of $\mathcal{J}_{\bar{\nu}}$, we will adopt the definition from [AR, Section 4.2.1].

Definition 3.10. Let $\bar{\nu} \in \bar{\mathbb{X}}_\bullet$. The *Wakimoto sheaf* $\mathcal{J}_{\bar{\nu}}$ is the object in $\mathcal{D}_{\text{cons}}(\text{Hk}_{\mathcal{I}})$ corepresenting the functor

$$\mathcal{F} \mapsto \operatorname{colim} \operatorname{Hom}(\nabla_{t_{\bar{\nu}_1}}, \mathcal{F} * \nabla_{t_{\bar{\nu}_2}}), \tag{3.26}$$

with the colimit running over all $\bar{\nu}_1, \bar{\nu}_2 \in \bar{\mathbb{X}}_\bullet^+$ such that $\bar{\nu} = \bar{\nu}_1 - \bar{\nu}_2$. The transition morphisms in the colimit are given by convolution (and using the canonical isomorphisms in Lemma 3.2).

Note that all the transition morphisms in the colimit are isomorphisms. In particular, we can conclude (by invertibility of $\nabla_{t_{\bar{\nu}_2}}$), cf. Lemma 3.3) that

$$\mathcal{J}_{\bar{\nu}} \cong \Delta_{t_{\bar{\nu}_2}} * \nabla_{t_{\bar{\nu}_1}} \tag{3.27}$$

as desired. More generally, we can use the fact that $\mathcal{D}_{\text{cons}}([*/S_k])$ acts on $\mathcal{D}_{\text{cons}}(\text{Hk}_{\mathcal{I}})$ to deduce a functor

$$\mathcal{J}_{\bar{\nu}} \colon \mathcal{D}_{\text{cons}}([*/S_k]) \to \mathcal{D}_{\text{cons}}(\text{Hk}_{\mathcal{I}}),\ M \mapsto \mathcal{J}_{\bar{\nu}} * M \tag{3.28}$$

between the two categories via evaluation at the Wakimoto sheaf. This will be called the Wakimoto functor and still be denoted by $\mathcal{J}_{\bar{\nu}}$ by abuse of notation.

Remark 3.11. The Wakimoto sheaves $\mathcal{J}_{\bar{\nu}}$ were introduced by Mirković for geometrizing Bernstein elements in the affine Hecke algebra, see [AR, Section 5.1].

Given a subset $\Omega \subset \bar{\mathbb{X}}_\bullet$, it will also be convenient to define the Ω-Wakimoto functor

$$\mathcal{J}_\Omega = \oplus_{\bar{\nu} \in \Omega} \mathcal{J}_{\bar{\nu}} \colon \mathcal{D}_{\text{cons}}([\underline{\Omega}/S_k]) \to \mathcal{D}_{\text{cons}}(\text{Hk}_{\mathcal{I}}) \tag{3.29}$$

as the direct sum of the $\mathcal{J}_{\bar{\nu}}$ for $\bar{\nu} \in \Omega$, where $\underline{\Omega} = \coprod_\Omega \operatorname{Spec} k$ regarded as an ind-scheme, so that complexes of étale sheaves have compact support. If $\Omega = \bar{\mathbb{X}}_\bullet$ is the total set, then we simply write $\mathcal{J}$ for $\mathcal{J}_{\bar{\mathbb{X}}_\bullet}$, which is monoidal by Lemma 3.2. Indeed, we can identify $\mathcal{D}_{\text{cons}}([\underline{\bar{\mathbb{X}}_\bullet}/S_k])$ with the full subcategory of compact objects of the product in Cat_∞ of the monoidal 1-category $\bar{\mathbb{X}}_\bullet$ with the stable ∞-category $\mathcal{D}_{\text{cons}}([*/S_k])$. We see that the first category maps monoidally to the abelian category $\mathcal{P}(\text{Hk}_{\mathcal{I}})$ via the Wakimoto sheaves $\mathcal{J}_{\bar{\nu}}$, see [AR, Section 4.2.3], whereas the second maps monoidally to the $\mathbb{E}_1$-center $\mathcal{D}_{\text{cons}}(\text{Hk}_{\mathcal{I}})$. This implies the claim by the universal property of centers, see [Lur17, Section 5.3.1].

Lemma 3.12. *The Wakimoto functors satisfy the following properties:*

(1) *For any $\bar{\nu} \in \bar{\mathbb{X}}_\bullet$, $\mathcal{J}_{\bar{\nu}}$ is t-exact for the perverse t-structure.*
(2) *For any $\bar{\nu} \in \bar{\mathbb{X}}_\bullet$, $\mathcal{J}_{\bar{\nu}}$ is supported on $\mathrm{Fl}_{\leq \bar{\nu}}$ and $j_{\bar{\nu}}^* \mathcal{J}_{\bar{\nu}} \simeq \Lambda[\langle 2\bar{\rho}, \bar{\nu}\rangle]$.*
(3) *For any $\bar{\mu}, \bar{\nu} \in \bar{\mathbb{X}}_\bullet$, there exists a canonical isomorphism $\mathcal{J}_{\bar{\mu}} * \mathcal{J}_{\bar{\nu}} \simeq \mathcal{J}_{\bar{\mu}+\bar{\nu}}$.*
(4) *For any $\bar{\mu}, \bar{\nu} \in \bar{\mathbb{X}}_\bullet$ with $t_{\bar{\nu}} \not\leq t_{\bar{\mu}}$, we have $R\mathrm{Hom}_{\mathcal{D}_{\text{ét}}(\mathrm{Hk}_{\mathcal{I}})}(\mathcal{J}_{\bar{\mu}}, \mathcal{J}_{\bar{\nu}}) = 0$.*

Proof. The first statement follows from 3.5, the second from 3.6 and the third is implicit in the discussion of monoidality of $\mathcal{J}$. Let us discuss the forth statement. Using (3), the invertibity of $\mathcal{J}_{\bar{\nu}}(\Lambda)$ and the definition of the semi-infinite Bruhat order $\preceq$ reduces by suitable convolution to the case that $\bar{\nu}, \bar{\mu} \in \bar{\mathbb{X}}_\bullet$ are dominant. Then $\mathcal{J}_{\bar{\mu}}(M) = \nabla_{t_{\bar{\mu}}}(M), \mathcal{J}_{\bar{\nu}}(N) = \nabla_{t_{\bar{\nu}}}(N)$ and thus by 2.2

$$R\mathrm{Hom}_{\mathcal{D}_{\text{ét}}(\mathrm{Hk}_{\mathcal{I}})}(\nabla_{t_{\bar{\mu}}}(M), \nabla_{t_{\bar{\nu}}}(N)) \cong R\mathrm{Hom}_{\mathcal{D}_{\text{ét}}(\mathrm{Hk}_{\mathcal{I}}}(j_{t_{\bar{\nu}}}^* \nabla_{t_{\bar{\nu}}}, N[\ell(t_{\bar{\nu}})]) = 0 \tag{3.30}$$

if $t_{\bar{\nu}} \not\leq t_{\bar{\mu}}$ (for the Bruhat order $\leq$ or equivalently the semi-infinite order $\preceq$ as $\bar{\nu}, \bar{\mu} \in \bar{\mathbb{X}}_\bullet^+$). □

For a stable ∞-category $\mathcal{D}$ and a set of objects $S \subset \mathrm{Ob}(\mathcal{D})$, let $\langle S \rangle$ be the smallest full subcategory of $\mathcal{D}$ whose objects include S and which is stable under cones and shifts.

Definition 3.13. Define the Wakimoto category as the full subcategory $\mathrm{Wak} := \langle \mathcal{J} \rangle$ of $\mathcal{D}_{\text{ét}}(\mathrm{Hk}_{\mathcal{I}})$ generated by the essential image of $\mathcal{J}$ under cones. An object $\mathcal{F} \in \mathrm{Ob}(\mathrm{Wak})$ is called Wakimoto filtered.

By 3.12 the category $\mathrm{Wak} \subseteq \mathcal{D}_{\text{ét}}(\mathrm{Hk}_{\mathcal{I}})$ is stable under convolution. As it contains $\delta_e \cong \mathcal{J}_{\bar{\nu}}$ it is thus itself monoidal.

Remark 3.14. In the works [AB09, AR], their respective authors do not define the full subcategory $\mathrm{Wak} \subset \mathcal{D}_{\text{ét}}(\mathrm{Hk}_{\mathcal{I}})$, but rather a full subcategory $\mathcal{P}(\mathrm{Wak}) \subset \mathcal{P}(\mathrm{Hk}_{\mathcal{I}})$ consisting of those perverse sheaves that admit a filtration by perverse sheaves with grading in the essential image of $\mathcal{J}$. Morally, one can try to think of $\mathcal{P}(\mathrm{Wak})$ as the heart of Wak, but it is not an abelian category, only exact, and it is not true that every perverse sheaf that is Wakimoto filtered as a complex actually lies in $\mathcal{P}(\mathrm{Wak})$. Indeed, pick ν dominant with respect to B and let $\Delta_0 \subset \Delta_{-\nu}$ be the socle by [BBM04, Lemma 2.1]. Then, the cokernel lies in Wak, but its 0-th graded piece equals $\Delta_0[1]$, which is not perverse.

We give the following simple criterion for determing whether an object of $\mathcal{D}_{\text{ét}}(\mathrm{Hk}_{\mathcal{I}})$ lies in Wak.

Proposition 3.15. *Let $\mathcal{F} \in \mathcal{D}_{\mathrm{cons}}(\mathrm{Hk}_{\mathcal{I}})$. Then, the following are equivalent:*

(1) *$\mathcal{F}$ is Wakimoto filtered;*
(2) $\mathrm{supp}(\mathcal{J}_{-\bar{\nu}} * \mathcal{F}) \subset \{t_{-\bar{\mu}} \colon \bar{\mu} \in \mathbb{X}_\bullet^+\}$ *for all $\bar{\nu} \ll 0$;*
(3) $\mathrm{cosupp}(\mathcal{J}_{\bar{\nu}} * \mathcal{F}) \subset \{t_{\bar{\mu}} \colon \bar{\mu} \in \mathbb{X}_\bullet^+\}$ *for all $\bar{\nu} \gg 0$.*

In particular, if $\mathcal{F}$ satisfies $\mathcal{J}_{\bar{\nu}} * \mathcal{F} \cong \mathcal{F} * \mathcal{J}_{\bar{\nu}}$ for all $\nu \in \bar{\mathbb{X}}_\bullet$, then it is Wakimoto filtered.

Here, the notation $\bar{\nu} \gg 0$ means that $\langle \bar{\nu}, \alpha \rangle \gg 0$ for all B-positive relative roots α of G, while $\bar{\nu} \ll 0$ means $-\bar{\nu} \gg 0$.

Proof. Assume that $\mathcal{F}$ is Wakimoto filtered, and let us check that it satisfies (2) and (3). We may then assume that $\mathcal{F} = \mathcal{J}_{\bar{\nu}'}$ for some $\nu' \in \bar{\mathbb{X}}_\bullet$. If now $\nu \gg 0$, then

$$\mathcal{J}_{-\bar{\nu}} * \mathcal{F} \cong \Delta_{t_{-\bar{\nu}+\nu'}} \tag{3.31}$$

and the support claim follows. Similarly, we can argue for (3). Let us now assume that $\mathcal{F}$ satisfies (2). We want to show that $\mathcal{F}$ is Wakimoto filtered. Replacing $\mathcal{F}$ by $\mathcal{J}_{-\bar{\nu}} * \mathcal{F}$ for some suitable $\nu \gg 0$, we may assume that

$$\mathrm{supp}(\mathcal{F}) \subseteq \{t_{-\bar{\mu}} : \mu \in \bar{\mathbb{X}}_\bullet^+\}. \tag{3.32}$$

It is then formal that $\mathcal{F}$ lies in $\langle \Delta_{t_{-\bar{\mu}}} : \mu \in \bar{\mathbb{X}}_\bullet^+ \rangle$, cf. [AR, Lemma 4.4.3]. But

$$\langle \Delta_{t_{-\bar{\mu}}} : \mu \in \bar{\mathbb{X}}_\bullet^+ \rangle \subseteq \mathrm{Wak} \tag{3.33}$$

by the construction of Wakimoto sheaves. The argument that (3) implies (1) is similar.

For the last claim, let $A_{\mathcal{F}} \subseteq W$ be as in Proposition 3.7, i.e.,

$$\mathrm{supp}(\Delta_w * \mathcal{F}) \subseteq w \cdot A_{\mathcal{F}}, \quad \mathrm{supp}(\mathcal{F} * \Delta_w) \subseteq A_{\mathcal{F}} \cdot w \tag{3.34}$$

for all $w \in W$. As $\mathcal{J}_{-\bar{\nu}} * \mathcal{F} \cong \mathcal{F} * \mathcal{J}_{-\bar{\nu}}$ for $\bar{\nu} \in \bar{\mathbb{X}}_\bullet$, we can conclude that for $\nu \gg 0$

$$\mathrm{supp}(\mathcal{J}_{-\bar{\nu}} * \mathcal{F}) \subset t_{-\bar{\nu}} A_{\mathcal{F}} \cap A_{\mathcal{F}} t_{-\bar{\nu}}. \tag{3.35}$$

Now, we claim that for $\nu \gg 0$

$$t_{-\bar{\nu}} A_{\mathcal{F}} \cap A_{\mathcal{F}} t_{-\bar{\nu}} \subseteq \{ t_{-\bar{\mu}} \colon \bar{\mu} \in \mathbb{X}_\bullet^+ \}, \tag{3.36}$$

which would finish the proof. To check the claim let us recall that

$$\bar{\mathbb{X}}_\bullet^+ \cong W_{\mathrm{fin}} \backslash W / W_{\mathrm{fin}}. \tag{3.37}$$

If now $w \in t_{-\bar{\nu}} A_{\mathcal{F}} \cap A_{\mathcal{F}} t_{-\bar{\nu}}$, then we can write $w = t_{-\bar{\nu}} w_1 = w_2 t_{-\bar{\nu}}$ for $w_1, w_2 \in A_{\mathcal{F}} \subseteq W$, i.e.,

$$t_{\bar{\nu}} = w_1^{-1} t_{\bar{\nu}} w_2. \tag{3.38}$$

Given (3.37) and writing w_1, w_2 as a product of a translation and an element in the finite Weyl group, we can conclude that $w_1, w_2 \in \bar{\mathbb{X}}_\bullet$, i.e.,

$$t_{-\bar{\nu}} A_{\mathcal{F}} \cap A_{\mathcal{F}} t_{-\bar{\nu}} \subseteq \bar{\mathbb{X}}_\bullet. \tag{3.39}$$

As $A_{\mathcal{F}}$ is finite, we can conclude that for $\nu \gg 0$ we even get

$$t_{-\bar{\nu}} A_{\mathcal{F}} \cap A_{\mathcal{F}} t_{-\bar{\nu}} \subseteq \{ t_{-\bar{\mu}} : \mu \in \bar{\mathbb{X}}_\bullet^+ \} \tag{3.40}$$

as desired. □

Remark 3.16. In [AB09, Proposition 5] and [AR, Proposition 4.4.1], it is shown that a central perverse sheaf whose convolution functor is perverse t-exact lies in the category $\mathcal{P}(\mathrm{Wak})$. The proof given in those references is considerably more complicated, because of the need to ensure that the graded sheaves are actually perverse. Our proof is much simpler due to taking place in the derived setting, and later we will see how to recover the extra degree information required for perversity for the essential image of the Gaitsgory's central functor $\mathcal{Z}$.

For an arbitrary subset $\Omega \subset \bar{\mathbb{X}}_\bullet$, we can also define the full subcategory $\mathrm{Wak}_\Omega = \langle \mathcal{J}_\Omega \rangle$.

Proposition 3.17. *If $\Omega \subset \bar{\mathbb{X}}_\bullet$ is a lower poset (for $\preceq$), the inclusion $\mathrm{Wak}_\Omega \to \mathrm{Wak}$ has a right adjoint $\mathrm{Wak} \to \mathrm{Wak}_\Omega$, $\mathcal{F} \mapsto \mathcal{F}_\Omega$ such that the cone $\mathcal{G}$ of the adjunction unit $\mathcal{F}_\Omega \to \mathcal{F}$ lies in Wak and satisfies $\mathcal{G}_\Omega = 0$.*

Proof. Given $\mathcal{F} \in \mathrm{Ob}(\mathrm{Wak})$, we show the existence of a final morphism $\mathcal{F}_\Omega \to \mathcal{F}$, such that $\mathcal{F}_\Omega \in \mathrm{Ob}(\mathrm{Wak}_\Omega)$. By induction on the length of the filtration, we can write $\mathcal{F}$ as an extension of $\mathcal{G} = \mathcal{G}_\Omega$ by $\mathcal{J}_{\bar{\nu}}(M)$ for some $\bar{\nu} \in \bar{\mathbb{X}}_\bullet$ and $M \in \mathcal{D}^b_{\mathrm{ét}}(\mathrm{Spec}\, k)$. If $\bar{\nu} \in \Omega$, we are done. Otherwise, we have $\bar{\mu} \not\preceq \bar{\mu}$ for all $\mu \in \Omega$ by the lower set hypothesis on Ω, so we get $R\mathrm{Hom}(\mathcal{J}_{\bar{\nu}}(M), \mathcal{G}) = 0$ by Lemma 3.12 and hence there exists a (unique) splitting $\mathcal{F} \simeq \mathcal{J}_{\bar{\nu}}(M) \oplus \mathcal{G}$. One concludes that $\mathcal{F}_\Omega$ exists and identifies with $\mathcal{G}$ via the given map to $\mathcal{F}$ (again by Lemma 3.12). □

If Ω equals $\{\bar{\nu} \preceq \bar{\mu}\}$ resp. $\{\bar{\nu} \prec \bar{\mu}\}$ for some $\bar{\mu} \in \bar{\mathbb{X}}_\bullet$, we simply write $\mathrm{Wak}_{\preceq \bar{\mu}}$ resp. $\mathrm{Wak}_{\prec \bar{\mu}}$, instead of Wak_Ω. We can now define the Wakimoto gradeds for $\mathcal{F} \in \mathrm{Wak}$.

Definition 3.18. For any $\bar{\mu} \in \bar{\mathbb{X}}_\bullet$, we define the endofunctor

$$\mathrm{gr}_{\bar{\mu}} \colon \mathrm{Wak} \to \mathrm{Wak}, \mathcal{F} \mapsto \mathrm{cone}(\mathcal{F}_{\prec \bar{\mu}} \to \mathcal{F}_{\preceq \bar{\mu}}). \tag{3.41}$$

We also define $\mathrm{gr} := \oplus_{\bar{\mu} \in \bar{\mathbb{X}}_\bullet} \mathrm{gr}_{\bar{\mu}} \colon \mathrm{Wak} \to \mathrm{Wak}$.

Note that $\mathrm{gr}_{\bar\nu}(\mathcal{F})$ lies in the essential image of the functor $\mathcal{J}_{\bar\nu}\colon \mathcal{D}_{\mathrm{cons}}([*/S_k]) \to \mathrm{Wak}$. In the next subsection, we will show that this functor is fully faithful by explicitly constructing an inverse via constant terms of the opposite Borel, see 3.21. In particular, we can essentially uniquely lift $\mathrm{gr}_{\bar\nu}(\mathcal{F})$ to an element of $\mathcal{D}_{\mathrm{cons}}([*/S_k])$ and can make the following definition.

Definition 3.19. Let $\mathcal{F} \in \mathrm{Wak}$, we define

$$\mathrm{Grad}_{\bar\nu}(\mathcal{F}) \in \mathcal{D}_{\mathrm{cons}}([*/S_k]) \tag{3.42}$$

to be the canonical object such that $\mathcal{J}_{\bar\nu}(\mathrm{Grad}_{\bar\nu}(\mathcal{F}))$ identifies with $\mathrm{gr}_{\bar\nu}(\mathcal{F})$.

3.3. **Cohomology of Wakimoto filtered objects.** We now analyze the cohomology of objects in Wak. First, we show that convolution with Wakimoto sheaves induces a shift.

Proposition 3.20. *For any $\bar\nu \in \bar{\mathbb{X}}_\bullet$ and $\mathcal{F} \in \mathcal{D}_{\mathrm{cons}}(\mathrm{Hk}_{\mathcal{I}})$, there is a canonical isomorphism*

$$R\Gamma(\mathrm{Fl}_{\mathcal{I}}, \mathcal{F} * \mathcal{J}_{\bar\nu}) \simeq R\Gamma(\mathrm{Fl}_{\mathcal{I}}, \mathcal{F})[\langle 2\rho, \nu\rangle]. \tag{3.43}$$

Proof. Let us first assume that $\bar\nu \in \bar{\mathbb{X}}_\bullet^+$, which implies

$$\mathcal{J}_{\bar\nu}(\Lambda) = \nabla_{t_{\bar\nu}} = Rj_{w,*}(\Lambda)[\ell(t_{\bar\nu})]. \tag{3.44}$$

The map $(\mathrm{pr}_1, m)\colon \mathrm{Fl}_{\mathcal{I}} \tilde\times \mathrm{Fl}_{\mathcal{I}} \to \mathrm{Fl}_{\mathcal{I}} \times \mathrm{Fl}_{\mathcal{I}}$ is an isomorphism. The second projection $\mathrm{Fl}_{\mathcal{I}} \tilde\times \mathrm{Fl}_{\mathcal{I}} \to \mathrm{Hk}_{\mathcal{I}}$ is transformed to the map

$$\pi\colon \mathrm{Fl}_{\mathcal{I}} \times \mathrm{Fl}_{\mathcal{I}} \to \mathrm{Hk}_{\mathcal{I}},\ (\bar g, \bar h) \mapsto \overline{g^{-1}h}. \tag{3.45}$$

By definition $\mathcal{F} * \mathcal{J}_{\bar\nu}(\Lambda) \cong R\mathrm{pr}_{2,*}(\mathrm{pr}_1^*(\mathcal{F}) \otimes^L_\Lambda \pi^* \nabla_{\bar\nu}(\Lambda))$ and thus we get

$$\begin{aligned} &R\Gamma(\mathrm{Fl}_{\mathcal{I}}, \mathcal{F} * \mathcal{J}_{\bar\nu}(\Lambda)) \\ \simeq &R\Gamma(\mathrm{Fl}_{\mathcal{I}} \times \mathrm{Fl}_{\mathcal{I}}, \mathrm{pr}_1^*(\mathcal{F}) \otimes^L_\Lambda \pi^* \nabla_{t_{\bar\nu}}) \\ \simeq &R\Gamma(\mathrm{Fl}_{\mathcal{I}}, \mathcal{F} \otimes^L_\Lambda R\mathrm{pr}_{1,*}\pi^* \nabla_{t_{\bar\nu}}) \end{aligned}$$

Using that $R\Gamma(\mathrm{Fl}_{\mathcal{G}, \le t_{\bar\nu}}, \nabla_{t_{\bar\nu}}) \cong \Lambda[\ell(t_{\bar\nu})]$ one checks that the pullback of $R\mathrm{pr}_{1,*}\pi^*\nabla_{t_{\bar\nu}}$ along $LG \to \mathrm{Fl}_{\mathcal{I}}$ is isomorphic to $\Lambda[\ell(t_{\bar\nu})]$. As the object $R\mathrm{pr}_{1,*}\pi^*\nabla_{t_{\bar\nu}} \in \mathcal{D}_{\mathrm{\acute{e}t}}(\mathrm{Fl}_{\mathcal{G}})$ is LG-equivariant (because π and pr_1 are) we can conclude that $R\mathrm{pr}_{1,*}(\pi^*\nabla_{t_{\bar\nu}}) \cong \Lambda[\ell(t_{\bar\nu})]$. Moreover, we normalize this isomorphism such that over $1 \cdot I \in \mathrm{Fl}_{\mathcal{I}}$ it reduces to the canonical isomorphism $R\Gamma(\mathrm{Fl}_{\mathcal{G}, \le t_{\bar\nu}}, \nabla_{t_{\bar\nu}}) \cong R\Gamma(\mathrm{Fl}_{\mathcal{I}, t_{\bar\nu}}, \Lambda[\ell(t_{\bar\nu})]) \cong \Lambda[\ell(t_{\bar\nu})]$ (induced by adjunction). With this convention, the resulting isomorphism

$$R\Gamma(\mathrm{Fl}_{\mathcal{I}}, \mathcal{F} * \mathcal{J}_{\bar\nu}) \cong R\Gamma(\mathrm{Fl}_{\mathcal{I}}, \mathcal{F})[\ell(t_{\bar\nu})] \tag{3.46}$$

for $\bar\nu \in \mathbb{X}_\bullet^+$ is additive in $\bar\nu$. Thus, it can be extended to the desired natural isomorphism

$$R\Gamma(\mathrm{Fl}_{\mathcal{I}}, \mathcal{F} * \mathcal{J}_{\bar\nu}) \cong R\Gamma(\mathrm{Fl}_{\mathcal{I}}, \mathcal{F})[\langle 2\bar\rho, \bar\nu\rangle], \tag{3.47}$$

using [Zhu14, Lemma 9.1.] to see that $\ell(t_{\bar\nu}) = \langle 2\rho, \nu\rangle$ if $\bar\nu$ is dominant. $\square$

We immediately deduce the following two corollaries.

Corollary 3.21. *There is a canonical isomorphism*

$$R\Gamma(\mathrm{Fl}_{\mathcal{I}}, \mathcal{J}_{\bar\nu}(M)) \simeq M[\langle 2\bar\rho, \bar\nu\rangle] \tag{3.48}$$

for $M \in \mathcal{D}([*/S_k])$ and $\bar\nu \in \bar{\mathbb{X}}_\bullet$.

Proof. This follows from Corollary 3.20 by setting $\mathcal{F} = \mathcal{J}_0(\Lambda) = \delta_e$. $\square$

Corollary 3.22. *For any $\mathcal{F} \in \mathcal{P}(\mathrm{Wak})$, there is a canonical isomorphism*

$$H^n(\mathrm{Fl}_{\mathcal{I}}, \mathcal{F}) \simeq \bigoplus_{\langle 2\bar\rho, \bar\nu\rangle = -n} \mathrm{Grad}_{\bar\nu}(\mathcal{F}). \tag{3.49}$$

Proof. This result is analogous to [AR, Proposition 4.5.4]. The existence of a canonical isomorphism follows from Corollary 3.21 by using a filtration of $\mathcal{F}$ by Wakimoto sheaves. Note that the associated graded of $\mathcal{F}$ is perverse, so we conclude that, whenever $\bar{\nu}_1 \preceq \bar{\nu}_2$, then the cohomology complexes $R\Gamma(\mathrm{Fl}_{\mathcal{I}}, \mathrm{gr}_{\bar{\nu}_i}(\mathcal{F}))$ sit in different degrees with the same parity. This implies that the connecting homomorphisms of the associated long exact sequences vanish. $\square$

We wish to determine the Λ-module $\mathrm{Grad}_{\bar{\mu}}(\mathcal{F})$ in a functorial manner. For this we calculate constant terms.

Proposition 3.23. *For any $\mathcal{F} \in \mathcal{D}_{\text{ét}}(\mathrm{Hk}_{\mathcal{I}})$, $w \in W$ and $\bar{\nu} \in \mathbb{X}_\bullet$, there is a canonical identification*

$$\mathrm{CT}_{B^-}(\mathcal{J}_{\bar{\nu}} * \mathcal{F})_{t_{\bar{\nu}}w} \simeq \mathrm{CT}_{B^-}(\mathcal{F})_w[\langle 2\rho, \nu\rangle] \tag{3.50}$$

between stalks of constant term complexes.

Proof. First, we assume $\bar{\nu} \in \bar{\mathbb{X}}_\bullet^+$, so that $\mathcal{J}_{\bar{\nu}}(\Lambda) = \nabla_{t_{\bar{\nu}}}(\Lambda)$. Notice that

$$\nabla_{t_{\bar{\nu}}}(\Lambda) * \mathcal{F} \simeq Rm_*(\Lambda \tilde{\boxtimes} \bar{\nu}(\pi)_* \mathcal{F})[\langle 2\rho, \nu\rangle].$$

By Braden's theorem, the left side of (3.50) naturally identifies with cohomology supported at the corresponding LU-orbit $\mathcal{S}_{t_{\bar{\nu}}w}$. Since $\mathcal{S}_{t_{\bar{\nu}}}$ contains $\mathrm{Fl}_{\mathcal{I},t(\lambda)}$ by Lemma 2.12, the pullback of $\mathcal{S}_{t_{\bar{\nu}}w}$ to $\mathrm{Fl}_{\mathcal{I},\leq t(\lambda)} \tilde{\times} \mathrm{Fl}_{\mathcal{I}}$ identifies with $L^+\mathcal{U}\bar{\nu}(\pi)L^+\mathcal{U} \times^{L^+\mathcal{U}} \mathcal{S}_w$. By abuse of notation, we denote the latter scheme by $\mathrm{Fl}_{t(\lambda)} \tilde{\times} \mathcal{S}_w$, even though the twisted product is not for the Iwahori group.

Amassing all this information, we get that

$$R\Gamma_c(\mathcal{S}^-_{t_{\bar{\nu}}w}, \mathcal{J}_{\bar{\nu}}(\Lambda) * \mathcal{F}) \simeq R\Gamma(\mathrm{Fl}_{\mathcal{I},t_{\bar{\nu}}} \tilde{\times} \mathcal{S}_w, \Lambda \tilde{\boxtimes} \bar{\nu}(\pi)_* Ri_w^! \mathcal{F})[\langle 2\rho, \nu\rangle], \tag{3.51}$$

where we also applied base change to commute $Ri_w^!$ and Rm_*. Because $\mathrm{Fl}_{\mathcal{I},t_{\bar{\nu}}}$ is an orbit under the pro-unipotent group $L^+\mathcal{U}$, see again Lemma 2.12, the twisted product does not alter the cohomology complex, thereby yielding the desired claim. $\square$

Corollary 3.24. *We have a canonical isomorphism* $\mathrm{Grad}_{\bar{\nu}}(\mathcal{F})[\langle 2\rho, \nu\rangle] \simeq \mathrm{CT}_{B^-}(\mathcal{F})_{t_{\bar{\nu}}}$.

Proof. This follows by induction on $\bar{\nu}$, by considering the filtration $\mathcal{F}_{\preceq \bullet}$ and applying Proposition 3.23. $\square$

Remark 3.25. The corollary above tells us when $\mathrm{Grad}_{\bar{\nu}}(\mathcal{F})$ is perverse with some ease for $\mathcal{F} \in \mathrm{Wak}$. This corollary together with geometric Satake and constant terms is what will allow us to show that the central functor $\mathcal{Z}$ actually factors through $\mathcal{P}(\mathrm{Wak})$, thus bypassing the strategy of [AB09, Theorem 4, Proposition 5] and [AR, Proposition 4.4.1]. Indeed, if we know that $\mathrm{CT}_{B^-}(\mathcal{F})$ is perverse, then its associated graded is perverse, and we can write the Wakimoto complex $\mathcal{F}$ as an extension of perverse Wakimoto sheaves, so it lies in $\mathcal{P}(\mathrm{Wak})$.

We end this section by discussing the monoidal structure of the functor $\mathrm{Grad} := \bigoplus_{\bar{\nu} \in \bar{\mathbb{X}}_\bullet} \mathrm{Grad}_{\bar{\nu}}$ restricted to $\mathcal{P}(\mathrm{Wak})$.[1]

Lemma 3.26. *The full subcategories* $\mathcal{P}(\mathrm{Wak}) \subset \mathrm{Wak}$ *are stable under convolution.*

Proof. By induction on the number of non-zero graded pieces, we reduce to the case of the convolution of two Wakimoto complexes, but this is Lemma 3.12. $\square$

Proposition 3.27. *For any $\mathcal{F}, \mathcal{G} \in \mathcal{P}(\mathrm{Wak})$, and $\bar{\nu}_1, \bar{\nu}_2 \in \mathbb{X}_\bullet$ with $\bar{\nu} := \bar{\nu}_1 + \bar{\nu}_2$, there is a canonical morphism*

$$\beta_{\bar{\nu}_1,\bar{\nu}_2} : \mathrm{gr}_{\bar{\nu}_1}(\mathcal{F}) * \mathrm{gr}_{\bar{\nu}_2}(\mathcal{G}) \to \mathrm{gr}_{\bar{\nu}}(\mathcal{F} * \mathcal{G}). \tag{3.52}$$

such that $\bigoplus_{\bar{\nu}_1+\bar{\nu}_2=\bar{\nu}} \beta_{\bar{\nu}_1,\bar{\nu}_2}$ is an isomorphism.

[1] We however don't discuss an $\mathbb{E}_1$-monoidal structure of the functor Grad on Wak.

Proof. The statement is proved by induction on the number of non-vanishing $\mathrm{Grad}_{\bar{\nu}}$, similarly to the equicharacteristic case [AB09, Proposition 6a)] and [AR, Lemma 4.7.4, Proposition 4.7.5]. $\square$

Corollary 3.28. *For any $\mathcal{F}, \mathcal{G} \in \mathcal{P}(\mathrm{Wak})$, and $\bar{\nu}_1, \bar{\nu}_2 \in \bar{\mathbb{X}}_\bullet$ with $\bar{\nu} := \bar{\nu}_1 + \bar{\nu}_2$, there is a canonical morphism*

$$\alpha_{\bar{\nu}_1,\bar{\nu}_2} : \mathrm{Grad}_{\bar{\nu}_1}(\mathcal{F}) \otimes^L_\Lambda \mathrm{Grad}_{\bar{\nu}_2}(\mathcal{G}) \to \mathrm{Grad}_{\bar{\nu}}(\mathcal{F} * \mathcal{G}). \tag{3.53}$$

such that $\bigoplus_{\bar{\nu}_1+\bar{\nu}_2=\bar{\nu}} \alpha_{\bar{\nu}_1,\bar{\nu}_2}$ is an isomorphism. Consequently, for any such $\mathcal{F}$ and $\mathcal{G}$, there is a canonical isomorphism

$$\mathrm{Grad}(\mathcal{F}) \otimes^L_\Lambda \mathrm{Grad}(\mathcal{G}) \cong \mathrm{Grad}(\mathcal{F} * \mathcal{G}) \tag{3.54}$$

inside the category $\mathcal{D}_{\text{ét}}(\underline{\bar{\mathbb{X}}_\bullet})$.

Proof. This follows directly from proposition 3.27 and lemma 3.12 (3). $\square$

4. Central functor

4.1. **Background.** We introduce the spaces that underlie the construction of the Gaitsgory's central functor $\mathcal{Z}$ in mixed characteristic. The spaces are not of classical nature, and live in the world of v-stacks created by Scholze [Sch17, SW20]. We recall their basic properties, following [FS21, AGLR22].

We use the notation introduced in 1.5, but additionally assume that G is residually split, which implies that each $L^+\mathcal{I}$-orbit in $\mathrm{Fl}_{\mathcal{G}}$ is already defined over $\mathrm{Spec}(k)$, cf. 2.1.

Definition 4.1. The Hecke stack $\mathrm{Hk}_{\mathcal{G},O}$ is the v-stack sending a characteristic p affinoid perfectoid space $\mathrm{Spa}(R, R^+)$ to the groupoid of

- untilts $\mathrm{Spa}(R^\sharp, R^{\sharp,+})$ of $\mathrm{Spa}(R, R^+)$ over $\mathcal{O}$,
- $\mathcal{G}$-torsors $\mathcal{P}_1$ and $\mathcal{P}_2$ on $\mathrm{Spec}(B^+_{\mathrm{dR}}(R^\sharp))$ together with an isomorphism

$$\gamma : \mathcal{P}_1|_{\mathrm{Spec}(B_{\mathrm{dR}}(R^\sharp))} \cong \mathcal{P}_2|_{\mathrm{Spec}(B_{\mathrm{dR}}(R^\sharp))}. \tag{4.1}$$

We refer to [SW20, Section 20.3] for the definition of the rings $B^+_{\mathrm{dR}}(R^\sharp)$ and $B_{\mathrm{dR}}(R^\sharp)$.

An alternative way to define $\mathrm{Hk}_{\mathcal{G},O}$ is as the v-stack quotient

$$\mathrm{Hk}_{\mathcal{G},O} = [L^+_O\mathcal{G} \backslash \mathrm{Gr}_{\mathcal{G},O}] \tag{4.2}$$

where $L^+_O\mathcal{G}$ is the jet group over O, i.e., the v-group sheaf over $\mathrm{Spd}(O)$ with value $\mathcal{G}(B^+_{\mathrm{dR}}(R^\sharp))$ on untilts $\mathrm{Spa}(R^\sharp, R^{\sharp,+})$ over O, and

$$\mathrm{Gr}_{\mathcal{G},O} := L_O\mathcal{G}/L^+_O\mathcal{G} \tag{4.3}$$

with $L_O\mathcal{G}$ the loop group over O, i.e., the v-group sheaf over $\mathrm{Spd}(O)$ with value $\mathcal{G}(B_{\mathrm{dR}}(R^\sharp))$ on $\mathrm{Spa}(R^\sharp, R^{\sharp,+})$, cf. [AGLR22, Lemma 4.10].

Note that over the generic fiber $\eta = \mathrm{Spd}(F)$, $\mathrm{Gr}_{\mathcal{G},O}$ identifies with the B_{dR}-affine Grassmannian $\mathrm{Gr}_{G,F}$. On the other hand, over the special fiber $s = \mathrm{Spd}(k)$, $\mathrm{Gr}_{\mathcal{G},O}$ becomes isomorphic to the analytification $\mathrm{Fl}^\diamondsuit_{\mathcal{G}}$ of the Witt vector affine flag variety $\mathrm{Fl}_{\mathcal{G}}$.

Now we pick a complete algebraically closed extension C of F with residue field $\bar{k}$, and let $\bar{\eta} = \mathrm{Spd}(C)$, $\bar{s} = \mathrm{Spd}(\bar{k})$. Consider the natural diagram

$$\mathrm{Hk}_{\mathcal{G},C} \overset{j}{\hookrightarrow} \mathrm{Hk}_{\mathcal{G},O_C} \overset{i}{\hookleftarrow} \mathrm{Hk}_{\mathcal{G},\bar{k}}, \tag{4.4}$$

where j is the open immersion of the generic fiber and i the closed immersion of the special fiber. This induces a nearby cycles functor, see [AGLR22, Section 6.5],

$$R\Psi := i^* Rj_* : \mathcal{D}_{\text{ét}}(\mathrm{Hk}_{\mathcal{G},C}) \longrightarrow \mathcal{D}_{\text{ét}}(\mathrm{Hk}_{\mathcal{G},\bar{k}}) \tag{4.5}$$

between the stable ∞-categories of derived étale sheaves in the sense of Scholze [Sch17, Definition 14.13, Lemma 17.1] with bounded support as in Fargues–Scholze [FS21, Chapter VI]. For ℓ-adic coefficients, we follow the same conventions of passing to the limit as in [Sch17, Section 27] and then inverting ℓ, compare with [AGLR22, Section 6.5].

An important property of $R\Psi$ is that it preserves universal local acyclicity in the sense of [FS21, Section IV.2.1], see also [AGLR22, Section 6] for our convention for non-torsion coefficients Λ. Below, we denote by $\mathcal{D}_{\mathrm{ula}}(X/S) \subset \mathcal{D}_{\mathrm{ét}}(X)$ the full subcategory of universally locally acyclic sheaves (or, if the base is understood, simply $\mathcal{D}_{\mathrm{ula}}(X)$).

Proposition 4.2. *Nearby cycles $R\Psi$ restrict to a functor*

$$\mathcal{D}_{\mathrm{ula}}(\mathrm{Hk}_{\mathcal{G},C}) \to \mathcal{D}_{\mathrm{ula}}(\mathrm{Hk}_{\mathcal{G},\bar{k}}). \tag{4.6}$$

Proof. This is [AGLR22, Corollary 6.14]. □

Recall that in the previous sections of the paper, we introduced a Hecke stack $\mathrm{Hk}_{\mathcal{G}}^{\mathrm{sch}}$ as a perfect k-stack. Its associated v-sheaf under the functor $\diamondsuit$ is the fiber over $\mathrm{Spd} k$ of the analytic Hecke stack $\mathrm{Hk}_{\mathcal{G}}^{\mathrm{an}}$ that we defined over $\mathrm{Spd} O$. There is a natural comparison map of sheaves due to [Sch17, Section 27]

Proposition 4.3. *The natural comparison functor*

$$c\colon \mathcal{D}_{\mathrm{ét}}(\mathrm{Hk}_{\mathcal{G},\bar{k}}^{\mathrm{sch}}) \to \mathcal{D}_{\mathrm{ét}}(\mathrm{Hk}_{\mathcal{G},\bar{k}}^{\mathrm{an}}) \tag{4.7}$$

is an equivalence carrying $\mathcal{D}_{\mathrm{cons}}(\mathrm{Hk}_{\mathcal{G},\bar{k}}^{\mathrm{sch}})$ to $\mathcal{D}_{\mathrm{ula}}(\mathrm{Hk}_{\mathcal{G},\bar{k}}^{\mathrm{an}})$.

Proof. For the definition of the comparison functor, we refer to [Sch17, Section 27] and [AGLR22, Appendix A]. The above result is [AGLR22, Propositions 6.7 and A.5]. □

This result also highlights the importance of ula sheaves as singling out constructible sheaves over a base field. From now on, we will not make a stark distinction between $\mathrm{Hk}_{\mathcal{G}}^{\mathrm{sch}}$ and $\mathrm{Hk}_{\mathcal{G},k}^{\mathrm{an}}$ and will simply omit the superscript when writing down its derived category of sheaves.

Definition 4.4. We define the central functor (for the Witt vector affine flag variety) as the composition

$$\mathcal{Z}\colon \mathrm{Rep}(\hat{G}) \xrightarrow{\sim} \mathcal{P}_{\mathrm{ula}}(\mathrm{Hk}_{G,C}) \xrightarrow{R\Psi} \mathcal{D}_{\mathrm{ula}}(\mathrm{Hk}_{\mathcal{G},\bar{k}}) \tag{4.8}$$

Here, the first arrow comes from the geometric Satake equivalence of [FS21, Chapter VI], with the Satake category consisting of ula perverse sheaves on $\mathrm{Hk}_{G,C}$. The second arrow are just nearby cycles which respect the ula property by Proposition 4.2. Often below, we will find it convenient to still abusively denote by $\mathcal{Z}$ the nearby cycles functor $R\Psi\colon \mathcal{D}_{\mathrm{ula}}(\mathrm{Hk}_{G,C}) \to \mathcal{D}_{\mathrm{ula}}(\mathrm{Hk}_{\mathcal{G},\bar{k}})$.

Remark 4.5. As explained in [AGLR22, Section 8] the nearby cycles functor is Galois equivariant. More precisely, given $A \in \mathcal{P}_{\mathrm{ula}}(\mathrm{Hk}_{G,E})$ for some finite extension E/F, then $R\Psi(A_C)$ has a natural Γ_E-action that is equivariant with respect to the residual action of Γ_{k_E}. Here, $\Gamma_E \subseteq \Gamma$ denotes the Galois group of E, and Γ_{k_E} the one for the residue field k_E of E.

4.2. **Convolution and fusion.** In this section, we are going to discuss in detail the convolution and fusion products.

Definition 4.6. Given a finite linearly ordered set $J = \{j_1 < \cdots < j_n\}$, we define the convolution Hecke stack $\mathrm{Hk}_{\mathcal{G}}^{J}$ to be the v-sheaf over $\mathrm{Spd}\, O$ which classifies successive modifications of $\mathcal{G}$-torsors over B_{dR}^{+}, indexed by the elements $j_i \in J$. More precisely, for a given $f\colon S \to \mathrm{Spd}\, O$ the groupoid $\mathrm{Hk}_{\mathcal{G}}^{J}(S)$ is given by $\mathcal{G}$-torsors $\mathcal{P}_{j_1}, \ldots, \mathcal{P}_{j_n}$ on $B_{\mathrm{dR}}^{+}(S)$ with modifications $\mathcal{P}_{j_1} \dashrightarrow \mathcal{P}_{j_2}, \ldots, \mathcal{P}_{j_{n-1}} \dashrightarrow \mathcal{P}_{j_n}$ defined on $B_{\mathrm{dR}}(S)$.

One often finds the expression $\mathrm{Hk}_{\mathcal{G}}^{J} = \mathrm{Hk}_{\mathcal{G}}^{j_1} \tilde{\times} \ldots \tilde{\times} \mathrm{Hk}_{\mathcal{G}}^{j_n}$ to denote the convolution Hecke stack. We have already seen that there is a natural correspondence with $n = 2$ inducing the convolution product $* : \mathcal{D}_{\text{ét}}(\mathrm{Hk}_{\mathcal{G},S}) \times \mathcal{D}_{\text{ét}}(\mathrm{Hk}_{\mathcal{G},S}) \to \mathcal{D}_{\text{ét}}(\mathrm{Hk}_{\mathcal{G},S})$ for any $S \to \operatorname{Spd} O$, see 3.1. We wish to enhance this operation to a monoidal structure of ∞-categories in the sense of [Lur17, Definition 4.1.1.10]. This will be quite technical, and we recommend the unaccostumed reader to try to ignore the heavy language at first, and focus on the geometry at hand. After each categorical proof, we also provide an explanation of our constructions at the level of 1-morphisms of correspondences, which should prove helpful.

Let us recall some of the notions from [Lur17, Section 4.1]. First, we have the (nerve of) the 1-category $\mathrm{Comm}^{\otimes}$ (also denoted by Fin_* or $\mathbb{E}_{\infty}^{\otimes}$ in [Lur17]) whose objects are finite pointed sets $\langle n \rangle = \{0, 1, \ldots, n\}$ with base point 0 and whose morphisms $\langle n \rangle \to \langle m \rangle$ preserve 0. A symmetric monoidal ∞-category $\mathcal{C}$ is given by a cocartesian fibration $\mathcal{C}^{\otimes} \to \mathrm{Comm}^{\otimes}$ of ∞-operads, see [Lur17, Example 2.1.2.18], [Lur17, Definition 2.1.1.10], in particular, $\mathcal{C}^{\otimes}_{[n]} \simeq \mathcal{C}^n$ in a natural manner. Similarly, we have the ∞-operad $\mathrm{Assoc}^{\otimes}$ (which is equivalent to some other common ∞-operads denoted by $\mathbb{A}_{\infty}^{\otimes}$ or $\mathbb{E}_1^{\otimes}$ in [Lur17]) given as the 1-category whose objects are pointed finite set $\langle n \rangle$ and morphisms $\langle n \rangle \to \langle m \rangle$ are pointed maps equipped with a total order on the non-pointed fibers and composition is given by the lexicographical order, see [Lur17, Remark 4.1.1.4]. A monoidal ∞-category is a cocartesian fibration $\mathcal{C}^{\otimes} \to \mathrm{Assoc}^{\otimes}$ of ∞-operads and its underlying ∞-category $\mathcal{C}$ is the fiber of the fibration over $\langle 1 \rangle$. When the cocartesian fibration is clear from the context, we will often abuse language and refer to $\mathcal{C}$ as a monoidal ∞-category. Such a datum induces by [Lur17, Propositions 2.4.1.7, 2.4.2.5] a map $\mathrm{Assoc}^{\otimes} \to \mathrm{Cat}_{\infty}^{\times}$ that preserves inert morphisms in the sense of [Lur17, Definitions 2.1.1.8, 2.1.2.3] and also cocartesian morphisms, in particular $\mathcal{C}^{\otimes} \to \mathrm{Assoc}^{\otimes}$ induces an associative algebra in Cat_{∞}. Here, $\mathrm{Cat}_{\infty}^{\times}$ denotes Cat_{∞} with its cartesian symmetric monoidal structure, cf. [Lur17, Section 2.4.1]. Note that in general maps of ∞-operads are not necessarily monoidal, but rather only lax-monoidal, see [Lur17, Definition 2.1.3.7].

In order to produce the desired map that will induce a monoidal structure on $\mathcal{D}_{\text{ét}}(\mathrm{Hk}_{\mathcal{G},S})$, we recall that following [Man22a, Definition A.5.2] we dispose of a symmetric monoidal ∞-category $\mathrm{Corr}(\mathrm{vSt})$ of correspondences on v-stacks. The 6-functor formalism defined in [Sch17] for torsion coefficients can be reinterpreted as in [Man22a, Definition A.5.6] thanks to [Man22b, Theorem 5.11] via an operadic map

$$\mathcal{D}_{\text{ét}}^{\otimes} \colon \mathrm{Corr}^{\otimes}(\mathrm{vSt})_{\ell\text{-fine}} \to \mathrm{Cat}_{\infty}^{\times}, \tag{4.9}$$

where the ℓ-fine subscript indicates that we restrict to the full subcategory of $\mathrm{Corr}(\mathrm{vSt})$ whose correspondences have ℓ-fine maps to the right. We extend it to ℓ-adic coefficients via the naive construction of taking limits and tensoring with $\mathbb{Q}$, instead of using nuclear ℓ-adic sheaves, compare with [Sch17, Section 26] and [Man22b, page 6]. Note also that $\mathcal{D}_{\text{ét}}$ is a map of ∞-operads, and not symmetric monoidal (only lax symmetric monoidal). Despite this, all the maps obtained below between ∞-operads of either correspondences or sheaves will turn out to be monoidal.

Now, we are going to enhance $*$ to a monoidal structure on the ∞-category, by constructing a map $\mathcal{H}_S^{\otimes} \colon \mathrm{Assoc}^{\otimes} \to \mathrm{Corr}^{\otimes}(\mathrm{vSt})$ of ∞-operads that commutes with the maps towards $\mathrm{Comm}^{\otimes}$ and recovers the convolution $*$ on $\mathrm{Hk}_{\mathcal{G},S}$ via evaluation on the active morphism $\langle 2 \rangle \to \langle 1 \rangle$ (with order $1 < 2$). We were crucially assisted in this task by discussions with Heyer, Mann, and Zhao. Note that there is an obvious isomorphism

$$\mathrm{Hk}_{\mathcal{G},S} := L_S^+\mathcal{G} \backslash L_S\mathcal{G} / L_S^+\mathcal{G} \simeq [*/L_S^+\mathcal{G}] \times_{[*/L_S\mathcal{G}]} [*/L_S^+\mathcal{G}] \tag{4.10}$$

One can therefore realize $\mathrm{Hk}_{\mathcal{G},S}$ as the internal endomorphism object of $[*/L_S^+\mathcal{G}]$ in the ∞-category $\mathrm{Corr}(\mathrm{vSt}_{/[*/L_S\mathcal{G}]})$, and hence it inherits a natural ∞-monoidal structure in the category of correspondences by forgetting the slice over $[*/L_S\mathcal{G}]$. In fact, the functor $\mathrm{Corr}(\mathrm{vSt}_{/[*/L_S\mathcal{I}]}) \to$

Corr(vSt) is naturally symmetric monoidal. Note that the associated planar ∞-operad in the sense of [Lur17, Definition 4.1.3.2] is nothing other than the Čech nerve of the natural map $[*/L_S^+\mathcal{G}] \to [*/L_S\mathcal{G}]$.

Let us try to understand more closely what the map $\mathcal{H}_S^{\otimes}\colon \mathrm{Assoc}^{\otimes} \to \mathrm{Corr}^{\otimes}(\mathrm{vSt})$ induced by the above monoidal structure on $\mathrm{Hk}_{\mathcal{G},S}$ looks like. We send an object $\langle n\rangle$ to the fiber product $\mathrm{Hk}_{\mathcal{G},S}^n$ over S and the morphism $\alpha\colon \langle n\rangle \to \langle m\rangle$ to the correspondence

$$\mathrm{Hk}_{\mathcal{G},S}^n \leftarrow \mathrm{Hk}_{\mathcal{G},S}^{\alpha} \to \mathrm{Hk}_{\mathcal{G},S}^m, \tag{4.11}$$

where the middle term is the m-indexed product of the convolution Hecke stacks in the sense of 4.6 with superscripts ranging over the ordered fibers of α, the left map is the natural projection and the right map is the product of the natural multiplication. Note that the left map is a torsor for a power of $L_S^+\mathcal{G}$, and thus pro-smooth, whereas the right map is fibered in powers of $\mathrm{Gr}_{\mathcal{G},S}$ and hence it is ind-proper. One can also write down the image under $\mathcal{H}_S^{\otimes}$ of arbitrary n-morphisms of the 1-category $\mathrm{Assoc}^{\otimes}$, which are in bijection with sequences of composable morphisms.

In order to be able to apply the functor $\mathcal{D}_{\text{ét}}^{\otimes}$, we have to replace the convolution Hecke stacks by finite-dimensional truncations so that the maps to the right become ℓ-fine, but here we will ignore this subtlety and refer to [XZ17, Definition 5.1.2, Subsection 5.1.7] for a detailed treatment. We denote by $\mathcal{D}_{\text{ét}}^{\otimes}(\mathrm{Hk}_{\mathcal{G},S})$ the monoidal ∞-category obtained from composing $\mathcal{H}_S^{\otimes}$ and $\mathcal{D}_{\text{ét}}^{\otimes}$ (after taking appropriate truncations, so that this becomes legitimate). This clearly refines the convolution product $*$, as seen by taking one of the two active maps $\langle 2\rangle \to \langle 1\rangle$.

Lemma 4.7. *The full subcategory $\mathcal{D}_{\mathrm{ula}}(\mathrm{Hk}_{\mathcal{G},S})$ is stable under convolution.*

Proof. The reason is ula sheaves are preserved under smooth pullback, exterior products, and proper pushforward, which are the operations involved in the convolution product. □

By [Lur17, Proposition 2.2.1.1] on full subcategories of ∞-operads, we have a monoidal ∞-category $\mathcal{D}_{\mathrm{ula}}(\mathrm{Hk}_{\mathcal{G},S})$ giving rise to convolution.

Lemma 4.8. *Given a map $f\colon T \to S$, the pullback functor f^* is monoidal, i.e., it enhances essentially uniquely to a $\mathbb{E}_1$-monoidal map $f^{*,\otimes}\colon \mathcal{D}_{\text{ét}}^{\otimes}(\mathrm{Hk}_{\mathcal{G},S}) \to \mathcal{D}_{\text{ét}}^{\otimes}(\mathrm{Hk}_{\mathcal{G},T})$ (and similarly for ula sheaves).*

Proof. This is immediate because the pullback functor $f^*\colon \mathrm{Corr}(\mathrm{vSt}_S) \to \mathrm{Corr}(\mathrm{vSt}_T)$ is symmetric monoidal. □

Proposition 4.9. *The functor $\mathcal{Z}\colon \mathcal{D}_{\mathrm{ula}}(\mathrm{Hk}_{G,C}) \to \mathcal{D}_{\mathrm{ula}}(\mathrm{Hk}_{\mathcal{G},k})$ is monoidal, i.e., it enhances essentially uniquely to a $\mathbb{E}_1$-monoidal map $\mathcal{Z}^{\otimes}\colon \mathcal{D}_{\mathrm{ula}}^{\otimes}(\mathrm{Hk}_{G,C}) \to \mathcal{D}_{\mathrm{ula}}^{\otimes}(\mathrm{Hk}_{\mathcal{G},k})$.*

Proof. Recall that $\mathcal{Z} = i^*Rj_*$, where j and i denote the inclusion of the generic and special fibers of $\mathrm{Hk}_{\mathcal{I},O_C}$. We have seen that both pullback functors j^* and i^* are monoidal, thanks to Lemma 4.8. We claim that on ULA objects, j^* induces an isomorphism of ∞-operads. This can be checked at the level of underlying ∞-categories by [Lur17, Remark 2.1.3.8], and that statement is [AGLR22, Proposition 6.12]. □

There is a more general version of the Hecke stack that can be obtained by not taking $\mathrm{Spd}\, O$ as the base, but allowing products with itself over $\mathrm{Spd}\, k$.

Definition 4.10. Let $S_i \to \mathrm{Spd}\, O$, $i = 1, \dots, d$ be finitely many v-sheaves over O. We define the Hecke stack $\mathrm{Hk}_{\mathcal{G},S}$ with $S = S_1 \times \cdots \times S_d$ as the classifying stack of modifications of $\mathcal{G}$-bundles over the completion of the relative curve $\mathcal{Y}_S$ at the union of the d Cartier divisors specified by the d projections $S \to S_i$, see [FS21, Definition VI.1.6].

A similar variant exists for the convolution Hecke stacks, where one allows compositions of several modifications instead of modifying simultaneously at several divisors. We are now able to recall the fusion interpretation from [FS21, Section VI.9] that refines the convolution product and induces symmetry constraints on perverse sheaves. Recall that a perverse t-structure on $\mathcal{D}_{\text{ét}}(\mathrm{Hk}_{G,C})$ was defined by Fargues–Scholze in [FS21, Definition/Proposition VI.7.1].

During the rest of this subsection and the next one, we are going to abbreviate the categories $\mathcal{D}_{\text{ula}}(\mathrm{Hk}_{\mathcal{G},S})$ by $\mathcal{C}_S$, where S is some v-sheaf over $(\operatorname{Spd} O)^n$. If S is the product of the v-sheaf associated with Huber rings (R_i, R_i°) over O, then we will write $\mathcal{C}_{R_1\times\cdots\times R_n}$ for $\mathcal{C}_S$ so as to highlight each of the factors. The full subcategory of perverse sheaves will be abbreviated by $\mathcal{P}_S$ and $\mathcal{P}_{R_1\times\cdots\times R_n}$, respectively.

Proposition 4.11. *The full subcategory $\mathcal{P}_C \subset \mathcal{C}_C$ of perverse sheaves is stable under convolution and it extends to a symmetric monoidal ∞-category.*

Proof. Stability under convolution can be found in [FS21, Proposition VI.8.1] and the symmetric monoidal structure follows from [FS21, Definition/Proposition VI.9.4]. We explain the second part, which will prove useful later on. We have to prove that the bifunctor

$$\mathcal{P}_C \times \mathcal{P}_C \to \mathcal{P}_C \tag{4.12}$$

is monoidal, where the left side carries the monoidal structure. This will endow $\mathcal{P}_C$ with the structure of a braided monoidal category (but we will not check explicitly that it is symmetric). To get the braiding, we extend the map into two commutative triangles in Cat_∞:

$$\begin{tikzcd} & \mathcal{P}_{C^2}^{\neq} \\ \mathcal{P}_C \times \mathcal{P}_C \arrow[ur] \arrow[r] \arrow[dr] & \mathcal{P}_{C^2} \arrow[u] \arrow[d] \\ & \mathcal{P}_C \end{tikzcd} \tag{4.13}$$

where the vertical arrows are the pullbacks to the obvious strata of $(\operatorname{Spd} C)^2$ given by the diagonal and its complement, and the middle map is induced by the fusion correspondence

$$\mathrm{Hk}_{G,C}^2 \leftarrow \mathrm{Hk}_{G,C} \tilde{\times} \mathrm{Hk}_{G,C} \to \mathrm{Hk}_{G,C^2}. \tag{4.14}$$

The vertical maps are clearly monoidal and the upper one is fully faithful by [FS21, Proposition VI.9.3], so it suffices to see that the upper diagonal map is monoidal. Indeed, the loop groups $L^+_{C^2}G$ and $L_{C^2}G$ naturally factor into a product away from the diagonal, so we get an induced map $\mathcal{H}_C^{\otimes} \times \mathcal{H}_C^{\otimes} \leftarrow \mathcal{H}_{C^2}^{\otimes,\neq}$ of functors $\mathrm{Assoc}^{\otimes} \to \mathrm{Corr}^{\otimes}(\mathrm{vSt})$ regarded as a correspondence to the left by functoriality of endomorphisms objects. This yields our desired monoidal map upon applying $\mathcal{D}^{\otimes}$ and restricting to the monoidal subcategories of perverse sheaves. □

For the reader's convenience, let us explain more informally what is happening in the above proof. Let $\alpha\colon \langle n\rangle \to \langle m\rangle$ be a 1-morphism in $\mathrm{Assoc}^{\otimes}$. Notice that we have a composition of two correspondences, namely the fusion and the diagonal ones:

$$\mathrm{Hk}_{G,C}^n \leftarrow \mathrm{Hk}_{G,C^n}^{\alpha} \to \mathrm{Hk}_{G,C^n}^m \leftarrow \mathrm{Hk}_{G,C}^m \tag{4.15}$$

where the first two maps are the natural pro-smooth projection and ind-proper multiplication, and the last is a diagonal closed immersion. It is clear that the fiber product is the usual correspondence defining the monoidal structure on $\mathcal{D}_{\text{ula}}(\mathrm{Hk}_{\mathcal{G},S})$. Now the advantage of the first correspondence lies in the fact that, after excluding the partial diagonals, the stack $\mathrm{Hk}_{\mathcal{G},S}^{\alpha}$ decomposes as a product of regular Hecke stacks, so that the order of the modifications (in other words the ordering on the fibers of α) no longer matters. If we restrict to the full subcategory $\mathcal{P}_C \subset \mathcal{C}_C$ of perverse sheaves, then pullback $\mathcal{P}_{C^n} \to \mathcal{P}^{\neq}_{C^n}$ away from the union of the partial diagonals of $(\operatorname{Spd} C)^n$ is fully faithful, see [FS21, Proposition VI.9.3]. This yields the various symmetry constraints, as desired.

4.3. **Associative center.** Let $\mathcal{C}$ be a monoidal ∞-category. One may attach to $\mathcal{C}$ another monoidal ∞-category called its associative center and denoted $\mathcal{Z}(\mathcal{C})$. Observe that the ∞-category $\operatorname{End}(\mathcal{C}) = \operatorname{Fun}(\mathcal{C},\mathcal{C})$ is left-tensored over $\mathcal{C}$ via the latter's monoidal structure. We define $\mathcal{Z}(\mathcal{C}) := \operatorname{End}_{\mathcal{C}\times\mathcal{C}}(\mathcal{C})$ of $\mathcal{C}$-bilinear endomorphisms in the sense of [Lur17, Definition 4.6.2.7]. Since these are monoidal ∞-categories with tensor structure given by composition, and $\mathcal{C}$-bilinearity is stable under composition, we see that the full subcategory $\mathcal{Z}(\mathcal{C})$ inherits a monoidal structure. It comes equipped with a natural monoidal map $\mathrm{ev}_1\colon \mathcal{Z}(\mathcal{C}) \to \mathcal{C}$ given by evaluation at the monoidal unit. Also note that this definition coincides by [Lur17, Theorem 4.4.1.28, Theorem 5.3.1.30] with the center of an associative algebra of Cat_∞ in the sense of [Lur17, Definition 5.3.1.12].

Theorem 4.12. *The monoidal functor $\mathcal{Z}\colon \mathcal{C}_C \to \mathcal{C}_k$ lifts monoidally to the center $\mathcal{Z}(\mathcal{C}_k)$.*

Proof. According to [Lur17, Definition 5.3.1.12], this amounts to showing that the left action morphism $\mathcal{Z}_{\mathrm{lc}} := * \circ (\mathcal{Z}, \mathrm{id})\colon \mathcal{C}_C \times \mathcal{C}_k \to \mathcal{C}_k$ is monoidal, where the left side is an associative algebra in Cat_∞ by multiplying coordinates separately. We consider the following union of two commutative squares in Cat_∞:

$$\begin{tikzcd}
\mathcal{C}_C \times \mathcal{C}_k \arrow[r] & \mathcal{C}_{C\times k} \\
\mathcal{C}_{O_C} \times \mathcal{C}_k \arrow[r] \arrow[u] \arrow[d] & \mathcal{C}_{O_C\times k} \arrow[u] \arrow[d] \\
\mathcal{C}_k \times \mathcal{C}_k \arrow[r] & \mathcal{C}_{k^2}
\end{tikzcd} \tag{4.16}$$

where the vertical maps are pullback functors and hence clearly monoidal, and the horizontal maps are given by the fusion product (and therefore are not a priori monoidal). Note also that $\mathcal{C}_{k^2} = \mathcal{C}_k$ as we take products over k itself and so the lower horizontal map is simply convolution. Since the left upper morphism is an equivalence, we recover $\mathcal{Z}_{\mathrm{lc}}$ by taking an inverse and composing across the left lower edge of the diagram. Since the right upper map is fully faithful by Lemma 4.13 below, it suffices to monoidally enhance the upper horizontal map. But this follows as in the case of the fusion map $\mathcal{P}^2_C \to \mathcal{P}^{\neq}_{C^2}$ of perverse sheaves away from the diagonal: indeed, the loop groups $L^+_{C\times k}\mathcal{G}$ and $L_{C\times k}\mathcal{G}$ split as a direct product of the loop groups over C and k, so we get an equivalence by functoriality of endomorphism objects. □

Again for the reader's convenience, we repeat our explanation of our reasoning in terms of 1-morphisms of $\mathrm{Assoc}^\otimes$. We have to see that morphisms $\alpha\colon \langle n\rangle \to \langle m\rangle$ in $\mathrm{Assoc}^\otimes$ are naturally intertwined with $\mathcal{Z}_{\mathrm{lc}}$, i.e., that the diagram below

$$\begin{tikzcd}
\mathcal{C}^n_C \times \mathcal{C}^n_k \arrow[r] \arrow[d] & \mathcal{C}^m_C \times \mathcal{C}^m_k \arrow[d] \\
\mathcal{C}^n_k \arrow[r] & \mathcal{C}^m_k
\end{tikzcd} \tag{4.17}$$

commutes, where the vertical maps are powers of $\mathcal{Z}_{\mathrm{lc}}$ and the horizontal ones are induced by α.

Notice that the composition across the right arises from the composition of correspondences

$$\mathrm{Hk}^n_{\mathcal{G},O_C} \times \mathrm{Hk}^n_{\mathcal{G},k} \leftarrow \mathrm{Hk}^{\gamma[m]\circ\alpha[2]}_{\mathcal{G},O^n_C\times k^n} \to \mathrm{Hk}^m_{\mathcal{G},O^n_C\times k^n} \tag{4.18}$$

where $\gamma : \langle 2\rangle \to \langle 1\rangle$ is active carrying the usual order, $\gamma[m]\colon \langle 2m\rangle \to \langle m\rangle$ denotes its concatenation, and similarly for $\alpha[2]\colon \langle 2n\rangle \to \langle 2m\rangle$. The composition across the left arises instead from the correspondence

$$\mathrm{Hk}^n_{\mathcal{G},O_C} \times \mathrm{Hk}^n_{\mathcal{G},k} \leftarrow \mathrm{Hk}^{\alpha\circ\gamma[n]}_{\mathcal{I},O_C^n\times k^n} \to \mathrm{Hk}^m_{\mathcal{G},O_C^n\times k^n}. \tag{4.19}$$

Indeed, we can invoke the monoidal equivalence $\mathcal{C}_C \simeq \mathcal{C}_{O_C}$ proved in Proposition 4.9 and [AGLR22, Proposition 6.12], apply the monoidal functor $\mathcal{D}$ to the previous correspondences, and then compose with the pullback i^*.

In order to verify that these maps are naturally isomorphic, we must be able to swap the contribution of each O_C-factor adjacent to a k-factor. Thanks again to the equivalence $\mathcal{C}_{O_C} \simeq \mathcal{C}_C$ of [AGLR22, Proposition 6.12] and the fully faithful embeddings $\mathcal{C}_{O_C^n\times k^n} \subset \mathcal{C}_{C^n\times k^n}$ of Lemma 4.13 proved below, we are reduced to comparing the maps after taking the pullback functor j^*. But since $\operatorname{Spd} C$ and $\operatorname{Spd} k$ map disjointly to $\operatorname{Spd} O$, both convolution Hecke stacks become isomorphic to $\mathrm{Hk}^\alpha_{G,C^n} \times \mathrm{Hk}^\alpha_{\mathcal{G},k^n}$, so the result is clear.

The following lemmas were used in the proof of Theorem 4.12:

Lemma 4.13. *The natural map $j^*\colon \mathcal{C}_{O_C^n\times k^m} \to \mathcal{C}_{C^n\times k^m}$ is fully faithful for any $n, m \geq 0$.*

Proof. We must show that the unit $A = Rj_*j^*A$ for every ULA sheaf. This follows from the ula property and the next lemma. □

Recall our shorthand notation for the various functors and categories defined over products of O_C, C, and k. In order to avoid cumbersome notation below involving Spd and lots of brackets, we apply this convention now to the point functor, so that $*_{O_C^n} := (\operatorname{Spd} O_C)^n$

Lemma 4.14. *If $j\colon *_{C^n} := (\operatorname{Spd} C)^n \to (\operatorname{Spd} O_C)^n =: *_{O_C^n}$, then $Rj_*\Lambda = \Lambda$.*

Proof. For reasons that will become clear during our induction argument, we replace the exponent n by a finite set J during our proof. If $|J| = 1$, this follows already from [GL24, Theorem 4.7] applied to the kimberlite $*_{O_C}$, since its reduction equals $*_k$ and hence nearby cycles are per definition algebraic, so they can be calculated via the étale site for kimberlites, which is trivial.

If $|J| = 2$, then we first compute the stalk of $Rj_*\Lambda$ at $*_{k\times C}$. We know that partially compactly supported cohomology vanishes by [FS21, Theorem IV.5.3], so $R\Gamma(*_{O_C\times C}, j_!\Lambda) = 0$, compare with [FS21, Proposition V.4.2, Remark V.4.3]. This means that our sought stalk is given by $R\Gamma(*_{C^2}, \Lambda)$ which coincides with Λ thanks to [Sch17, Theorem 19.5]. It remains to compute the stalk at the reduction $*_k$ of the kimberlite $*_{O_C^2}$, so we apply [GL24, Theorem 4.7] once again.

Finally, in the general case, we can stratify $*_{O_C^J}$ by locally closed subsets of the form $*_{C^K}$ where $K \subset J$. We prove the equality $Rj_*\Lambda = \Lambda$ on the analytic strata (i.e., with K being non-empty) by descending induction on the cardinality of K. If $K = J$, there is nothing to show. Otherwise, consider the open set $*_{O_C^{J\setminus K}\times C^K}$ and observe again by [FS21, Theorem IV.5.3] that $R\pi_*i_!\Lambda = 0$ where

$$*_{O_C^{J\setminus K'}\times C^{K'}} \xrightarrow{i} *_{O_C^{J\setminus K}\times C^K} \xrightarrow{\pi} *_{O_C^{J\setminus K'}\times C^K}, \tag{4.20}$$

and $K \subset K'$ has singleton complement. This implies the claim regarding the stratum $*_{C^K}$ again thanks to [Sch17, Theorem 19.5]. As for the non-analytic point $*_k$ of $*_{O_C^J}$, we invoke [GL24, Theorem 4.7] again for the last time. □

Next, we prove that the symmetry constraints that appear in the full subcategory $\mathcal{P}_C \subset \mathcal{C}_C$ of perverse sheaves are compatible with the braidings in the associative center $\mathcal{Z}(\mathcal{C}_k)$. While $\mathcal{P}_C$ is symmetric monoidal, $\mathcal{Z}(\mathcal{C}_k)$ is not. Instead, the associative center carries a structure over the ∞-operad $\mathbb{E}_2^\otimes$ of little squares, see [Lur17, Definition 5.1.0.2]. This arises more formally as the tensor product in Op_∞ of $\mathbb{E}_1^\otimes$ with itself, see [Lur17, Theorem 5.1.2.2]. Here, we identify $\mathbb{E}_1^\otimes$ with $\mathrm{Assoc}^\otimes$ via [Lur17, Example 5.1.0.7]. Our assertion that associative centers carry an

$\mathbb{E}_2^{\otimes}$-structure is [Lur17, Remark 5.3.1.13], which explains that they can be regarded as associative algebras in the category of associative algebras of Cat_∞, the extra associative structure arising by bilinearity.

Theorem 4.15. *The composite $\mathcal{P}_C \subset \mathcal{C}_C \xrightarrow{\mathcal{Z}} \mathcal{Z}(\mathcal{C}_k)$ is an $\mathbb{E}_2$-monoidal map.*

Proof. Our goal is verifying that the monoidal map $\mathcal{P}_C \to \mathcal{Z}(\mathcal{C}_k)$ actually respects the extra monoidal structures on both sides in the ∞-category of associative algebras. This amounts to checking by the universal property of the center that the following commutative square

$$\begin{tikzcd} \mathcal{P}_C \times \mathcal{P}_C \arrow[r] \arrow[d] & \mathcal{P}_C \times \mathcal{C}_k \arrow[d] \\ \mathcal{P}_C \arrow[r] & \mathcal{C}_k \end{tikzcd} \tag{4.21}$$

in Cat_∞ is actually a commutative square in $\mathrm{Alg}_{\mathbb{E}_1^{\otimes}}(\mathrm{Cat}_\infty^{\times})$, where the maps are the obvious ones induced by convolution or $\mathcal{Z}$ and their monoidal enhancements were defined in Proposition 4.11 and Theorem 4.12.

Let us recapitulate how the braiding isomorphisms were constructed. For $\mathcal{P}_C$, we saw during Proposition 4.11 how to define a monoidal structure on the left vertical map via a pair of commuting triangles. In the special fiber, we saw during Theorem 4.12 how to define a monoidal structure on the right vertical map (actually, we took the larger category $\mathcal{C}_C$ instead of just $\mathcal{P}_C$) via a pair of commuting squares.

We must now perform these constructions at once in such a way that they are intertwined. Indeed, we have the following pair of commuting triangles

$$\begin{tikzcd} & \mathcal{C}^{\neq}_{O_C^2} \\ \mathcal{C}_{O_C} \times \mathcal{C}_{O_C} \arrow[ur] \arrow[r] \arrow[dr] & \mathcal{C}_{O_C^2} \arrow[u] \arrow[d] \\ & \mathcal{C}_{O_C} \end{tikzcd} \tag{4.22}$$

which relate to the previously constructed diagrams via natural pullback functors and passing to certain full subcategories. More precisely, restricting to C^2 and to perverse sheaves recovers the diagram (4.13), while restricting to $O_C \times k$ recovers the diagram (4.16) up to composing across the upper left and the lower left corners. Now, the upper vertical map is not fully faithful, and so we need to restrict to a full subcategory of sheaves where that happens. It suffices to take the category $\mathcal{E}_{O_C^2}$ of sheaves which are perverse over C^2 by Lemma 4.16 below. $\square$

Lemma 4.16. *Denote by $\mathcal{E}_{O_C^n}$ the ∞-category given as the fiber product $\mathcal{C}_{O_C^n} \times_{\mathcal{C}_{C^n}} \mathcal{P}_{C^n}$. Then, the pullback functor $\mathcal{E}_{O_C^n} \to \mathcal{E}^{\neq}_{O_C^n}$ is fully faithful.*

Proof. We have a commutative diagram

$$\begin{tikzcd} \mathcal{E}_{O_C^n} \arrow[r] \arrow[d] & \mathcal{E}^{\neq}_{O_C^n} \arrow[d] \\ \mathcal{P}_{C^n} \arrow[r] & \mathcal{P}^{\neq}_{C^n} \end{tikzcd} \tag{4.23}$$

The left arrow is fully faithful, because it is base changed from $\mathcal{C}_{O_C^n} \subset \mathcal{C}_{C^n}$ along $\mathcal{P}_{C^n} \to \mathcal{C}_{C^n}$ as proved in Lemma 4.13. The bottom arrow is fully faithful by [FS21, Lemma VI.9.3]. To show full faithfulness of the right arrow, it suffices to handle the map $\mathcal{C}^{\neq}_{O_C^n} \to \mathcal{C}^{\neq}_{C^n}$. By the ula property, we are reduced to showing that the derived pushforward of the constant sheaf along $*^{\neq}_{C^n} \to *^{\neq}_{O_C^n}$ is constant, which is also a consequence of Lemma 4.14. In particular, the top arrow is fully faithful. $\square$

4.4. **Perversity.** Recall that for every algebraically closed field C, we have a perverse t-structure on $\mathcal{D}_{\text{ét}}(\mathrm{Hk}_{\mathcal{G},C})$ given by strata dimension, see [FS21, Section VI.7] and [AGLR22]. This restricts to a t-structure on the full subcategory $\mathcal{D}_{\mathrm{ula}}(\mathrm{Hk}_{\mathcal{G},C})$ of ula sheaves, since Λ is a field[2]. It would be possible to define a relative perverse t-structure as in [FS21, Definition/Proposition VI.7.1], at least after restricting to ula sheaves, but we will not pursue this avenue here.

Our main result is the perverse t-exactness of $\mathcal{Z}$ at Iwahori level.

Theorem 4.17. *Assume $\mathcal{G} = \mathcal{I}$ is Iwahori. Let $B \subset G$ be an arbitrary Borel subgroup. The complex $\mathcal{Z}(V)$ is a Wakimoto-filtered perverse sheaf with graded isomorphic to $\mathcal{J}(V|_{\hat{T}^\Gamma})$.*

Proof. By Theorem 4.12, we see that $\mathcal{Z}(V)$ lies in the essential image of the obvious evaluation functor coming from the associative center $\mathcal{Z}(\mathcal{D}_{\mathrm{ula}}(\mathrm{Hk}_{\mathcal{I},k}))$. By Proposition 3.15, this implies that $\mathcal{Z}(V)$ lies in the full subcategory Wak for our choice of Borel subgroup $B \subset G$. It remains to see that there is a canonical isomorphism

$$\mathrm{CT}_{B^-}(\mathcal{Z}(V))_{\bar{\nu}} \simeq V(w_0\bar{\nu})[\langle 2\rho, \nu\rangle], \tag{4.24}$$

where w_0 denotes the longest element of the finite absolute Weyl group of G. Indeed, we would then know by Corollary 3.24 that $\mathcal{Z}(V)$ is a perverse sheaf, because the same would hold for its Wakimoto grading. But notice that constant terms of $\mathcal{Z}(V)$ can be calculated applying geometric Satake in the generic fiber, see [AGLR22, Corollary 6.14, Equation (6.32)], which yields the desired answer. $\square$

Remark 4.18. There appears to be a discrepancy between the isomorphism $\mathrm{grad} \circ \mathcal{Z}(V) \simeq V|_{\hat{T}^\Gamma}$ and (4.24) due to the appearance of the longest element w_0 in the latter formula. However, this is due to the fact that we were implicitly using an identification of T with the universal Cartan of G, compare with [AR, Remark 1.1.10]. Conjugating the identification by w_0 will not change the $\hat{T}^I$-grading coming from geometric Satake, but will change the one coming from the Wakimoto filtration, thereby fixing the issue.

Next, we deduce a few important consequences from this theorem. We start by proving that $\mathcal{Z}(V)$ is perverse for general parahorics. This is based on a suggestion of Achar to Cass–van den Hoven–Scholbach, see [CvdHS24, Theorem 5.30].

Corollary 4.19. *Let $\mathcal{G}$ be an arbitrary parahoric. Then, $\mathcal{Z}(V)$ is a perverse sheaf.*

Proof. Pick a Borel subgroup $T \subset B \subset G$ such that the underlying euclidean roots of the affine roots vanishing on the facet $\mathbf{f}$ fixed by $\mathcal{G}(O)$ are positive with respect to B. One can easily check that $t_{\bar{\mu}}$ is right $W_{\mathbf{f}}$-minimal for all B-dominant $\bar{\mu}$, compare with [CvdHS24, Lemma 5.28], so the map $\mathrm{Fl}_{\mathcal{I},t_{\bar{\mu}}} \to \mathrm{Fl}_{(\mathcal{I},\mathcal{G}),t_{\bar{\mu}}}$ is an isomorphism under the same assumption. Let now $\bar{\nu}$ be an arbitrary coweight and write it as the difference $\bar{\nu}_1 - \bar{\nu}_2$ of two B-dominant coweights. Collecting the previous facts, we deduce that $R\pi_*\mathcal{J}^B_{\bar{\nu}} = \Delta_{\mathcal{I},t_{\bar{\nu}_2}} * \nabla_{(\mathcal{I},\mathcal{G}),t_{\bar{\nu}_1}}$, and hence $\mathcal{Z}(V)$ lies in non-negative perverse degrees by Lemma 3.5. Similarly, after replacing B by its inverse, we can see that $\mathcal{Z}(V)$ lies in non-positive degrees. $\square$

[2]Otherwise, the truncation functors do not generally preserve perfect complexes, an issue that already arises for the natural t-structure.

From now on, we always assume that the parahoric level $\mathcal{G} = \mathcal{I}$ is Iwahori. We say that a central perverse sheaf A is convolution exact if its left (equivalently right) convolution functor $\ell_A \colon \mathcal{D}_{\mathrm{ula}}(\mathrm{Hk}_{\mathcal{I},k}) \to \mathcal{D}_{\mathrm{ula}}(\mathrm{Hk}_{\mathcal{I},k})$ is t-exact for the perverse t-structure.

Corollary 4.20. *The central perverse sheaf $\mathcal{Z}(V)$ is convolution exact.*

Proof. Given an element w of the affine Weyl group, we can find a Borel $B \subset G$ such that $\ell(t_{\bar{\nu}}w) = \ell(t_{\bar{\nu}}) + \ell(w)$ for all $\bar{\nu} \gg 0$ with respect to B. Indeed, we can consider a minimal gallery from the $\mathcal{I}(O)$-stable alcove $\mathbf{a}$ to its Weyl translate $w\mathbf{a}$, and simply take B as the Borel corresponding to a Weyl chamber containing the vector given as the difference of the barycenters of $\mathbf{a}$ and $w\mathbf{a}$.

Now consider the complexes $\mathcal{J}_{\bar{\nu}} * \nabla_w$ for arbitrary $\bar{\nu}$, and notice that it equals the perverse sheaf $\Delta_{t_{-\bar{\nu}'}} * \nabla_{t_{\bar{\nu}''}w}$ if we choose $\bar{\nu} = \bar{\nu}'' - \bar{\nu}'$ and $\bar{\nu}'' \gg 0$. Here, we applied Lemma 3.12 and Lemma 3.8. Now, Theorem 4.17 states that the perverse sheaf $\mathcal{Z}(V)$ admits a filtration with subquotients isomorphic to a direct sum of $\mathcal{J}_{\bar{\nu}}$, hence implying that $\mathcal{Z}(V) * \nabla_w$ is perverse for any V. By a dual argument, the same result holds for Δ_w. Finally, we apply the fact that the iterated extensions of the non-positive shifts of ∇_w (resp. non-negative shifts of Δ_w) span the non-negative part ${}^p\mathcal{D}^{\geq 0}_{\mathrm{ula}}(\mathrm{Hk}_{\mathcal{I},k})$ (resp. the non-positive part ${}^p\mathcal{D}^{\leq 0}_{\mathrm{ula}}(\mathrm{Hk}_{\mathcal{I},k})$) of the perverse t-structure to deduce that $\ell_{\mathcal{Z}(V)}$ is indeed perverse t-exact. $\square$

In the following, we say that an endomorphism $\varphi : A \to A$ of an object A in an abelian category $\mathcal{C}$ is unipotent if $(\varphi - 1)^n = 0$ for some positive integer n. We say φ is quasi-unipotent if a power of φ is unipotent. Recall that $\mathcal{Z}(V)$ carries a natural I_E-action, where E is the reflex field of the representation V and $I_E \subseteq \Gamma_E$ the inertia subgroup, see Remark 4.5.

Corollary 4.21. *The I_E-action on the perverse sheaf $\mathcal{Z}(V)$ is given by quasi-unipotent automorphisms. Moreover, there exists a finite index subgroup $I' \subset I_E$ such that the action factors through its maximal pro-ℓ quotient. If G is split, then $I' = I_E = I$ acts unipotently on $\mathcal{Z}(V)$.*

Proof. Since I_E fixes a Borel subgroup $B \subset G$ defined over F, we conclude the I_E-action on $\mathcal{Z}(V)$ preserves the Wakimoto filtration and it acts on $\mathrm{Grad}_{\bar{\nu}}(\mathcal{Z}(V)) \simeq V(w_0\bar{\nu})$, compare with Theorem 4.17, via its natural action on the given weight space. Since $V(w_0\bar{\nu})$ equals the sum of the $V(w_0\nu)$ for all lifts ν of $\bar{\nu}$, we see that I_E acts on the Wakimoto sheaves by permuting those weight spaces. Let F' be a splitting field of G and note that its absolute Galois group I' acts trivially on V. In particular, the I'-action on $\mathcal{Z}(V)$ is unipotent. Note, moreover, that both the pro-p wild inertia, and the remaining prime-to-ℓ tame quotient must map trivially to an unipotent ℓ-adic group, so the I'-action factors through its maximal pro-ℓ quotient. $\square$

In particular, if Λ is an algebraic extension of $\mathbb{Q}_\ell$ and given an isomorphism between $\mathbb{Z}_\ell$ and the maximal pro-ℓ quotient of I', we deduce the existence of a canonical nilpotent morphism

$$\mathbf{n}_V : \mathcal{Z}(V) \longrightarrow \mathcal{Z}(V) \tag{4.25}$$

such that the action of $\gamma' \in I'$ on $\mathcal{Z}(V)$ is given by $\exp(t_\ell(\gamma')\mathbf{n}_V)$, where $t_\ell : I' \to \mathbb{Z}_\ell$ is the natural quotient map.

Corollary 4.22. *The isomorphism of functors* $\mathrm{Grad} \circ \mathcal{Z}(V) \simeq V|_{\hat{T}^I}$ *is monoidal.*

Proof. We first explain how to construct the monoidal structure of the restriction functor $V \mapsto V_{|\hat{T}^I}$ geometrically using constant terms following [Yu22, §6] and [ALRR24, §4]. Namely, for any

$\mathcal{A}, \mathcal{B} \in \mathcal{P}(\mathrm{Hk}_{G,C})$, we have isomorphisms

$$\mathrm{CT}_{B^-}(\mathcal{A} * \mathcal{B})_\nu \cong R\Gamma_c(S_\nu^-, \mathcal{A} * \mathcal{B}) \tag{4.26}$$
$$\cong \bigoplus_{\nu_1+\nu_2=\nu} R\Gamma_c(S_{\nu_1}^- \tilde{\times} S_{\nu_2}^-, \mathcal{A} \tilde{\boxtimes} \mathcal{B}) \tag{4.27}$$
$$\cong \bigoplus_{\nu_1+\nu_2=\nu} R\Gamma_c(S_{\nu_1}^-, \mathcal{A}) \otimes R\Gamma_c(S_{\nu_2}^-, \mathcal{B})) \tag{4.28}$$
$$\cong \bigoplus_{\nu_1+\nu_2=\nu} \mathrm{CT}_{B^-}(\mathcal{A})_{\nu_1} \otimes \mathrm{CT}_{B^-}(\mathcal{B})_{\nu_2}. \tag{4.29}$$

by combining [BR18, Theorem 5.9], [Yu22, Lemma 6.1], and [ALRR24, Corollary 4.16]. This monoidal structure coincides with the natural one on $V \mapsto V_{\hat{T}^I}$ under geometric Satake. However, strictly speaking, this construction is not quite complete, because on the one hand [Zhu17, Yu22] works with the Witt vector affine Grassmannian instead of the B_{dR}^+-affine Grassmannian, and moreover it would not be immediate that this monoidal structure is compatible with that of [FS21]. In order to fill this gap, we must use the equivalence between the Satake categories over C and $\bar{k}$ for the split group G_C in [FS21, VI.6.7] and a theorem of Bando [Ban22, Ban23] showing that it is monoidal, i.e., that pulling back the Satake category of [Zhu17, Yu22] to the context of [FS21] yields the same monoidal structure on perverse sheaves.

As nearby cycles commute with constant terms [FS21, Proposition IV.6.12], we get an equivalence $\mathcal{Z}_T \circ \mathrm{CT}_{B^-} \simeq \mathrm{CT}_{B^-} \circ \mathcal{Z}_G$, where the indices T and G denote the underlying group of the Hecke stack for which we take nearby cycles. In particular, we obtain a monoidal structure on the functor $\mathrm{CT}_{B^-} \circ \mathcal{Z}_G$ by composing the above monoidal structure on CT_{B^-} with the one on $\mathscr{Z}_T$ (which clearly coincides with the restriction along $\hat{T}^I \subseteq \hat{T}$ under geometric Satake for T resp. $\mathcal{T}$). Looking back to the construction of the monoidal structure on Wakimoto gradeds in Corollary 3.28, it made inductive use of the isomorphisms from Corollary 3.22 and Corollary 3.24. These resulted as well from decomposing twisted products of semi-infinite orbits, and so this monoidal structure on $\mathrm{CT}_{B^-} \circ \mathscr{Z}_G$ must coincide with the previous one above, which was constructed using geometric Satake. $\square$

4.5. **Highest weight arrows.** Let $\mu \in \bar{\mathbb{X}}_\bullet$ be a dominant coweight with respect to B. For a $\hat{G}_\Lambda$-representation V with a single highest weight μ, we see that $\mathcal{Z}(V)$ is supported on the μ-admissible locus $\mathcal{A}_{\mathcal{I},\mu}$, cf. [AGLR22, Theorem 6.16], which equals the union of the $\mathcal{I}(O)$-orbits of the translations $t(\bar{\nu})$ associated with weights ν of V. We are going to define a canonical map

$$\mathfrak{f}_V : \mathcal{Z}(V) \to \mathrm{gr}_{\bar{\mu}}\, \mathcal{Z}(V) \tag{4.30}$$

called the *highest weight arrow*, which geometrizes the projection onto the $\bar{\mu}$-weight space.

First, observe that we have the adjunction unit

$$\mathcal{Z}(V) \to Rj_{\bar{\mu},*} j_{\bar{\mu}}^* \mathcal{Z}(V). \tag{4.31}$$

But the restriction of $\mathcal{Z}(V)$ to the I-orbit $\mathrm{Fl}_{\mathcal{I},\bar{\mu}}$ is isomorphic to the local system with value $R\Gamma(\mathrm{Fl}_{\mathcal{I},\bar{\mu}}, \mathcal{Z}(V))$. On the other hand, we know by Lemma 2.12 that $\mathrm{Fl}_{\mathcal{I},\bar{\mu}}$ coincides with the intersection $\mathcal{A}_{\mathcal{I},\mu} \cap \mathcal{S}_{t(\bar{\mu})} = \mathrm{Fl}_{\mathcal{I},\leq\bar{\mu}} \cap \mathcal{S}_{t(\bar{\mu})}$. Therefore, Corollary 3.24 tells us that $j_{\bar{\mu}}^* \mathcal{Z}(V) \simeq \mathrm{Grad}_{\bar{\mu}}\, \mathcal{Z}(V)[\langle 2\rho, \mu\rangle]$ in natural fashion. In particular, we get $Rj_{\bar{\mu},*} j_{\bar{\mu}}^* \mathcal{Z}(V) \simeq \mathrm{gr}_{\bar{\mu}}\, \mathcal{Z}(V)$ and we obtain the desired highest weight arrow.

Proposition 4.23. *The highest weights arrows are symmetric monoidal, i.e., for V (resp. W) a representation of $\hat{G}_\Lambda$ with a single highest weight μ (resp. ν), there are natural identifications*

$\mathfrak{f}_V * \mathfrak{f}_W \simeq \mathfrak{f}_{V\otimes W} \simeq \mathfrak{f}_W * \mathfrak{f}_V$ *in the sense that the diagram*

$$\begin{tikzcd} \mathcal{Z}(V) * \mathcal{Z}(W) \arrow[r, "\sim"] \arrow[d, "\mathfrak{f}_V * \mathfrak{f}_W"] & \mathcal{Z}(V \otimes W) \arrow[d, "\mathfrak{f}_{V\otimes W}"] & \mathcal{Z}(W) * \mathcal{Z}(V) \arrow[l, "\sim"'] \arrow[d, "\mathfrak{f}_W * \mathfrak{f}_V"] \\ \mathrm{gr}_{\bar{\mu}} \mathcal{Z}(V) * \mathrm{gr}_{\bar{\nu}} \mathcal{Z}(W) \arrow[r, "\sim"] & \mathrm{gr}_{\overline{\mu+\nu}} \mathcal{Z}(V \otimes W) & \mathrm{gr}_{\bar{\nu}} \mathcal{Z}(W) * \mathrm{gr}_{\bar{\mu}} \mathcal{Z}(V) \arrow[l, "\sim"'] \end{tikzcd} \tag{4.32}$$

is commutative, where the horizontal isomorphisms in the first row stem from Proposition 4.9, and the isomorphisms in the second row are given by

$$\mathrm{gr}_{\bar{\mu}} \mathcal{Z}(V) * \mathrm{gr}_{\bar{\nu}} \mathcal{Z}(W) \simeq \mathrm{gr}_{\overline{\mu+\nu}}(\mathcal{Z}(V) * \mathcal{Z}(W)) \simeq \mathrm{gr}_{\overline{\mu+\nu}}(\mathcal{Z}(V \otimes W)) \tag{4.33}$$

with the first isomorphism given by Proposition 3.27.

Proof. By 4.22 we know that the composition $\mathrm{Grad} \circ \mathcal{Z}$ identifies with the restriction functor from $\mathrm{Rep}_\Lambda(\hat{G})$ to $\mathrm{Rep}_\Lambda(\hat{T}^I)$ as a tensor functor, so it is symmetric monoidal. Indeed, the monoidal structure of Sat comes from the monoidality of constant terms in the generic fiber Gr_G which is compatible with the one in the special fiber $\mathrm{Fl}_{\mathcal{I}}$, which was used in Corollary 3.28. Finally, we just have to remark that the adjunction unit is naturally symmetric monoidal as are the isomorphisms $Rj_{\bar{\mu},*}j^*_{\bar{\mu}}\mathcal{Z}(V) \simeq \mathrm{gr}_{\bar{\mu}} \mathcal{Z}(V)$. □

We also have the relation of $\mathfrak{f}_V$ with the monodromy operator.

Lemma 4.24. *Let V be a representation of $\hat{G}_\Lambda$, then we have*

$$\mathfrak{f}_V \circ \mathbf{n}_V = 0. \tag{4.34}$$

Proof. By definition, $\mathfrak{f}_V$ is the quotient map of $\mathcal{Z}(V)$ towards the final subquotient of the Wakimoto filtration, upon which I' acts trivially by geometric Satake, see 4.21. □

Moreover, we also have that n_V is monoidal with respect to V. Note that here the tensor product of two nilpotent operators n_A and n_B of objects A and B of a monoidal category is given by $n_A \otimes 1 + 1 \otimes n_B$.

Lemma 4.25. *The nilpotent endomorphisms $\mathbf{n}_V$ for $V \in \mathrm{Rep}_\Lambda(\hat{G})$ form a nilpotent monoidal endomorphism $\mathbf{n}$ of $\mathcal{Z}\colon \mathrm{Rep}_\Lambda(\hat{G}) \to \mathcal{P}(\mathrm{Hk}_{\mathcal{I},k})$.*

Proof. It is enough to observe that the monoidal structure in Proposition 4.9 is I'-equivariant, but this follows directly from the construction. □

4.6. **Mixed variant.** In this subsection, we are going to upgrade our previous work to the setting of mixed sheaves. We consider a p-adic field F with ring of integers O, a finite residue field k of cardinality q, and an absolute Galois group Γ. We continue to fix a quasi-split and residually split F-group G with a Iwahori O-model $\mathcal{I}$. In this subsection, we assume furthermore that Λ is an algebraic extension of $\mathbb{Q}_\ell$ and contains a preferred choice of square-root $\sqrt{q}$.

We need to introduce the Γ-equivariant derived category of étale sheaves on our preferred spaces. Note that the Deligne topos $X \times_s \eta$ for a finite type k-scheme X with compatible Γ-action defined in [SGA73], see also [HZ23], is the same as the étale topos of the stack $[\Gamma\backslash X_{\bar{k}}]$. We usually consider its stable derived category $\mathcal{D}_{\text{ét}}([\Gamma\backslash X_{\bar{k}}])$ which is equivalent to the stable derived category $\mathcal{D}_{\text{ét}}(X \times_s \eta)$ of Deligne defined in [SGA73], compare with the definitions in [HZ23, Appendix A]. Recall that we have a decisive notion of a mixed complex $A \in \mathcal{D}_{\text{ét}}([\Gamma\backslash X_{\bar{k}}])$ of weight $\leq w$ (resp. $\geq w$) in the sense of [HZ23, Definition 2.4.4]. The condition $\leq w$ is defined by requiring that $\mathcal{H}^i(\sigma^* A)$ have weights bounded by w in the sense of Deligne, where $\sigma\colon X \to [\Gamma\backslash X_{\bar{k}}]$ is induced by a section of the morphism of sets $\Gamma \to \mathrm{Gal}_k$. The weight bound is ultimately independent from σ, see [HZ23, Section 2.4] for a discussion. The condition $\geq w$ is defined in terms of $\leq w$ and Verdier duality for $X_{\bar{k}}$.

Again, we can define the mixed standard functor from $\mathcal{D}_{\text{ét}}([\Gamma\backslash *])$ towards $\mathcal{D}_{\text{ét}}([\Gamma\backslash \mathrm{Hk}_{\mathcal{I},\bar{k}}])$

$$\Delta_w^{\mathrm{mix}}\colon M \mapsto j_{w!}M\langle \ell(w)\rangle, \tag{4.35}$$

where $\langle d\rangle$ denotes the shift-twist operator $[d](\frac{d}{2})$, and the mixed costandard functor

$$\nabla_w^{\mathrm{mix}}\colon M \mapsto Rj_{w*}M\langle \ell(w)\rangle. \tag{4.36}$$

both of which preserve mixed perverse sheaves by Weil II. Lemma 3.2 and Lemma 3.3 generalize to the current setting, so that we can define the mixed Wakimoto functor $\mathcal{J}_{\bar{\nu}}^{\mathrm{mix}}\colon \mathcal{D}_{\text{ét}}([\Gamma\backslash *]) \to \mathcal{D}_{\text{ét}}([\Gamma\backslash \mathrm{Hk}_{\mathcal{I},\bar{k}}])$ mapping a weighted complex M of Λ-modules to the object representing

$$\mathcal{F} \mapsto \operatorname{colim} \operatorname{Hom}(\nabla_{t_{\bar{\nu}_1}}^{\mathrm{mix}}(M), \mathcal{F} * \nabla_{t_{\bar{\nu}_2}}^{\mathrm{mix}}(\Lambda)), \tag{4.37}$$

where $\bar{\nu}_1, \bar{\nu}_2 \in \bar{\mathbb{X}}_\bullet^+$ run over all those elements such that $\bar{\nu} = \bar{\nu}_1 - \bar{\nu}_2$. Again this sends a mixed weighted Λ-module to a mixed perverse sheaf, since the mixedness property is preserved under derived pushforward and pullback, whereas perversity was already verified in Lemma 3.6. We can define a notion of a mixed Wakimoto complex, as lying in the full subcategory $\mathrm{Wak}^{\mathrm{mix}}$ spanned by the essential image of $\mathcal{J}_{\bar{\nu}}^{\mathrm{mix}}$ on mixed Λ-modules. Similarly, we can generalize Proposition 3.23 to the mixed setting, in such a way that it allows us to determine the Wakimoto grading of such an object. The full subcategory $\mathcal{P}(\mathrm{Wak}^{\mathrm{mix}})$ consists of the objects in $\mathrm{Wak}^{\mathrm{mix}}$ whose gradeds are all perverse.

Note that by [AGLR22, Section 8], the functor of nearby cycles upgrades to the mixed setting

$$R\Psi^{\mathrm{mix}} := (i^{\mathrm{mix}})^* R(j^{\mathrm{mix}})_*\colon \mathcal{D}_{\text{ét}}([\Gamma\backslash \mathrm{Hk}_{G,C}]) \to \mathcal{D}_{\text{ét}}([\Gamma\backslash \mathrm{Hk}_{\mathcal{I},\bar{k}}]) \tag{4.38}$$

and composition with the functor $\mathrm{Rep}_\Lambda({}^L G) \to \mathcal{P}(\mathrm{Hk}_{G,F})$ defines the mixed central functor $\mathcal{Z}^{\mathrm{mix}}(-)$.

Theorem 4.26. *The mixed central functor $\mathcal{Z}^{\mathrm{mix}}$ lands in $\mathcal{P}(\mathrm{Wak}^{\mathrm{mix}})$. Concretely, the Wakimoto gradeds of $\mathcal{Z}^{\mathrm{mix}}(V)$ are canonically isomorphic to $\mathcal{J}_{\bar{\nu}}^{\mathrm{mix}}(V(w_0\bar{\nu}))$.*

Proof. The arguments of Theorem 4.17 apply in this case as well. □

Recall that there exists a unique exhaustive and separated filtration $\mathrm{Fil}_i^{\mathrm{M}}\mathcal{Z}(V)$ (called the monodromy filtration) on the perverse sheaf $\mathcal{Z}(V)$ such that $\mathbf{n}_V$ is a filtered operator of degree -2 inducing isomorphisms $\mathbf{n}_V^i\colon \mathrm{Gr}_i^{\mathrm{M}}\mathcal{Z}(V) \simeq \mathrm{Gr}_{-i}^{\mathrm{M}}\mathcal{Z}(V)$. This filtration descends by functoriality to the corresponding mixed object $\mathcal{Z}^{\mathrm{mix}}(V)$. On the other hand, the mixed perverse sheaf $\mathcal{Z}^{\mathrm{mix}}(V)$ admits a filtration $\mathrm{Fil}_i^{\mathrm{W}}\mathcal{Z}^{\mathrm{mix}}(V)$ in mixed perverse sheaves whose weights are at most i and whose gradeds $\mathrm{Gr}_i^{\mathrm{W}}\mathcal{Z}^{\mathrm{mix}}(V)$ are purely of weight i, see [BBDG18, Théorème 5.3.5] and [HZ23, Theorem 2.6.8]. We say following [HZ23] that $\mathcal{Z}^{\mathrm{mix}}(V)$ is monodromy-pure of weight 0 if these two filtrations coincide. We have the following local weight-monodromy conjecture:

Conjecture 4.27. *The mixed perverse sheaf $\mathcal{Z}^{\mathrm{mix}}(V)$ is monodromy-pure of weight* 0.

For finite-type schemes over a field, it is known that nearby cycles send pure sheaves of weight 0 to monodromy-pure sheaves of weight 0, by a theorem of Gabber [BB93, Theorem 5.1.2]. In mixed characteristic, this was partially generalized by Hansen–Zavyalov [HZ23] assuming the existence of an étale cover by rigid-analytic tubes that admit an étale map to a disk.

Proposition 4.28. *If G is split and every non-zero weight of V is minuscule, then Conjecture 4.27 holds true for $\mathcal{Z}(V)$.*

Proof. By semi-simplicity of the Satake category in characteristic 0, we may assume $V = V_\mu$ is the simple representation with highest weight μ. In particular, we know by the proof of [AGLR22, Theorem 7.21, 7.23], that the local model $M_{\mathcal{I},\mu}$ -defined as the v-sheaf closure of the Schubert cell for μ- is representable by a flat projective scheme $M^{\mathrm{sch}}_{\mathcal{I},\mu}$ over O. By functoriality, it also maps to the local model $\mathcal{G}/\mathcal{P}^-_\mu$ at hyperspecial level $\mathcal{G}$, which is smooth over O. Since the transition map is an isomorphism in the generic fiber, we deduce by pull-back an étale cover of G/P^-_μ by rigid-analytic tubes admitting étale maps to a disk. Therefore, we can apply [HZ23, Theorem 4.4.4]. □

5. Coherent functor

In this section, we assume that G is split, that Λ is an algebraic extension of $\mathbb{Q}_\ell$, and that $\mathcal{I}$ is the Iwahori O-model obtained as the dilatation of a split model G_O along the closed subgroup $B_k \to G_k$. Consider the Springer resolution

$$p_{\mathrm{Spr}} : \hat{\mathcal{N}}_{\mathrm{Spr}} = \hat{G} \times^{\hat{B}} \mathrm{Lie}\, \hat{U} \to \hat{\mathcal{N}} \subset \mathrm{Lie}\, \hat{G} \tag{5.1}$$

of the nilpotent cone $\hat{\mathcal{N}}$ defined over Λ. Observe that there are natural functors $\mathrm{Rep}_\Lambda \hat{G} \to \mathrm{Coh}([\hat{G}\backslash\hat{\mathcal{N}}_{\mathrm{Spr}}])$ given by $V \mapsto \mathcal{O} \boxtimes V$ and $\mathrm{Rep}_\Lambda \hat{T} \to \mathrm{Coh}([\hat{G}\backslash\hat{\mathcal{N}}_{\mathrm{Spr}}])$ given by $\nu \mapsto \mathcal{O}(\nu)$, where $[\hat{G}\backslash\hat{\mathcal{N}}_{\mathrm{Spr}}]$ denotes the quotient stack. We aim to construct a monoidal functor

$$\mathcal{F} \colon \mathrm{Perf}(\hat{G}\backslash\hat{\mathcal{N}}_{\mathrm{Spr}}) \to \mathcal{D}_{\mathrm{\acute{e}t}}(\mathrm{Hk}_{\mathcal{I}}) \tag{5.2}$$

of monoidal, stable ∞-categories. Here, the domain of $\mathcal{F}$ is the category of perfect complex on a smooth Artin Λ-stack, thus equivalently, the ∞-derived category of coherent sheaves, and the source of $\mathcal{F}$ is the ∞-derived category of étale Λ-sheaves on a perfect Artin k-stack. The functor $\mathcal{F}$ is supposed to extend both the Wakimoto functor $\mathcal{J}$ and the central functor $\mathcal{Z}$ in the sense that the composition of $\mathcal{F}$ with the functor $V \mapsto V \boxtimes \mathcal{O}$ on $\mathrm{Rep}_\Lambda(\hat{G})$ resp. the functor $\nu \mapsto \mathcal{O}(\nu)$ on $\mathrm{Rep}_\Lambda(\hat{T})$ is equivalent to $\mathcal{Z}$ resp. $\mathcal{J}$.

5.1. **Generalities on coherent sheaves.** Throughout this section, we continue to assume Λ is an algebraic extension of $\mathbb{Q}_\ell$ and we let $X = Y/H$ be the quotient stack of a finitely presented quasi-affine Λ-scheme acted upon by a reductive group H over Λ. Let us recall how to define the derived category $\mathcal{D}_{\mathrm{qc}}(X)$ of quasi-coherent sheaves on X. Recall that the category Mod_Y of $\mathcal{O}_Y$-module sheaves is Grothendieck abelian in the sense of [Lur17, Definition 1.3.5.1]. By [Lur17, Definition 1.3.5.8], this abelian category induces a stable ∞-category $\mathcal{D}(\mathrm{Mod}_Y)$ of $\mathcal{O}_Y$-modules on Y. It is naturally endowed with a t-structure in the sense of [Lur17, Definition 1.2.1.4] defined by non-vanishing degrees of its cohomology functors, see [Lur17, Definition 1.3.5.16]. Hence, we can define $\mathcal{D}_{\mathrm{qc}}(Y)$ (resp. $\mathcal{D}_{\mathrm{coh}}(Y)$) as the full subcategory spanned by complexes whose cohomologies are quasi-coherent (resp. coherent) $\mathcal{O}_Y$-modules. We now define $\mathcal{D}_{\mathrm{qc}}(X)$ (resp. $\mathcal{D}_{\mathrm{coh}}(X)$) as the limit of the simplicial object in ∞-categories $[n] \mapsto \mathcal{D}_{\mathrm{qc}}(Y^n_X)$ (resp. $[n] \mapsto \mathcal{D}_{\mathrm{coh}}(Y^n_X)$). The resulting ∞-categories are stable and carry natural t-structures, whose hearts will be denoted $\mathrm{QCoh}(X)$, resp. $\mathrm{Coh}(X)$.

We will decorate the derived categories by the superscripts $^b, ^+, ^-$ to denote the full subcategories of bounded, left-bounded, and right-bounded complexes. The full subcategory $\mathrm{Perf}(Y)$ of perfect complexes is spanned by bounded complexes with finite Tor-amplitude (i.e., those which are represented by finite complexes of vector bundles as Y is quasi-affine), and we define $\mathrm{Perf}(X)$ again by descent.

Notice that the Grothendieck abelian category $\mathrm{QCoh}(X)$ induces a stable ∞-category $\mathcal{D}(\mathrm{QCoh}(X))$ again by an application of [Lur17, Definition 1.3.5.8] with a natural t-structure. It will be often useful to relate this to $\mathcal{D}_{\mathrm{qc}}(X)$. There is an induced t-exact functor $\mathcal{D}(\mathrm{QCoh}(X)) \to \mathcal{D}_{\mathrm{qc}}(X)$ and under our assumptions, we get:

Proposition 5.1. *The functor $\mathcal{D}(\mathrm{QCoh}(X)) \to \mathcal{D}_{\mathrm{qc}}(X)$ is an equivalence.*

Proof. In virtue of the equivalence of [Lur17, Remark 1.2.1.18], it suffices to verify that the functor induces an equivalence of bounded categories. Essential surjectivity can be tested at the triangulated level, i.e., by taking homotopy categories. Similarly, full faithfulness amounts to checking isomorphism of homotopy groups of mapping spaces, which can be expressed in terms of Ext groups by [Lur17, Notation 1.1.2.17], so we can also verify it at the triangulated level. Since $\mathcal{D}_{\mathrm{qc}}(X)$ is compactly generated by [HR17, Theorem B], the claim now follows from [HNR19, Theorem 1.2]. □

In order to understand right-bounded complexes in the affine case, the following lemma is decisive.

Lemma 5.2. *If Y is affine, then the abelian category $\mathrm{Coh}(X)$ has enough projectives. In particular, the t-exact functor $\mathcal{D}^-(\mathrm{Coh}(X)) \to \mathcal{D}^-_{\mathrm{coh}}(X)$ is an equivalence.*

Proof. Let R, resp. A be the ring of global section on Y resp. H. The category of finitely generated R-modules (which is equivalent to $\mathrm{Coh}(Y)$) has enough projectives by considering the collection of free R-modules. Notice that the functor of taking H-invariants on R-modules is exact by assumption on H. We can deduce that the H-equivariant free R-module $V \otimes M$ with V being a finite dimensional representation of H is projective in $\mathrm{Coh}(X)$. This collection of projectives is enough, as each coherent sheaf on X is surjected upon by the n-fold sum of the regular representation $A \otimes R$ for $n \gg 0$, and we can find a finite representation $V \subset A$ which completes the job, by finiteness of the underlying R-module of the initial coherent sheaf on X. □

If the stack X is smooth, then we actually get an equality $\mathcal{D}^b_{\mathrm{coh}}(X) = \mathrm{Perf}(X)$ of full subcategories. This motivates our construction of the AB functor via the following equivariant analogue of the localization theorem originally due to Thomason–Trobaugh [TT90] and Neeman [Nee92].

Proposition 5.3. *Let $U \subset X$ be an open immersion with closed complement Z. Then $\mathrm{Perf}(U)$ is the idempotent-completion of the quotient $\mathrm{Perf}(X)/\mathrm{Perf}(X)_Z$, where the denominator indicates the full subcategory spanned by complexes supported in Z.*

Proof. This is [KR18, Theorem 3.4, Equation (3.6)] for the underlying triangulated categories, which implies the statement in general. Let us explain how one obtains the result. First, it is clear that $\mathcal{D}_{\mathrm{qc}}(U)$ is a localization of $\mathcal{D}_{\mathrm{qc}}(X)$ with kernel $\mathcal{D}_{\mathrm{qc}}(X)_Z$, because restriction admits a right adjoint given by pushforward with unit being an equivalence. Finally, since each of the categories involved are compactly generated by [HR17, Theorem B] with compact objects given exactly by perfect complexes by [HR17, Lemma 4.4], we can apply the localization theorem, see [HR17, Theorem 3.12], to obtain the claim. □

5.2. **Coherent sheaves on the Springer variety.** Recall that the variety $\hat{G}/\hat{U}$, which is a $\hat{T}$-torsor over $\hat{G}/\hat{B}$, is quasi-affine[3] , so it embeds openly in the spectrum $\hat{\mathcal{X}}$ of its global sections $\mathcal{O}(\hat{G}/\hat{U})$. In turn, these admit the following explicit description as a graded Λ-algebra

$$\mathcal{O}(\hat{G}/\hat{U}) = \bigoplus_{\mu \in \mathbb{X}_\bullet^+} V_\mu \tag{5.3}$$

where V_μ denotes the highest weight representation of highest weight μ and multiplication is given by the obvious maps $V_{\mu_1} \otimes V_{\mu_2} \to V_{\mu_1+\mu_2}$, see [AR, Lemma 6.2.1]. In particular, the above Λ-algebra is finitely generated.

[3]By the construction of quotients via fixed vectors in representations, any quotient of an affine scheme of finite type over a field by a unipotent group scheme is quasi-affine. For details on $\hat{G}/\hat{U}$ see [AR, Subsection 6.2.1].

Similarly, we can define the following $\hat{T}$-torsor

$$\hat{\mathcal{N}}_{\mathrm{Spr}}^{\mathrm{qaf}} = \hat{G} \times^{\hat{U}} \mathrm{Lie}(\hat{U}) \tag{5.4}$$

over the Springer resolution, which is a quasi-affine scheme with an action of $\hat{G}' := \hat{G} \times \hat{T}$. The Lie algebra $\hat{\mathfrak{g}}$ of $\hat{G}$ acts naturally via derivations on the structure sheaf of $\hat{G}/\hat{U}$, see [AR, Equation (6.2.8)] and we can associate to it the so-called infinitesimal universal stabilizer $\hat{\mathcal{N}}_{\mathrm{Spr}}^{\mathrm{af}}$ as the closed subscheme of $\hat{\mathfrak{g}} \times \hat{\mathcal{X}}$ given by the image of the derivation map. Note that, even though the intersection of $\hat{\mathcal{N}}_{\mathrm{Spr}}^{\mathrm{af}}$ with $\hat{\mathfrak{g}} \times \hat{G}/\hat{U}$ is exactly $\hat{\mathcal{N}}_{\mathrm{Spr}}^{\mathrm{qaf}}$, it is not generally true that $\hat{\mathcal{N}}_{\mathrm{Spr}}^{\mathrm{af}}$ coincides with the scheme-theoretic closure of the locally closed immersion $\hat{\mathcal{N}}_{\mathrm{Spr}}^{\mathrm{qaf}} \to \hat{\mathfrak{g}} \times \hat{\mathcal{X}}$. The latter is an integral variety admitting $\hat{\mathcal{N}}_{\mathrm{Spr}}^{\mathrm{qaf}}$ as a dense open subset, with ideal of definition given by the kernel of $\mathcal{O}(\hat{\mathfrak{g}}) \times \mathcal{O}(\hat{\mathcal{X}}) \to \mathcal{O}(\hat{\mathcal{N}}_{\mathrm{Spr}}^{\mathrm{qaf}})$.

We have two distinguished collections of generators for the derived category of $[\hat{G}\backslash\hat{\mathcal{N}}_{\mathrm{Spr}}]$.

Lemma 5.4. *The derived category $\mathcal{D}^b_{\mathrm{coh}}([\hat{G}\backslash\hat{\mathcal{N}}_{\mathrm{Spr}}])$ is spanned by the set of the line bundles $\mathcal{O}(\nu)$ for $\nu \in \mathbb{X}_\bullet$, or by the set of the vector bundles $V \otimes \mathcal{O}(\nu)$ for $V \in \mathrm{Rep}\,\hat{G}$ and $\nu \in \mathbb{X}_\bullet^+$.*

Proof. This is [Bez09, Lemma 21]. See also [AR, Lemma 6.2.7]. □

Originally, it was claimed in [AB09, Lemma 20] that the triangulated category $\mathcal{D}^b_{\mathrm{coh}}([\hat{G}\backslash\hat{\mathcal{N}}_{\mathrm{Spr}}])$ is a Verdier quotient of $\mathrm{Perf}([\hat{G}'\backslash\hat{\mathcal{N}}_{\mathrm{Spr}}^{\mathrm{af}}])$. Upon expanding the argument in [AR, Proposition 6.2.8], we noticed that it seemed to rely on density of $\hat{\mathcal{N}}_{\mathrm{Spr}}^{\mathrm{qaf}} \subset \hat{\mathcal{N}}_{\mathrm{Spr}}^{\mathrm{af}}$, which unfortunately fails in general. Instead, we will argue below via [AR, Remark 6.3.10].

5.3. **Construction of the AB functor.** As in this whole section, we assume that $\mathcal{I}$ is the standard Iwahori attached to the fixed Borel B of the pinned split group G. We recall also the notation $G' = G \times T$ and $\hat{G}' = \hat{G} \times \hat{T}$. First, we start with the functor

$$\mathcal{Z}' := \mathcal{Z} \times \mathcal{J} \colon \mathrm{Rep}_\Lambda(\hat{G}') \to \mathcal{D}_{\mathrm{ula}}(\mathrm{Hk}_{\mathcal{I}}) \tag{5.5}$$

which has a natural monoidal structure[4] and factors through the full subcategory of $\mathcal{P}(\mathrm{Wak})$ consisting of Wakimoto-filtered perverse sheaves. However, this is still not good enough, because the convolution of Wakimoto-filtered perverse sheaves is not *symmetric* in general.

In order to fix this, we consider the (non-full!) subcategory $\mathcal{C}$ of $\mathcal{P}(\mathrm{Hk}_{\mathcal{I}})$ whose objects are those in the image of $\mathcal{Z}'$ and whose morphisms commute with the images along $\mathcal{Z}'$ of the symmetry isomorphisms of $\mathrm{Rep}_\Lambda(\hat{G}')$. This is a symmetric monoidal category by definition, see [AR, Lemma 6.3.3]. Consider the following Λ-algebra

$$A = \mathrm{Hom}_{\mathrm{Ind}\mathcal{C}}(1_{\mathcal{C}}, \mathcal{Z}'(\mathcal{O}(\hat{G}'))) \tag{5.6}$$

where the multiplication is induced by that of the group $\hat{G}'$, and $\mathcal{O}(\hat{G}')$ is a $\hat{G}'$-representation via conjugation. By [AR, Proposition 6.3.5], this defines an identification between $\mathcal{C}$ and the category of free A-modules with $\hat{G}'$-equivariant structure of the form $V \otimes_\Lambda A$ where V is a finite dimensional Λ-representation of $\hat{G}'$.

Next, we construct a Λ-algebra homomorphism

$$\mathcal{O}(\hat{\mathcal{N}}_{\mathrm{Spr}}^{\mathrm{af}}) \to A \tag{5.7}$$

that is equivariant with respect to the $\hat{G}'$-module structures. Via the $\hat{G}'$-equivariant embedding $\mathcal{O}(\hat{\mathcal{N}}_{\mathrm{Spr}}^{\mathrm{af}}) \to \mathfrak{g} \times \hat{\mathcal{X}}$, we start by handling each of these two factors separately (following closely the respective part in [AR, Section 6.3]).

[4]Even in the $\mathbb{E}_1$-monoidal sense: The functors $\mathcal{Z}$ and $\mathcal{J}$ are $\mathbb{E}_1$-monoidal, and $\mathcal{Z}$ is central, see 4.12. This implies the existence of $\mathcal{Z}'$ by the definition of $\mathbb{E}_1$-centers, see [Lur17, Definition 5.3.1.2].

For any $\hat{G}$-representation V, we extend it to a $\hat{G}'$-representation $V' = V \boxtimes 1$ by letting $\hat{T}$ act trivially and consider the logarithm of the monodromy n_V acting on $\mathcal{Z}(V) = \mathcal{Z}'(V')$. The collection of these endomorphisms defines a map of Λ-algebras $\mathcal{O}(\hat{\mathfrak{g}}) \to A$ which is $\hat{G}'$-equivariant. For details we refer to [AR, Example 6.3.1] and Lemma 4.25.

Next, we need to define a map of $\hat{G}'$-modules $V_\nu \boxtimes -\nu \to A$ and the natural source for this is the highest weight arrow f_ν provided by the Wakimoto filtration, see 4.5. It defines a morphism in $\mathcal{C}$ by the already checked compatibilities, so applying the description of $\mathcal{C}$ in terms of A yields a map $V'_\nu \otimes A \to 1 \boxtimes \nu \otimes A$ which corresponds to our goal after twisting by ν and restricting the domain on the left.

In total, we have thus constructed a $\hat{G}'$-module homomorphism

$$\mathcal{O}(\hat{\mathfrak{g}} \times \hat{\mathcal{X}}) \to A. \tag{5.8}$$

However, we are still left with the task of showing that this factors over the coordinate ring of the affine enlargement $\hat{\mathcal{N}}^{\mathrm{af}}_{\mathrm{Spr}}$, which is a closed subscheme of $\hat{\mathfrak{g}} \times \hat{\mathcal{X}}$ of the Springer bundle.

Lemma 5.5. *The $\hat{G}'$-equivariant map (5.8) factors uniquely through a $\hat{G}'$-equivariant map*

$$\mathcal{O}(\hat{\mathcal{N}}^{\mathrm{af}}_{\mathrm{Spr}}) \to A. \tag{5.9}$$

Proof. Here, we follow [AR, Lemma 6.3.7]. We know that the highest weight arrow is equivariant with respect to the monodromy operator. Passing to the logarithm, we see as in 4.24 that $f_\nu \circ \mathbf{n}_\nu = 0$. This equality holds true in the auxiliary category $\mathcal{C}$ (in fact, the monodromy action on $\mathcal{Z}$ factors through $\mathcal{C}$). Comparing with [AR, Example 6.3.1], we conclude from this identity that the definition ideal of $\hat{\mathcal{N}}^{\mathrm{af}}_{\mathrm{Spr}}$ inside $\mathfrak{g} \times \hat{\mathcal{X}}$ vanishes under the map to A. □

So far, we have arrived at a functor

$$\tilde{F} \colon \mathrm{Coh}_{\mathrm{fr}}([\hat{G}' \backslash \hat{\mathcal{N}}^{\mathrm{af}}_{\mathrm{Spr}}]) \to \mathcal{C}, \tag{5.10}$$

where $\mathrm{Coh}_{\mathrm{fr}}$ denotes the full subcategory of $\mathrm{Coh}([\hat{G}' \backslash \hat{\mathcal{N}}^{\mathrm{af}}_{\mathrm{Spr}}])$ spanned by the objects $V \otimes \mathcal{O}$ for $V \in \mathrm{Rep}_\Lambda(\hat{G})$. Now we are going to show that this functor passes to the actual Springer resolution $\hat{\mathcal{N}}_{\mathrm{Spr}}$.

Lemma 5.6. *The functor $\tilde{F}$ composed with the Wakimoto grading functor* gr *from 3.18 identifies with the pullback functor of coherent sheaves along the morphism*

$$[\hat{T} \backslash e] \to [\hat{G} \backslash \hat{\mathcal{N}}_{\mathrm{Spr}}] \cong [\hat{G}' \backslash \hat{\mathcal{N}}^{\mathrm{qaf}}_{\mathrm{Spr}}] \subseteq [\hat{G}' \backslash \hat{\mathcal{N}}^{\mathrm{qaf}}_{\mathrm{Spr}}].$$

Here, $e := \mathrm{Spec}(\Lambda) \to \hat{\mathcal{N}}_{\mathrm{Spr}} = \hat{G} \times^{\hat{B}} \mathrm{Lie}(\hat{U})$ denotes the point $[(1, 0)]$.

Proof. We follow the proof in [AR, Lemma 6.3.8]. It suffices to understand the corresponding $\hat{T}$-equivariant map of Λ-algebras $\mathcal{O}(\hat{\mathcal{N}}^{\mathrm{af}}_{\mathrm{Spr}}) \to \Lambda$. But the monodromy acts trivially on the Wakimoto grading as we saw in 4.21, and the highest weight arrow is projects to V_λ to the highest weight space $V_\lambda(\lambda)$. Hence the sought homomorphism is just evaluation at the origin e. □

Proposition 5.7. *There is a unique monoidal functor of stable ∞-categories up to equivalence*

$$\mathcal{F} \colon \mathrm{Perf}([\hat{G} \backslash \hat{\mathcal{N}}_{\mathrm{Spr}}]) \to \mathcal{D}_{\mathrm{cons}}(\mathrm{Hk}_{\mathcal{I}}) \tag{5.11}$$

extending $\tilde{F}$.

Proof. This is [AR, Proposition 6.3.9, Remark 6.3.10] in the triangulated setting and we follow their argument.

Since $\mathrm{Coh}_{\mathrm{fr}}([\tilde{G}'\backslash\hat{\mathcal{N}}^{\mathrm{af}}_{\mathrm{Spr}}])$ consists of compact, projective generators of $\mathrm{Coh}([\tilde{G}'\backslash\hat{\mathcal{N}}^{\mathrm{af}}_{\mathrm{Spr}}])$ by Lemma 5.2, left Kan extension of the composition $\mathrm{Coh}_{\mathrm{fr}}([\tilde{G}'\backslash\hat{\mathcal{N}}^{\mathrm{af}}_{\mathrm{Spr}}]) \xrightarrow{\tilde{F}} \mathcal{C} \to \mathcal{D}_{\text{ét}}(\mathrm{Hk}_{\mathcal{I}})$ yields the exact "left-derived" functor $L\tilde{F}\colon \mathcal{D}^{\leq 0}(\mathrm{Coh}([\tilde{G}'\backslash\hat{\mathcal{N}}^{\mathrm{af}}_{\mathrm{Spr}}])) \to \mathcal{D}_{\text{ét}}(\mathrm{Hk}_{\mathcal{I}})$ as in [Lur17, Theorem 1.3.3.2]. This functor formally extends to an exact functor on $\mathcal{D}^{-}(\mathrm{Coh}([\tilde{G}'\backslash\hat{\mathcal{N}}^{\mathrm{af}}_{\mathrm{Spr}}]))$. Thanks to the equivalence from Proposition 5.1 and after restricting to perfect complexes, we get a functor

$$\tilde{\mathcal{F}} : \mathrm{Perf}([\hat{G}'\backslash\hat{\mathcal{N}}^{\mathrm{af}}_{\mathrm{Spr}}]) \to \mathcal{D}_{\mathrm{cons}}(\mathrm{Hk}_{\mathcal{I}}). \tag{5.12}$$

This functor is monoidal because it can be written as the composition of the monoidal functor $\mathcal{C}^b(\tilde{F})$, where $\mathcal{C}^b$ denotes the associated ∞-category of complexes, followed by the restricted realization functor $\mathcal{C}^b(\mathcal{P}(\mathrm{Wak})) \to \mathcal{D}_{\mathrm{cons}}(\mathrm{Hk}_{\mathcal{I}})$, which is monoidal because $\mathcal{P}(\mathrm{Wak})$ is a full subcategory of $\mathcal{D}_{\mathrm{cons}}(\mathrm{Hk}_{\mathcal{I}})$ closed under convolution. Proposition 5.3 implies that the category $\mathrm{Perf}([\hat{G}\backslash\hat{\mathcal{N}}_{\mathrm{Spr}}])$ is the idempotent-completion of the quotient of $\mathrm{Perf}([\hat{G}'\backslash\hat{\mathcal{N}}^{\mathrm{af}}_{\mathrm{Spr}}])$ by the full subcategory of those perfect complexes supported on the complement. Since $\mathcal{D}_{\mathrm{cons}}(\mathrm{Hk}_{\mathcal{I}})$ is idempotent-complete, we are reduced to showing that such a perfect complex lies in the kernel of $\tilde{\mathcal{F}}$. Since the image of $\tilde{\mathcal{F}}$ lies in Wak (this reduces to the same statement for $\tilde{F}$ as Wak is idempotent-complete), we can check acyclicity after passing to gradeds by first taking Wakimoto filtrations termwise and then inducting. But the grading functor corresponds at the coherent level to restriction to the origin of $\hat{\mathcal{N}}_{\mathrm{Spr}}$ by Lemma 5.6, hence the desired vanishing holds. □

6. Iwahori–Whittaker averaging

We continue to work with a pinned split F-group G with a fixed maximal torus T and a Borel B containing T (in particular, we do not regard them as being defined over O, unless indexed by O). We let $\mathcal{I}$ be the standard Iwahori O-model of G, i.e., such that $\mathcal{I}(O)$ fixes the origin of the apartment $\mathcal{A}(G,T)$ induced by the pinning and is contained in the B-dominant Weyl chamber.

We let $\mathcal{I}^{\mathrm{op}}$ denote the parahoric O-model opposite to $\mathcal{I}$ with respect to the origin of the apartment and the Borel, and simply call it the *opposite Iwahori*. In other words, $\mathcal{I}^{\mathrm{op}}(O)$ fixes the alcove opposite to the one fixed by $\mathcal{I}(O)$. We define likewise the *pro-p Iwahori* $\mathcal{I}_{\mathrm{u}}$ as the unique smooth affine O-model of G with connected geometric fibers whose O-valued points are the kernel of $\mathcal{I}(O) \to \mathcal{I}^{\mathrm{red}}_k(k)$, where the $\mathcal{I}^{\mathrm{red}}_k$ is the reductive quotient of the special fiber of $\mathcal{I}$.

Our next task is to choose a Whittaker datum. Assume Λ is an algebraic field extension of $\mathbb{Q}_\ell(\zeta)$ where $\zeta \in \bar{\mathbb{Q}}_\ell$ is a choice of a primitive p-th root of unity. We get the Artin–Schreier étale sheaf $\mathcal{L}_{\mathrm{AS}}$ on $\mathbb{G}_{a,k}$: this is the rank 1 direct summand of the pushforward $\pi_*\Lambda$ of the constant sheaf along the Artin–Schreier cover $\pi\colon \mathbb{G}_{a,k} \to \mathbb{G}_{a,k}$ arising as the ζ-eigenspace for the Galois action of $\mathbb{Z}/p\mathbb{Z}$. It is a character sheaf in the sense of Lusztig–Yun [LY20], i.e., we have isomorphisms $m^*\mathcal{L}_{\mathrm{AS}} \simeq \mathcal{L}_{\mathrm{AS}} \boxtimes \mathcal{L}_{\mathrm{AS}}$ and $e^*\mathcal{L}_{\mathrm{AS}} \cong \Lambda$ with respect to the multiplication m and unit e of $\mathbb{G}_{a,k}$, that satisfy associativity constraints (this is equivalent to the corresponding ∞-enhancement, because $\mathcal{P}(\mathbb{G}_{a,k})$ is an abelian category). Besides, it satisfies the following cohomological vanishing

$$R\Gamma(\mathbb{G}_{a,k}, \mathcal{L}_{\mathrm{AS}}) = R\Gamma_c(\mathbb{G}_{a,k}, \mathcal{L}_{\mathrm{AS}}) = 0, \tag{6.1}$$

which will turn out to be important later on.

Let U^{op} be the unipotent radical of the opposite Borel B^{op}. Consider the homomorphism $U^{\mathrm{op}}_k \to \mathbb{G}_{a,k}$ induced by the sum of the negative simple roots and let $\chi : L^+\mathcal{I}^{\mathrm{op}}_u \to \mathbb{G}_{a,k}$ be the homomorphism resulting from pre-composing the first one with the natural projection $L^+\mathcal{I}^{\mathrm{op}}_u \to U^{\mathrm{op}}_k$. Taking the pullback of $\mathcal{L}_{\mathrm{AS}}$ along χ, we get a character sheaf on $\mathcal{L}_{\mathcal{IW}} \in \mathcal{D}_{\mathrm{cons}}(L^+\mathcal{I}^{\mathrm{op}}_u)$. Indeed, this is the character sheaf attached to the cover $\pi_\chi\colon (L^+\mathcal{I}^{\mathrm{op}}_u)_{\mathrm{AS}} \to L^+\mathcal{I}^{\mathrm{op}}_u$ deduced from $\pi\colon \mathbb{G}_{a,k} \to \mathbb{G}_{a,k}$ by pullback along χ.

Definition 6.1. The derived category $\mathcal{D}_{\text{ét}}(\mathrm{Hk}_{\mathcal{IW}})$ of *Iwahori-Whittaker sheaves* is the ζ-isotypical component of the stable ∞-category $\mathcal{D}_{\text{ét}}([(L^+\mathcal{I}_u^{\text{op}})_{\text{AS}}\backslash \mathrm{Fl}_{\mathcal{I}}])$.

In the above definition, we are using the fact that the Λ-linear stable ∞-category $\mathcal{D}_{\text{ét}}([(L^+\mathcal{I}_u^{\text{op}})_{\text{AS}}\backslash \mathrm{Fl}_{\mathcal{I}}])$ has a $\mathbb{Z}/p\mathbb{Z}$-action coming from $\mathbb{Z}/p\mathbb{Z} \simeq \ker(\pi_\chi)$ and that it decomposes as a direct sum of full subcategories where $\mathbb{Z}/p\mathbb{Z}$ acts via a $\Lambda^\times$-valued character, since Λ has characteristic 0 and $\mu_p \subseteq \Lambda$. Note that no underlying stack $\mathrm{Hk}_{\mathcal{IW}}$ seems to exist, but we find this shorthand notation useful, and hope it does not cause any confusion to the reader. We could also define $\mathcal{D}_{\text{ét}}(\mathrm{Hk}_{\mathcal{IW}})$ as the ∞-category of $(L^+\mathcal{I}_{\mathrm{u}}^{\text{op}}, \mathcal{L}_{\mathcal{IW}})$-equivariant étale sheaves on $\mathrm{Fl}_{\mathcal{I}}$, obtained by twisting by the character sheaf $\mathcal{L}_{\mathcal{IW}}$ the cosimplicial diagram obtained by applying $\mathcal{D}_{\text{ét}}$ to the Čech complex of $\mathrm{Fl}_{\mathcal{I}} \to [L^+\mathcal{I}_{\mathrm{u}}^{\text{op}}\backslash \mathrm{Fl}_{\mathcal{I}}]$. In the end, it turns out that all of this is unnecessarily complicated, because:

Proposition 6.2. *The forgetful functor $\mathcal{D}_{\text{ét}}(\mathrm{Hk}_{\mathcal{IW}}) \to \mathcal{D}_{\text{ét}}(\mathrm{Fl}_{\mathcal{I}})$ is fully faithful.*

Proof. This is essentially [ARW16, Proposition A.5] and follows from the fact that $L^+\mathcal{I}_u^{\text{op}}$ is pro-unipotent and hence so is its Artin–Schreier cover. Thus, we can apply [FS21, Proposition VI.4.1]. □

The category of Iwahori–Whittaker sheaves inherits a perverse t-structure from its fully faithful embedding into $\mathcal{D}_{\text{ét}}(\mathrm{Fl}_{\mathcal{I}})$, so that one can consider its heart $\mathcal{P}(\mathrm{Hk}_{\mathcal{IW}})$, called the category of Iwahori–Whittaker perverse sheaves. The ∞-category $\mathcal{D}_{\text{ét}}(\mathrm{Hk}_{\mathcal{IW}})$ does not appear to be monoidal, but it is a right module of $\mathcal{D}_{\text{ét}}(\mathrm{Hk}_{\mathcal{I}})$ in the sense of [Lur17, Definition 4.2.1.13]. Indeed, we invoke the natural isomorphism

$$[(L^+\mathcal{I}_{\mathrm{u}}^{\text{op}})_{\text{AS}}\backslash \mathrm{Fl}_{\mathcal{I}}] \simeq [*/(L^+\mathcal{I}_{\mathrm{u}}^{\text{op}})_{\text{AS}}] \times_{[*/LG]} [*/L^+\mathcal{I}] \tag{6.2}$$

to identify our stack as a homomorphism object in $\mathrm{Corr}(\mathrm{vSt}_{/[*/LG]})$ with a natural right module structure under the endomorphism object $\mathrm{Hk}_{\mathcal{I}} \cong [*/L^+\mathcal{I}] \times_{[*/LG]} [*/L^+\mathcal{I}]$. Taking the symmetric monoidal forgetful functor $\mathrm{Corr}(\mathrm{vSt}_{[*/LG]}) \to \mathrm{Corr}(\mathrm{vSt})$ and applying $\mathcal{D}_{\text{ét}}^{\otimes}$, we deduce a right module structure on $\mathcal{D}_{\text{ét}}([L^+\mathcal{I}_{\mathrm{u}}^{\text{op}}\backslash \mathrm{Fl}_{\mathcal{I}}])$ under $\mathcal{D}_{\text{ét}}(\mathrm{Hk}_{\mathcal{I}})$ and this module structure is preserved under passing to direct summands. We begin our study of this ∞-category by classifying Iwahori–Whittaker local systems on $L^+\mathcal{I}_u^{\text{op}}$-orbits in $\mathrm{Fl}_{\mathcal{I}}$. Note that the latter are exactly the $L^+\mathcal{I}^{\text{op}}$-orbits and hence are in bijection with the Iwahori–Weyl group W as in Section 2. We will denote the corresponding $L^+\mathcal{I}^{\text{op}}$-orbit of w by $\mathrm{Fl}_{\mathcal{I},w}^{\text{op}} := L^+\mathcal{I}^{\text{op}} w L^+\mathcal{I}/L^+\mathcal{I}$.

Lemma 6.3. *The orbit $\mathrm{Fl}_{\mathcal{I},w}^{\text{op}}$ carries a Iwahori–Whittaker local system if and only if w has minimal length in its W_{fin}-left coset, i.e., $\ell(w_{\text{fin}}w) \geq \ell(w)$ for all $w_{\text{fin}} \in W_{\text{fin}}$. If the latter condition holds, then the rank 1 Iwahori–Whittaker local system on $\mathrm{Fl}_{\mathcal{I},w}^{\text{op}}$ is unique up to isomorphism.*

Proof. The Iwahori–Whittaker equivariant condition forces the stabilizer of the point $\dot{w}$ to be contained in the kernel of χ, and conversely such a containment would allow us to pullback the Artin–Schreier sheaf along χ to the desired orbit. This inclusion happens if and only if $w(\alpha_s)$ is a positive affine root where α_s is the positive simple affine root attached to any positive simple reflection $s \in W_{\text{fin}}$. But this is equivalent to $sw > w$, i.e., that w is the minimal length representative of its W_{fin}-left coset. □

Since the set of left W_{fin}-cosets of the Iwahori–Weyl group is enumerated by $\mathbb{X}_\bullet$, we will call $\mathcal{L}_\nu$ the unique Iwahori–Whittaker local system supported on the $L^+\mathcal{I}_u^{\text{op}}$-orbit of w_ν, the minimal length element in $W_{\text{fin}}t_\nu$, according to the preceding statement. We also obtain the standard Iwahori–Whittaker equivariant sheaf

$$\Delta_\nu^{\mathcal{IW}} := (j_{w_\nu}^{\text{op}})_! \mathcal{L}_\nu[\ell(w_\nu)], \tag{6.3}$$

where $j^{\mathrm{op}}_{w_\nu}$ is the inclusion of the $L^+\mathcal{I}^{\mathrm{op}}$-orbit and likewise the costandard Iwahori–Whittaker equivariant sheaf

$$\nabla^{\mathcal{IW}}_\nu := R(j^{\mathrm{op}}_{w_\nu})_*\mathcal{L}_\nu[\ell(w_\nu)], \tag{6.4}$$

both of which are supported on $\mathrm{Fl}^{\mathrm{op}}_{\mathcal{I},\leq w_\nu}$ and are perverse because orbits of solvable groups are affine, so we can invoke Artin vanishing, compare with [BBDG18, Corollaire 4.1.10]. We also have access to IC sheaves $\mathrm{IC}^{\mathcal{IW}}_\nu$ by taking the image of the natural map $\Delta^{\mathcal{IW}}_\nu \to \nabla^{\mathcal{IW}}_\nu$. Recall that in [BGS96, Subsection 3.2] a sufficient criterion for the existence of projective covers, injective hulls and tilting modules was given. We call an abelian category satisfying these axioms a highest weight category, see also [BR18, Subsection 1.12.3].

Proposition 6.4. *The category $\mathcal{P}(\mathrm{Hk}_{\mathcal{IW}})$ of Iwahori–Whittaker equivariant perverse sheaves is a highest weight category, whose underlying poset equals $\mathbb{X}_\bullet \simeq W_{\mathrm{fin}}\backslash W$ ordered by the quotient Bruhat order.*

Proof. The first part is a standard consequence of arguments by Beilinson–Ginzburg–Soergel, see [BGS96, Theorems 3.2.1 and 3.3.1]. As for the second claim, it suffices to identify the closure relations. It can be easily checked that the opposite Schubert variety $\mathrm{Fl}^{\mathrm{op}}_{\mathcal{I},\leq w_\nu}$ equals the $(G,\mathcal{I})$-Schubert variety $\mathrm{Fl}_{(G,\mathcal{I}),\leq\nu}$ in the notation of [AGLR22, Section 3] (up to the order of action), which in turn coincides with $\mathrm{Fl}_{\mathcal{I},\leq w_0 w_\nu}$, where $w_0 \in W_{\mathrm{fin}}$ is the longest element (there is a notational clash here, because w_ν evaluated at $\nu = 0$ is simply the identity). Indeed, they have the same dimension and there is an obvious inclusion $\mathrm{Fl}^{\mathrm{op}}_{\mathcal{I},\leq w_\nu} \subset \mathrm{Fl}_{(G,\mathcal{I}),\leq\nu}$, because $L^+G \supset L^+\mathcal{I}^{\mathrm{op}}W_{\mathrm{fin}}$. The closure relations follow then from the usual combinatorics of flag varieties as in [AGLR22, Section 3]. Indeed, the variety $\mathrm{Fl}_{(G,\mathcal{I}),\leq\nu_1}$ is contained in $\mathrm{Fl}_{(G,\mathcal{I}),\leq\nu_2}$ if and only if $\nu_1 \leq \nu_2$ for the quotient Bruhat order. □

Notice that $\mathrm{Fl}^{\mathrm{op}}_{\mathcal{I},\leq 1} = \mathrm{Fl}_{\mathcal{I},\leq w_0} = G/B \subset \mathrm{Fl}_{\mathcal{I}}$, with opposite cells indexed by W_{fin}. This implies the equalities $\Delta^{\mathcal{IW}}_0 = \nabla^{\mathcal{IW}}_0 = \mathrm{IC}^{\mathcal{IW}}_0$ and we will denote this simple Iwahori-Whittaker perverse sheaf by Ξ. It allows us to define the Iwahori–Whittaker averaging functor

$$\mathrm{av}_{\mathcal{IW}} \colon \mathcal{D}_{\text{ét}}(\mathrm{Hk}_{\mathcal{I}}) \to \mathcal{D}_{\text{ét}}(\mathrm{Hk}_{\mathcal{IW}}) \tag{6.5}$$

given by $A \mapsto \Xi * A$. To get a better understanding of this functor, we start by the following calculation:

Lemma 6.5. *The sheaf $\mathrm{av}_{\mathcal{IW}}(\mathrm{IC}_w)$ equals $\mathrm{IC}^{\mathcal{IW}}_\nu$ if $w = w_\nu$ for some ν and vanishes otherwise.*

Proof. If $sw < w$ for some simple reflection $s \in W_{\mathrm{fin}}$, we conclude that IC_w is equivariant for the left action of $L^+\mathcal{P}_s$, where $\mathcal{P}_s$ is the minimal standard parahoric with respect to $\mathcal{I}$ and the simple reflection s. Notice that by [BGM+19, Lemma 2.5] we have a natural isomorphism

$$\Xi *^{\mathcal{I}} \mathrm{IC}_w \simeq (R\pi_{s,*}\Xi) *^{\mathcal{J}_s} \mathrm{IC}_w \tag{6.6}$$

where the exponent stands for the fact that the first convolution is induced by contracting the $L^+\mathcal{I}$-action, and the second one by contracting the $L^+\mathcal{J}_s$-action. Here, $\pi_s : \mathrm{Fl}_{\mathcal{I}} \to \mathrm{Fl}_{\mathcal{P}_s}$ is the projection. In particular, it suffices to check the vanishing of $R\pi_{s,*}\Xi$. Note that Ξ is supported on $U^{\mathrm{op}}w_0 \subset G/B$ as the complement cannot support a non-zero Iwahori-Whittaker sheaf by 6.3. We can now see that the fiber of π_s over the image of $\mathrm{supp}\,\Xi$ is isomorphic to U^{op}_a, where a is the positive root associated with s. Since χ does not vanish on U^{op}_a, it follows by proper base change that $R\pi_{s,*}\Xi$ identifies with the cohomology $R\Gamma(\mathbb{G}_{a,k}, \mathcal{L}_{\mathrm{AS}})$ of the Artin–Schreier sheaf, i.e., it vanishes.

If $w = w_\nu$ is the minimal length representative of $W_{\mathrm{fin}}t_\nu$, we can check that the map $\mathrm{Fl}^{\mathrm{op}}_{\mathcal{I},1}\tilde{\times}\mathrm{Fl}_{\mathcal{I},w} \to \mathrm{Fl}^{\mathrm{op}}_{\mathcal{I},w}$ is an isomorphism. Indeed, both are affine spaces with the same dimension, and thus the given map is universally bijective by basic properties of Tits systems. In particular, we conclude that $\mathrm{av}_{\mathcal{IW}}(\mathrm{IC}_w)$ identifies with $\mathrm{IC}^{\mathcal{IW}}_\nu$. □

Proposition 6.6. *The functor* $\mathrm{av}_{\mathcal{IW}}$ *is perverse t-exact.*

Proof. Since each half of the t-structure on $\mathcal{D}_{\text{ét}}(\mathrm{Hk}_{\mathcal{I}})$ is spanned under extensions by appropriate shifts of the standard or costandard sheaves, it will suffice by symmetry to show that

$$\mathrm{av}_{\mathcal{IW}}(\Delta_w) = \Delta_\nu^{\mathcal{IW}} \tag{6.7}$$

for all $w \in W$, where $\nu \in \mathbb{X}_\bullet$ is in the same left W_{fin}-coset. When $w = w_\nu$ is the minimal length representative, this follows from the same argument of the previous lemma for IC sheaves. In general, consider an injection of perverse sheaves $\Delta_{w_\nu} \to \Delta_w$ as in [AB09, Lemma 3b], whose cone is spanned under extensions by $\mathrm{IC}_y * \Delta_{w_\nu}$ for non-trivial $y \in W_{\text{fin}}$. The latter sheaves vanish under $\mathrm{av}_{\mathcal{IW}}$ by the previous lemma and we get the desired conclusion. □

We are now ready to prove the main result regarding Iwahori–Whittaker averaging.

Theorem 6.7. *The averaging functor restricted to perverse sheaves factors through a fully faithful functor*

$$\mathrm{av}^{\text{as}}_{\mathcal{IW}} \colon \mathcal{P}_{\text{as}}(\mathrm{Hk}_{\mathcal{I}}) \to \mathcal{P}(\mathrm{Hk}_{\mathcal{IW}}) \tag{6.8}$$

where the left side is the Serre quotient of $\mathcal{P}(\mathrm{Hk}_{\mathcal{I}})$ *obtained by modding out the sheaves* IC_w *for all* $w \in W$ *which are not minimal in their left* W_{fin}*-cosets.*

Proof. Let $\mathcal{G}_{\geq 1} \to G$ be the dilatation along the identity subgroup in G_k so that $L^+\mathcal{G}_{\geq 1} = L^{\geq 1}G_O \subset L^+G_O$ is the first congruence subgroup of G. We have a natural map $\alpha \colon [L^+\mathcal{G}_{\geq 1}\backslash \mathrm{Fl}_{\mathcal{I}}] \to \mathrm{Hk}_{\mathcal{I}}$ and similarly a forgetful functor between stable ∞-categories $\beta^* \colon \mathcal{D}_{\text{ét}}(\mathrm{Hk}_{\mathcal{IW}}) \to \mathcal{D}_{\text{ét}}([L^+\mathcal{G}_{\geq 1}\backslash \mathrm{Fl}_{\mathcal{I}}])$ because the kernel of χ contains $L^{\geq 1}G_O$. We consider the induced functor

$$\mathrm{ind}_{\mathcal{IW}} := {}^pH^{-\mathrm{rkG}} \circ R\alpha_* \circ \beta^* \colon \mathcal{P}(\mathrm{Hk}_{\mathcal{IW}}) \to \mathcal{P}(\mathrm{Hk}_{\mathcal{I}}) \tag{6.9}$$

and claim that its composition with the quotient map $\mathcal{P}(\mathrm{Hk}_{\mathcal{I}}) \to \mathcal{P}_{\text{as}}(\mathrm{Hk}_{\mathcal{I}})$ yields a right inverse to $\mathrm{av}^{\text{as}}_{\mathcal{IW}}$. First notice that $\mathrm{ind}_{\mathcal{IW}}(\Xi)$ is an extension of negative shifts of IC_w for $w \in W_{\text{fin}}$, with the local system IC_1 appearing with multiplicity 1 (see [AR, Lemma 6.4.8]). If w is non-trivial and $\mathcal{F} \in \mathcal{P}(\mathrm{Hk}_{\mathcal{I}})$, we can show that $\mathrm{IC}_w[n] * \mathcal{F}$ is $L^+\mathcal{P}_s$-equivariant for some simple reflection s, hence its perverse cohomology groups die under the quotient map $\mathcal{P}(\mathrm{Hk}_{\mathcal{I}}) \to \mathcal{P}_{\text{as}}(\mathrm{Hk}_{\mathcal{I}})$. If $w = 1$ and $n \neq 0$, then $\mathrm{IC}_1[n] * \mathcal{F}$ sits in non-zero perverse degree. In total, this yields an equivalence of functors $\mathrm{ind}^{\text{as}}_{\mathcal{IW}} \circ \mathrm{av}^{\text{as}}_{\mathcal{IW}} \simeq \mathrm{id}$. This implies that $\mathrm{av}^{\text{as}}_{\mathcal{IW}}$ is injective on Ext groups. To see that $\mathrm{av}^{\text{as}}_{\mathcal{IW}}$ is fully faithful, we argue by induction on the length of the objects being considered: the simple case is a consequence of Lemma 6.5, implying bijectivity of simple objects along $\mathrm{av}^{\text{as}}_{\mathcal{IW}}$; the induction step follows from the 5-lemma and the injectivity on Ext-groups. □

7. Tilting modules

We proved in Proposition 6.4 that the category of Iwahori-Whittaker perverse sheaves admits a highest weight category structure. It then makes sense to discuss *tilting objects* in this category. The aim of the current section is to show that the functor

$$\mathcal{Z}_{\mathcal{IW}} := \mathrm{av}_{\mathcal{IW}} \circ \mathcal{Z} : \mathrm{Rep}_\Lambda(\hat{G}) \to \mathcal{P}(\mathrm{Hk}_{\mathcal{IW}}) \tag{7.1}$$

lands on the full subcategory of tilting objects. This is related to the 'parabolic-singular' Koszul duality phenomenon studied by Beilinson–Ginzburg–Soergel [BGS96] for finite flag varieties and Bezrukavnikov–Yun [BY13] for Kac–Moody flag varieties. We continue to assume that Λ is an algebraic field extension of $\mathbb{Q}_\ell(\zeta_\ell)$ for a fixed primitive p-th root of unity $\zeta \in \bar{\mathbb{Q}}_\ell$.

We recall the following useful property.

Proposition 7.1. *For any* $\mathcal{F} \in \mathcal{D}_{\text{cons}}(\mathrm{Hk}_{\mathcal{IW}})$*,* $\mathcal{F}$ *is tilting if and only if* $(j^{\text{op}}_{w_\nu})^*\mathcal{F}$ *and* $R(j^{\text{op}}_{w_\nu})^!\mathcal{F}$ *are both concentrated in degree* $-\ell(w_\nu)$ *for all* $\nu \in \mathbb{X}_\bullet$*.*

Proof. This follows from [BBM04, Proposition 1.3]. □

7.1. **Multiplicities of tilting objects.** Let $\mathcal{F} \in \mathcal{P}(\mathrm{Hk}_{\mathcal{IW}})$ be a tilting object. Recall that the multiplicity of the standard (resp. costandard) objects $\Delta_\nu^{\mathcal{IW}}$ (resp. $\nabla_\nu^{\mathcal{IW}}$) in $\mathcal{F}$ is well-defined and we denote it by $(\mathcal{F} : \Delta_\nu^{\mathcal{IW}})$(resp. $(\mathcal{F} : \nabla_\nu^{\mathcal{IW}})$. It follows by orthogonality of $\Delta_\nu^{\mathcal{IW}}$ and $\nabla_\nu^{\mathcal{IW}}$ that

$$(\mathcal{F} : \Delta_\nu^{\mathcal{IW}}) \simeq \dim \mathrm{Hom}(\mathcal{F}, \nabla_\nu^{\mathcal{IW}}), \tag{7.2}$$

$$(\mathcal{F} : \nabla_\nu^{\mathcal{IW}}) \simeq \dim \mathrm{Hom}(\Delta_\nu^{\mathcal{IW}}, \mathcal{F}). \tag{7.3}$$

where the Hom spaces are taken inside $\mathcal{D}_{\text{ét}}(\mathrm{Hk}_{\mathcal{IW}})$, compare with [BGS96, Theorem 3.2.1]. We use the same notation for $\mathcal{F} \in \mathcal{D}_{\text{ét}}(\mathrm{Hk}_{\mathcal{IW}}$ as well.

Proposition 7.2. *For any $V \in \mathrm{Rep}_\Lambda(\hat{G})$ and any $\mu \in \mathbb{X}_\bullet$, we have*

$$\sum_{i \geq 0} (-1)^i (\mathcal{Z}_{\mathcal{IW}}(V)[-i] : \nabla_\nu^{\mathcal{IW}}) = \dim(V(\nu)) \tag{7.4}$$

$$\sum_{i \leq 0} (-1)^i (\mathcal{Z}_{\mathcal{IW}}(V)[i] : \Delta_\nu^{\mathcal{IW}}) = \dim(V(\nu)) \tag{7.5}$$

where $V(\nu)$ denotes the ν-weight space of V.

Proof. The proof follows the strategy of [AB09, Lemma 27] and [AR, Proposition 6.5.4] by Corollary 3.24, and Equation (6.7). □

Corollary 7.3. *For any $V \in$ $\mathrm{Rep}_\Lambda(\hat{G})$,*

(1) *if $\mathcal{Z}_{\mathcal{IW}}(V)$ is tilting, then*

$$(\mathcal{Z}_{\mathcal{IW}}(V) : \Delta_\mu^{\mathcal{IW}}) = (\mathcal{Z}_{\mathcal{IW}}(V) : \nabla_\mu^{\mathcal{IW}}) = \dim(V_\mu); \tag{7.6}$$

(2) *if V is the highest weight representation of highest weight μ, then $\mathcal{Z}_{\mathcal{IW}}(V)$ is supported on $\mathrm{Fl}^{\mathrm{op}}_{\mathcal{I},w_\mu}$.*

Proof. Statements follow from [AR, Remark 6.5.5, Corollary 6.5.6]. □

The work that now follows will eventually lead to proving that $\mathcal{Z}_{\mathcal{IW}}(V)$ is tilting for almost all groups G, which we explain below in Theorem 7.9. But first, we handle minuscule representations, and for that we require the next lemma.

Lemma 7.4. *For any $V \in \mathrm{Rep}(\hat{G})$, $\nu \in \mathbb{X}_\bullet$, $x \in W_{\mathrm{fin}}$, and $n \in \mathbb{Z}$, we have isomorphisms*

$$\mathrm{Ext}^n(\Delta_\nu^{\mathcal{IW}}, \mathcal{Z}_{\mathcal{IW}}(V)) \simeq \mathrm{Ext}^n(\Delta_{x(\nu)}^{\mathcal{IW}}, \mathcal{Z}_{\mathcal{IW}}(V)),$$

$$\mathrm{Ext}^n(\nabla_\nu^{\mathcal{IW}}, \mathcal{Z}_{\mathcal{IW}}(V)) \simeq \mathrm{Ext}^n(\nabla_{x(\nu)}^{\mathcal{IW}}, \mathcal{Z}_{\mathcal{IW}}(V)).$$

Proof. The proof is similar to the arguments in [AR, Lemma 6.5.11] and we sketch it here. Without loss of generality, we assume ν to be dominant so that $w_\nu = t_\nu$, as the statement only depends on its W_{fin}-orbit. We can find $y \in W_{\mathrm{fin}}$ with minimal length such that $t_\nu = w_{x(\nu)} y^{-1}$ is a minimal length decomposition, and $xy(\nu) = \nu$. Then

$$\Delta_\nu^{\mathcal{IW}} = \Delta_0^{\mathcal{IW}} * \Delta_{w_\nu} \simeq \Delta_0^{\mathcal{IW}} * \Delta_{w_{x(\nu)}} * \Delta_{y^{-1}} \simeq \Delta_{x(\nu)}^{\mathcal{IW}} * \Delta_{y^{-1}},$$

by Proposition 6.6 and Lemma 3.2. Then

$$\begin{aligned}
\mathrm{Ext}^n(\Delta_\nu^{\mathcal{IW}}, \mathcal{Z}_{\mathcal{IW}}(V)) &\simeq \mathrm{Ext}^n(\Delta_\nu^{\mathcal{IW}} * \Delta_y, \mathcal{Z}_{\mathcal{IW}}(V) * \Delta_y) \\
&\simeq \mathrm{Ext}^n(\Delta_{x(\nu)}^{\mathcal{IW}}, \mathcal{Z}_{\mathcal{IW}}(V) * \Delta_y) \\
&\simeq \mathrm{Ext}^n(\Delta_{x(\nu)}^{\mathcal{IW}}, \Delta_0^{\mathcal{IW}} * \mathcal{Z}(V) * \Delta_y) \\
&\simeq \mathrm{Ext}^n(\Delta_{x(\nu)}^{\mathcal{IW}}, \Delta_0^{\mathcal{IW}} * \Delta_y * \mathcal{Z}(V)) \\
&\simeq \mathrm{Ext}^n(\Delta_{x(\nu)}^{\mathcal{IW}}, \mathcal{Z}_{\mathcal{IW}}(V))
\end{aligned}$$

again by Proposition 6.6 and Lemma 3.2. The second isomorphism is proved analogously. □

Proposition 7.5. *Let V be a simple representation of $\hat{G}$ with highest weight μ being a minuscule dominant coweight of G, then $\mathcal{Z}_{\mathcal{IW}}(V)$ is tilting.*

Proof. Because of Proposition 7.1, it suffices to show that $(j^{\mathrm{op}}_{w_\nu})^*(\mathcal{Z}_{\mathcal{IW}}(V))$ and $(j^{\mathrm{op}}_{w_\nu})^!(\mathcal{Z}_{\mathcal{IW}}(V))$ are both perverse sheaves. Since μ is minuscule, the only weights we have to check are in the W_{fin}-orbit of μ. By adjunction and Lemma 7.4, we are reduced to showing the statements above for $\nu = \mu$. Note that the support of $\mathcal{Z}(V)$ equals the μ-admissible locus $A_{I,\mu}$ by [AGLR22, Theorem 6.16], whose open $L^+\mathcal{I}$-orbits are indexed by $W_{\mathrm{fin}}\mu$. We deduce that the support of $\mathcal{Z}_{\mathcal{IW}}(V)$ equals $\mathrm{Fl}_{\mathcal{I},\leq w_0 t_\mu}$ and hence the locally closed immersion $j^{\mathrm{op}}_{w_\mu}$ is actually open and dense. In particular, it is clear that $(j^{\mathrm{op}}_{w_\mu})^*(\mathcal{Z}_{\mathcal{IW}}(V))$ and $(j^{\mathrm{op}}_{w_\mu})^!(\mathcal{Z}_{\mathcal{IW}}(V))$ are both perverse. □

Now, we deduce Theorem 7.9 by propagating the result via convolution.

Proposition 7.6. *Let $V, W \in \mathrm{Rep}(\hat{G})$ such that $\mathcal{Z}_{\mathcal{IW}}(V)$ and $\mathcal{Z}_{\mathcal{IW}}(W)$ are both tilting. Then so is $\mathcal{Z}_{\mathcal{IW}}(V \otimes W)$.*

Proof. It suffices to prove that $(j^{\mathrm{op}}_{w_\nu})^*\mathcal{Z}_{\mathcal{IW}}(V\otimes W)$ and $(j^{\mathrm{op}}_{w_\nu})^!\mathcal{Z}_{\mathcal{IW}}(V\otimes W)$ are both perverse for any $\nu \in \mathbb{X}_\bullet$. Since $\mathcal{Z}_{\mathcal{IW}}(V\otimes W)$ is perverse, $(j^{\mathrm{op}}_{w_\nu})^*\mathcal{Z}_{\mathcal{IW}}(V\otimes W)$ concentrates in perverse degress ≤ 0, and $(j^{\mathrm{op}}_{w_\nu})^!(\mathcal{Z}_{\mathcal{IW}}(V \otimes W))$ concentrates in perverse degrees ≥ 0. Note that if $\mathcal{Z}_{\mathcal{IW}}(V)$ is tilting, the object $\mathcal{Z}_{\mathcal{IW}}(V \otimes W) \cong \mathcal{Z}_{\mathcal{IW}}(V) * \mathcal{Z}(W)$ admits a filtration with subquotients $\Delta^{\mathcal{IW}}_\mu * \mathcal{Z}(W)$. By Proposition 6.6 and Theorem 4.12, $\Delta^{\mathcal{IW}}_\mu * \mathcal{Z}(W) \cong \mathcal{Z}_{\mathcal{IW}}(W) * \Delta_{w_\mu}$. Since $\mathcal{Z}_{\mathcal{IW}}(W)$ is tilting, then $\Delta^{\mathcal{IW}}_\mu * \mathcal{Z}(W)$ admits a filtration with subquotients $\mathrm{av}_{\mathcal{IW}}(\Delta_{w_\nu} * \Delta_{w_\mu})$. Lemma 3.8 and Proposition 6.6 imply that $(j^{\mathrm{op}}_{w_\nu})^*(\mathcal{Z}_{\mathcal{IW}}(V \otimes W))$ concentrates in non-negative perverse degrees. The statement for $(j^{\mathrm{op}}_{w_\nu})^!(\mathcal{Z}_{\mathcal{IW}}(V \otimes W))$ is proved similarly. □

Now, we need to describe when minuscule representations form a class of Karoubi generators for the symmetric monoidal category $\mathrm{Rep}(\hat{G})$. It will be enough for us to restrict to adjoint G. The argument below was partially suggested to us by Jeremy Taylor.

Lemma 7.7. *Assume G is adjoint. The following are equivalent:*

(1) $\mathrm{Rep}(\hat{G})$ *is generated by minuscule representations under sums, retracts and tensor products;*
(2) $\hat{G}$ *admits a faithful minuscule representation;*
(3) *Every simple adjoint factor of G contains a minuscule coweight (i.e., it is not of type E_8, F_4 nor G_2).*

Proof. Recall that the generation property when including quasi-minuscules was observed by Ngô–Polo, see [NP01, Lemma 10.3], and we could read this off their proof. It is clear that (1) implies (3). Also, (2) implies (1) because we are working over a characteristic 0 field Λ and hence we can invoke the Peter–Weyl theorem: explicitly, taking global sections along the faithful representation $\rho\colon \hat{G} \to \mathrm{GL}(V)$ over Λ, we see that the regular $\hat{G}$-representation is spanned by minuscule representations under sums, retracts and tensor products, so by semisimplicity the same holds for any representation of G. Finally, we show that (3) implies (2). It is enough to assume G is almost simple and let μ be a minuscule coweight. The representation $\hat{G} \to \mathrm{GL}(V_\mu)$ has finite central kernel, which is shared by any non-trivial representation whose weights differ from μ by an element of the coroot lattice. Varying the minuscule μ, we get every single element in $\mathbb{X}_\bullet$ by [Bou68, Exercices 24–25, p. 225], which implies that the direct sum of all minuscule representations is faithful. □

Definition 7.8. We say that G has *enough minuscules* if its adjoint quotient satisfies the equivalent conditions of Lemma 7.7.

Now, we can prove the tilting property under a mild assumption involving exceptional groups. As explained during the introduction, this stems from the lack of the rotation $\mathbb{G}_{m,k}^{\mathrm{pf}}$-action in our p-adic context.

Theorem 7.9. *If G has enough minuscules, then $\mathcal{Z}_{\mathcal{IW}}(V)$ is tilting for all $V \in \mathrm{Rep}(\hat{G})$.*

Proof. First of all, we perform a reduction to the adjoint case. Recall that the adjoint map $M_{\mathcal{I},\mu} \to M_{\mathcal{I}_{\mathrm{ad}},\mu_{\mathrm{ad}}}$ is an isomorphism, so if V is simple (which we may and do assume), we can naturally identify the central sheaf $\mathcal{Z}(V)$ with the central sheaf $\mathcal{Z}(V_{\hat{G}_{\mathrm{sc}}})$ constructed in $\mathrm{Hk}_{\mathcal{I}_{\mathrm{ad}}}$, see also [FS21, Section VI.11] to see that geometric Satake is compatible with adjoint quotients. The natural functor $\mathcal{P}(\mathrm{Hk}_{\mathcal{IW}}) \to \mathcal{P}(\mathrm{Hk}_{\mathcal{IW}_{\mathrm{ad}}})$ also becomes an equivalence when restricted to a single connected component of the Hecke stack, identifying standard and costandard objects in the obvious way. In particular, the assertion can be read off the adjoint case.

Now, if G is adjoint, we apply Proposition 7.5, Proposition 7.6 and Lemma 7.7 in combination to arrive at the desired result. □

8. Regular quotient

During this section, we assume $\Lambda = \bar{\mathbb{Q}}_\ell$ is algebraically closed. Consider the Serre subcategory $\mathcal{P}_{>0}(\mathrm{Hk}_{\mathcal{I}}) \subset \mathcal{P}(\mathrm{Hk}_{\mathcal{I}})$ generated by IC sheaves with positive dimensional support and denote by Π^0 the natural quotient functor

$$\Pi^0 : \mathcal{P}(\mathrm{Hk}_{\mathcal{I}}) \to \mathcal{P}_0(\mathrm{Hk}_{\mathcal{I}}) := \mathcal{P}(\mathrm{Hk}_{\mathcal{I}})/\mathcal{P}_{>0}(\mathrm{Hk}_{\mathcal{I}}) \tag{8.1}$$

to the Serre quotient.

Therefore the simple objects in $\mathcal{P}_0(\mathrm{Hk}_{\mathcal{I}})$ are precisely given by the IC_τ where $\tau \in \Omega_{\mathbf{a}}$ stabilizes the base alcove. In particular, if G is semi-simple, $\mathcal{P}_0(\mathrm{Hk}_{\mathcal{I}})$ has only finitely many simple objects.

Proposition 8.1. *The monoidal structure on $\mathcal{P}(\mathrm{Hk}_{\mathcal{I}})$ given by perverse truncated convolution ${}^pH^0((-)*(-))$ descends to an exact monoidal structure $\circledast$ on $\mathcal{P}_0(\mathrm{Hk}_{\mathcal{I}})$.*

Proof. The proof follows the idea in [AR, Proposition 6.5.14] and we sketch it here. Let $A_1 = \mathrm{IC}_w$ for some $w \in W$ with $\ell(w) > 0$. Then there exists a simple reflection s such that $\ell(sw) < \ell(w)$. Let $\mathcal{J}_s$ be the minimal parahoric containing $\mathcal{I}$ associated with s. Then A_1 is $\mathcal{J}_s$-equivariant. It follows that $A_1 * A_2$ is also $\mathcal{J}_s$-equivariant for any $A_2 \in \mathcal{P}(\mathrm{Hk}_{\mathcal{I}})$, and so are its perverse cohomology sheaves. But a $\mathcal{J}_s$-equivariant perverse sheaf has equivariant composition factors, hence lies in $\mathcal{P}_{>0}(\mathrm{Hk}_{\mathcal{I}})$. Varying w and by symmetry, we conclude that the monoidal structure given by ${}^pH^0(*)$ on $\mathcal{P}(\mathrm{Hk}_{\mathcal{I}})$ descends to a monoidal structure $\circledast$ on $\mathcal{P}_0(\mathrm{Hk}_{\mathcal{I}})$. In order to check exactness of $\circledast$, we must see that for arbitrary $x, y \in W$ the perverse cohomology sheaves in non-zero degree of a convolution product $\mathrm{IC}_x * \mathrm{IC}_y$ lie in $\mathcal{P}_{>0}(\mathrm{Hk}_{\mathcal{I}})$. The only remaining case to analyze is when both elements have length 0, but in this case $\mathrm{IC}_x * \mathrm{IC}_y = \mathrm{IC}_{xy}$. □

We have the following important result:

Lemma 8.2. *The functor $\mathcal{Z}_0 := \Pi^0 \circ \mathcal{Z} \colon \mathrm{Rep}(\hat{G}) \to \mathcal{P}_0(\mathrm{Hk}_{\mathcal{I}})$ is monoidal and central.*

Proof. By Proposition 8.1, we can construct the monoidality and centrality isomorphisms by applying those of Proposition 4.9 and Theorem 4.12 and then projecting towards $\mathcal{P}_0(\mathrm{Hk}_{\mathcal{I}})$, compare with [AR, Lemma 6.5.15]. □

Note that for every $\hat{G}$-representation V, we have a nilpotent operator $\mathbf{n}_V^0 \colon \mathcal{Z}_0(V) \to \mathcal{Z}_0(V)$ arising from the logarithm of the monodromy of $\mathcal{Z}(V)$. Denote by $\mathcal{P}_0^{\mathrm{c}}(\mathrm{Hk}_{\mathcal{I}})$ the full subcategory of $\mathcal{P}_0(\mathrm{Hk}_{\mathcal{I}})$ whose objects are the subquotients of $\mathcal{Z}^0(V), V \in \mathrm{Rep}(\hat{G})$. The exactness of $\circledast$ and monoidality of $\mathcal{Z}_0$ imply that $\mathcal{P}_0^{\mathrm{c}}(\mathrm{Hk}_{\mathcal{I}})$ is closed under the monoidal structure. By definition, the functor $\mathcal{Z}_0$ naturally factors through a functor $\mathcal{Z}_0^{\mathrm{c}} : \mathrm{Rep}(\hat{G}) \to \mathcal{P}_0^{\mathrm{c}}(\mathrm{Hk}_{\mathcal{I}})$.

Proposition 8.3. *There exists a closed subgroup $H \subset \hat{G}$ such that we have*

(1) *an equivalence of monoidal categories*
$$\Phi^0 : (\mathcal{P}_0^c(\mathrm{Hk}_{\mathcal{I}}), \circledast) \simeq (\mathrm{Rep}(H), \otimes). \tag{8.2}$$
(2) *a nilpotent element $n_0 \in \hat{\mathfrak{g}}$ such that $H \subset Z_{\hat{G}}(n_0)$.*
(4) *an isomorphism of functors $\alpha : \Phi^0 \circ \mathcal{Z}_0^c \simeq \mathrm{For}_H^{\hat{G}}$, carrying the monodromy operators $\mathbf{n}_V^0$ to the natural action of n_0 on V.*

Remark 8.4. If G has enough minuscules, then $\mathcal{P}_0^c(\mathrm{Hk}_{\mathcal{I}}) = \mathcal{P}_I^0$, and $H = Z_{\hat{G}}(n_0)$. We do not need this in the proof of the main theorem and will postpone the discussion of this fact to §10 (cf. Proposition 10.8).

Proof. The above proposition is the mixed characteristic analogue of a particular case of [Bez04, Proposition 1, Theorem 3]. We sketch the proof here and refer further details to *loc.cit.* Note that we can regard the regular representation $\mathcal{O}(\hat{G})$ of the dual group as a ring object in $\mathrm{Ind}(\mathrm{Rep}(\hat{G}))$. Then $\mathcal{Z}_0(\mathcal{O}(\hat{G}))$ is a ring object in $\mathrm{Ind}(\mathcal{P}_0^c(\mathrm{Hk}_{\mathcal{I}}))$. Zorn's lemma implies that there exists a maximal left ideal subobject $\mathcal{J} \subset \mathcal{Z}_0(\mathcal{O}(\hat{G}))$, whose quotient will be denoted by $\mathcal{O}(H)$. The centrality of $\mathfrak{Z}^0$ (cf. Lemma 8.2) implies that $\mathcal{O}(H)$ is also a ring object. Thus, we define $\mathcal{O}(H)$-Mod as the category of left $\mathcal{O}(H)$-modules in $\mathrm{Ind}(\mathcal{P}_0^c(\mathrm{Hk}_{\mathcal{I}}))$. Clearly $\mathcal{O}(H)$ is a simple object in the abelian category $\mathcal{O}(H)$-Mod. Hence, its endomorphism ring $K := \mathrm{End}_{\mathcal{O}(H)}(\mathcal{O}(H))$ is a division algebra, and $V \mapsto V \otimes K$ defines an equivalence between the category of right finite K-modules and the full subcategory in $\mathcal{O}(H)$-Mod generated by $\mathcal{O}(H)$ under finite direct sums and subquotients. Now, we deduce that
$$K \simeq \mathrm{Hom}_{\mathrm{Ind}(\mathcal{P}_0^c(\mathrm{Hk}_{\mathcal{I}}))}(\delta_0, \mathcal{O}(H)) \simeq \bar{\mathbb{Q}}_\ell. \tag{8.3}$$
because the left hand side is a countable $\bar{\mathbb{Q}}_\ell$-vector space and hence it must be algebraic. Now, we construct a monoidal fiber functor to invoke the Tannakian formalism.

Lemma 8.5. (1) *For any $A \in \mathcal{P}_0^c(\mathrm{Hk}_{\mathcal{I}})$, there exists a finite-dimensional vector space V such that $\mathcal{O}(H) \circledast A \simeq \mathcal{O}(H) \otimes V$ is an isomorphism of $\mathcal{O}(H)$-modules, where we endow V with the trivial $\mathcal{O}(H)$-action.*
(2) *The functor $\Phi_G : \mathcal{P}_0^c(\mathrm{Hk}_{\mathcal{I}}) \to \mathrm{Vect}_\Lambda$ defined by $A \mapsto \mathrm{Hom}(\mathcal{O}(H), \mathcal{O}(H) \circledast A)$ is an exact, faithful, and monoidal functor. In addition, $\Phi_G \circ \mathcal{Z}_0^c \simeq \mathrm{For}^{\hat{G}} : \mathrm{Rep}(\hat{G}) \to \mathrm{Vect}_{\bar{\mathbb{Q}}_\ell}$.*

Proof. To prove statement (1) in the above lemma, we first note that there is a canonical isomorphism $\mathcal{Z}_0(\mathcal{O}(\hat{G})) \circledast \mathcal{Z}_0(V) \simeq \mathcal{Z}_0(\mathcal{O}(\hat{G})) \otimes_\Lambda V$ of $\mathcal{Z}_0(\mathcal{O}(\hat{G}))$-modules for any $V \in \mathrm{Rep}(\hat{G})$. Quotienting out the maximal left idea $\mathcal{J}$, we conclude that $\mathcal{O}(H) \circledast \mathcal{Z}_0(V) \simeq \mathcal{O}(H) \otimes V$. The general situation follows from taking subquotients from both sides and we thus settle statement (1).

The exactness of Φ_G follows from that of $\mathcal{Z}_0$ and statement (1). Also, Equation (8.3) and statement (1) imply that $\mathcal{O}(H) \circledast A \cong \mathcal{O}(H) \otimes \Phi_G(A)$ for any $A \in \mathcal{P}_0^c(\mathrm{Hk}_{\mathcal{I}})$. Then for any $A_1, A_2 \in \mathcal{P}_0^c(\mathrm{Hk}_{\mathcal{I}})$, we have
$$\begin{aligned}\Phi_G(A_1 \circledast A_2) &\simeq \mathrm{Hom}(\mathcal{O}(H), \mathcal{O}(H) \otimes \Phi_G(A_1 \circledast A_2)) \simeq \mathrm{Hom}(\mathcal{O}(H), \mathcal{O}(H) \circledast (A_1 \circledast A_2)) \\ &\simeq \mathrm{Hom}(\mathcal{O}(H), \Phi_G(A_1) \otimes (\mathcal{O}(H) \circledast A_2)) \simeq \Phi_G(A_1) \otimes \Phi_G(A_2).\end{aligned}$$
Finally, it suffices to check that Φ_G sends non-zero objects to non-zero objects since it is exact. This can be checked on all simple objects $\Pi^0(\mathrm{IC}_\tau)$. The faithfulness then follows from the dualizability, in fact invertibility, of $\Pi^0(\mathrm{IC}_\tau)$ and the monoidal structure of Φ_G. ☐

Lemma 8.5 allows us to apply the Tannakian formalism and obtain an equivalence of monoidal categories
$$\Psi : \mathcal{P}_0^c(\mathrm{Hk}_{\mathcal{I}}) \simeq \mathrm{Comod}_{\mathcal{A}(H)}, \tag{8.4}$$

where $\mathcal{A}(H)$ is a Λ-bialgebra and $\mathrm{Comod}_{\mathcal{A}(H)}$ is the category of Λ-finite $\mathcal{A}(H)$-comodules. In addition, the composition of this equivalence with the natural forgetful functor $\mathrm{Comod}_{\mathcal{A}(H)} \to \mathrm{Vect}_\Lambda$ equals Φ_G. By the Tannakian construction, the functor $\Psi \circ \mathcal{Z}_0^c : \mathrm{Rep}(\hat{G}) \to \mathrm{Comod}_{\mathcal{A}(H)}$ induces a surjective morphism of bialgebras $\mathcal{O}(G) \to \mathcal{A}(H)$. It follows from [Bez04, Lemma 3] that $\mathcal{A}(H)$ is commutative and $\mathrm{Spec}\mathcal{A}(H)$ is the desired group scheme H.

Now we construct n_0. Recall our construction of the nilpotent endomorphism $\mathbf{n}_V^0$ of $\mathcal{Z}_0(V)$ for any $V \in \mathrm{Rep}(\hat{G})$. By naturality and compatibility with the monoidal structure as in Lemma 4.25, we deduce a tensor endomorphism of the functor $\Phi_G \circ \mathcal{Z}_0^c \simeq \mathrm{For}^{\hat{G}}$. In particular, this gives rise to an element n_0 of $\hat{\mathfrak{g}}$ by the Tannakian formalism. On the other hand, $\Phi_G \circ \mathcal{Z}_0^c \simeq \mathrm{For}^H \circ \Psi \circ \mathcal{Z}_0^c \simeq \mathrm{For}^H \circ \mathrm{For}^{\hat{G}}_H$, and $(\mathbf{n}_V^0)_V$ induces an automorphism of $\mathrm{For}^{\hat{G}}_H$. Hence, $H \subset Z_{\hat{G}}(n_0)$. □

Proposition 8.6. *If G has enough minuscules, then the nilpotent element n_0 is regular.*

Proof. Our argument is similar to the one in [AR, Section 6.5.8] and uses weight theory. Recall that in Conjecture 4.27, we posited that the mixed sheaves $\mathcal{Z}_{\mathrm{mix}}(V)$ ought to be monodromy-pure of weight 0, as this is the case in equicharacteristic due to a theorem of Gabber whose proof was written up by Beilinson–Bernstein [BB93], compare with [BB93, Theorem 5.1.2]. By Proposition 4.28, we know that this holds for minuscule representations. Note that the functor $\Pi^0 \colon \mathcal{P}(\mathrm{Hk}_{\mathcal{I}}) \to \mathcal{P}_0(\mathrm{Hk}_{\mathcal{I}})$ in (8.1) admits a mixed variant Π^0_{mix}, namely the quotient of mixed perverse sheaves by the ones with positive dimensional support. We claim that the images under Π^0_{mix} are monodromy-pure of weight 0. In other words, we want to show that the weight filtration obtained on $\mathcal{Z}_0(V)$ via push-pull coincides with the monodromy filtration induced by n_0. It suffices to prove this when G is adjoint, and then we can check that both filtrations are monoidal on V, see [BB93, Lemma 4.1.2], and also respect splittings, so we can propagate the claim starting from the minuscule case by Lemma 7.7.

Now, we can check whether n_0 is regular by calculating the dimension of $\hat{\mathfrak{g}}^{n_0}$. Reading off the weight filtration $\mathcal{Z}_0(\hat{\mathfrak{g}})$ on the Iwahori–Hecke algebra, one sees that its i-th graded has dimension equal to that of the sum of the weight spaces $\hat{\mathfrak{g}}(\nu)$ with $\langle 2\rho, \nu\rangle = i$. Since the weights of $\hat{\mathfrak{g}}$ are roots of $\hat{G}$, its non-zero gradeds are even integers, and hence $\dim(\hat{\mathfrak{g}}^{n_0}) = \dim(\hat{\mathfrak{g}}(0)) = \mathrm{rk}(G)$. □

9. Proof of the AB equivalence

At this point, we consider the composition of the two functors

$$\mathcal{F}_{\mathcal{IW}} := \mathrm{av}_{\mathcal{IW}} \circ \mathcal{F} \colon \mathrm{Perf}([\hat{G}\backslash\hat{\mathcal{N}}_{\mathrm{Spr}}]) \to \mathcal{D}_{\mathrm{cons}}(\mathrm{Hk}_{\mathcal{IW}}) \tag{9.1}$$

that we have extensively studied thus far. Our goal is to prove the Arkhipov–Bezrukavnikov equivalence below:

Theorem 9.1. *If G has enough minuscules, then the functor $\mathcal{F}_{\mathcal{IW}}$ is an equivalence.*

One can immediately draw the following conclusion:

Corollary 9.2. *If G has enough minuscules, then the functor $\mathrm{av}^{\mathrm{as}}_{\mathcal{IW}}$ from 6.7 is an equivalence of abelian categories.*

The strategy behind the proof of the theorem is as usual based on generators and relations. We start with the following lemma.

Lemma 9.3. *The ∞-category $\mathcal{D}_{\mathrm{cons}}(\mathrm{Hk}_{\mathcal{IW}})$ is spanned by $\mathrm{av}_{\mathcal{IW}}(\mathcal{J}_\nu)$ for all $\nu \in \mathbb{X}_\bullet$ under cones and extensions.*

Proof. As in Lemmas 3.1 and 3.6, we can check that $\mathrm{av}_{\mathcal{IW}}(\mathcal{J}_\nu)$ has the same class in the Grothendieck group as $\Delta_\nu^{\mathcal{IW}}$. Taking its Euler characteristic, we deduce that it is supported on $\mathrm{Fl}_{\mathcal{I},\leq w_0 w_\nu}$ and has generic rank 1. A standard induction argument now implies the spanning assertion. $\square$

Lemma 9.4. *For any $V \in \mathrm{Rep}_\Lambda \hat{G}$, the map*

$$\mathrm{Hom}(\mathcal{O}_{\hat{\mathcal{N}}_{\mathrm{Spr}}}, V \otimes \mathcal{O}_{\hat{\mathcal{N}}_{\mathrm{Spr}}}) \to \mathrm{Hom}(\Xi, \mathcal{Z}_{\mathcal{IW}}(V)) \tag{9.2}$$

induced by $\mathcal{F}_{\mathcal{IW}}$ is injective.

Proof. Since $\mathrm{av}^{\mathrm{as}}_{\mathcal{IW}}$ is fully faithful, it suffices to check the injectivity on the anti-spherical category $\mathcal{P}_{\mathrm{as}}(\mathrm{Hk}_{\mathcal{I}})$. We can also further reduce to verifying injectivity after passing to the quotient $\mathcal{P}_0(\mathrm{Hk}_{\mathcal{I}})$ defined in the previous section. Now, we use the regular orbit $\hat{G}/Z_{\hat{G}}(n_0) \simeq \mathcal{O}_r \subset \hat{\mathcal{N}}$, together with the compatible isomorphism $\mathcal{P}_0(\mathrm{Hk}_{\mathcal{I}}) \cong \mathrm{Rep}_\Lambda(H)$ for a certain subgroup $H \subset Z_{\hat{G}}(n_0)$. In terms of these data, the homomorphism of Hom-groups identifies with $V^{Z_{\hat{G}}(n_0)} \to V^H$, which is clearly injective. $\square$

We deduce our last key calculation:

Corollary 9.5. *For any $V \in \mathrm{Rep}_\Lambda \hat{G}$, any $n \in \mathbb{Z}$ and $\nu \in \mathbb{X}_+$, the natural map*

$$\mathrm{Ext}^n(\mathcal{O}, V \otimes \mathcal{O}(\nu)) \to \mathrm{Ext}^n(\Xi, \mathcal{Z}_{\mathcal{IW}}(V) * \mathcal{J}_\nu) \tag{9.3}$$

is injective.

Proof. The left side identifies with $(V \otimes H^n(\hat{\mathcal{N}}_{\mathrm{Spr}}, \mathcal{O}(\lambda)))^{\hat{G}}$. The higher cohomology of $\mathcal{O}_{\hat{\mathcal{N}}_{\mathrm{Spr}}}(\lambda)$ vanishes, meaning we only need to consider the right side when $n = 0$. Since there exists an equivariant embedding $\mathcal{O}_{\hat{\mathcal{N}}_{\mathrm{Spr}}}(\lambda) \to W \otimes \mathcal{O}_{\hat{\mathcal{N}}_{\mathrm{Spr}}}$ for a certain $W \in \mathrm{Rep}_\Lambda \hat{G}$, the claim reduces to the preceding lemma. $\square$

Finally, we can prove our main theorem, the AB equivalence.

Proof of Theorem 9.1. Applying [AR, Lemma 6.2.6] and the 5-lemma, fully faithfulness will follow from seeing that the injection (9.3) is bijective. Since both sides are finite dimensional Λ-vector spaces, it will be enough to check their dimensions match. Furthermore, once we know $\mathcal{F}_{\mathcal{IW}}$ is fully faithful, we conclude it is an equivalence as its image spans the Iwahori–Whittaker category.

Let us compute the dimension of the right side. After convolution on the right with $\mathcal{J}_{-\nu}(\Lambda)$, it vanishes if $n \neq 0$ by the tilting property of $\mathcal{Z}_{\mathcal{IW}}(V)$, see 7.9, and has otherwise dimension equal to that of the weight space $V(-\nu)$. As for the left side, we have already checked its vanishing if $n \neq 0$ and it has otherwise dimension equal to that of the weight space $V(\nu)$, as one checks via the cohomology of $\mathcal{O}(\nu)$ on the Springer resolution $\hat{\mathcal{N}}_{\mathrm{Spr}}$, compare with [AR, Subsection 6.6.3]. $\square$

10. Exotic t-structure on the Springer resolution

The equivalence in Theorem 9.1 allows us to transport the perverse t-structure on $\mathcal{D}_{\mathrm{cons}}(\mathrm{Hk}_{\mathcal{IW}})$ to a t-structure which we call the *exotic t-structure* on $\mathrm{Perf}([\hat{G}\backslash\hat{\mathcal{N}}_{\mathrm{Spr}}])$, at least when G has enough minuscules. The exotic t-structure has been intrinsically studied in [Bez06, MR16]. In this section, we discuss the exotic t-structure obtained via our object $\mathrm{Fl}_{\mathcal{I}}$, and explain how it will be used to prove the assertions in Remark 8.4 for groups with enough minuscules.

Recall the partial order $\leq$ on $\mathbb{X}_\bullet$ given by $\nu \leq \mu$ if and only if $\mu - \nu$ is a linear combination of positive roots. Note that the ∞-category $\mathrm{Perf}([\hat{G}\backslash\hat{\mathcal{N}}_{\mathrm{Spr}}])$ has finite cohomological dimension by either [BGS96, Corollary 3.2.2] or [DG13, Theorem 1.4.2], i.e., for any objects $\mathcal{A}, \mathcal{B}$, the vector space $\oplus_i \mathrm{Ext}^i(\mathcal{A}, \mathcal{B})$ is finite-dimensional. Then [Bez06, Lemma 5] implies that the line bundles

$\mathcal{O}(\nu)$ form an *exceptional collection* indexed by $\nu \in \mathbb{X}_\bullet$ in the sense of *loc.cit* and generate $\mathcal{D}^b_{\mathrm{coh}}([\hat{G}\backslash\hat{\mathcal{N}}_{\mathrm{Spr}}])$ under shifts and cones.

Choose a refinement of the Bruhat partial order $\leq$ on $\mathbb{X}_\bullet$ to a total order $\leq'$. Now, we can define the *exotic exceptional collection*

$$\{\nabla^{\mathrm{ex}}_\nu \colon \nu \in \mathbb{X}_\bullet\} \tag{10.1}$$

of $\mathcal{D}^b_{\mathrm{coh}}([\hat{G}\backslash\hat{\mathcal{N}}_{\mathrm{Spr}}])$ as the collection of objects produced by *mutation* of $\{\mathcal{O}(\nu)|\nu \in \mathbb{X}_\bullet)\}$ in the sense of [AR, §7.1.2]. By [Bez06, Proposition 3], it in turn gives rise to the dual exotic exceptional collection

$$\{\Delta^{\mathrm{ex}}_\nu \colon \nu \in \mathbb{X}_\bullet\} \tag{10.2}$$

in the sense of *loc.cit.* Define ${}^{\mathrm{ex}}\mathcal{D}^{b,\geq 0}_{\mathrm{coh}}([\hat{G}\backslash\hat{\mathcal{N}}_{\mathrm{Spr}}])$ (resp. ${}^{\mathrm{ex}}\mathcal{D}^{b,\leq 0}_{\mathrm{coh}}([\hat{G}\backslash\hat{\mathcal{N}}_{\mathrm{Spr}}])$) as the full subcategory generated under extensions by objects $\nabla^{\mathrm{ex}}_\nu[n]$ (resp. $\Delta^{\mathrm{ex}}_\nu[n]$) with $\nu \in \mathbb{Z}_{\geq 0}$ and $n \in \mathbb{Z}_{\leq 0}$ (resp. $n \in \mathbb{Z}_{\geq 0}$). Then [Bez06, Proposition 4] shows that the above pair of full subcategories forms a bounded t-structure and we call it the *exotic t-structure*. We denote the heart of this t-structure by $\mathrm{ExCoh}([\hat{G}\backslash\hat{\mathcal{N}}_{\mathrm{Spr}}])$.

Proposition 10.1. *If G has enough minuscules, there are isomorphisms*

$$\mathcal{F}_{\mathcal{IW}}(\nabla^{\mathrm{ex}}_\nu) \cong \nabla^{\mathcal{IW}}_\nu, \tag{10.3}$$

$$\mathcal{F}_{\mathcal{IW}}(\Delta^{\mathrm{ex}}_\nu) \cong \Delta^{\mathcal{IW}}_\nu, \tag{10.4}$$

for any $\nu \in \mathbb{X}_\bullet$.

Proof. The proof follows the idea of [AR, Proposition 7.1.5] in the equicharacteristic situation and we sketch it here. In view of Theorem 9.1, it amounts to prove that the collection $\{\nabla^{\mathcal{IW}}_\nu|\nu \in \mathbb{X}_\bullet\}$ coincide with the collection of exceptional objects that come from the mutation of $\{\mathrm{av}_{\mathcal{IW}}(\mathcal{J}_\nu) \colon \nu \in \mathbb{X}_\bullet\}$ with respect to the Bruhat order on $\mathbb{X}_\bullet$. This is shown by the closure relation of affine Schubert varieties proved in Lemma 2.2. The second isomorphism follows from the uniqueness of the dual exceptional collection. □

We have the following immediate corollary.

Corollary 10.2. *Assume G has enough minuscules. Then, the following hold:*

(1) *The functor $\mathcal{F}_{\mathcal{IW}}$ is t-exact with respect to the exotic t-structure on $\mathcal{D}^b_{\mathrm{coh}}([\hat{G}\backslash\hat{\mathcal{N}}_{\mathrm{Spr}}])$ and the perverse t-structure on $\mathcal{D}_{\mathrm{cons}}(\mathrm{Hk}_{\mathcal{IW}})$.*
(2) *In addition, the functor $\mathcal{F}_{\mathcal{IW}}$ restricts to an equivalence of abelian categories*

$$ {}^pH^0(\mathcal{F}_{\mathcal{IW}}) \colon \mathrm{ExCoh}([\hat{G}\backslash\hat{\mathcal{N}}_{\mathrm{Spr}}]) \xrightarrow{\sim} \mathcal{P}(\mathrm{Hk}_{\mathcal{IW}}). \tag{10.5}$$

We have already seen in §6 that the simple objects of $\mathcal{P}(\mathrm{Hk}_{\mathcal{IW}})$ denoted $\mathrm{IC}^{\mathcal{IW}}_\nu$ are in bijection with $\mathbb{X}_\bullet$. On the coherent side, the space $\mathrm{Hom}(\Delta^{\mathrm{ex}}_\nu, \nabla^{\mathrm{ex}}_\nu)$ is one-dimensional and the image of Δ^{ex}_ν under any non-zero map is a simple object in $\mathrm{ExCoh}([\hat{G}\backslash\hat{\mathcal{N}}_{\mathrm{Spr}}])$. We denote this simple object by L^{ex}_ν.

Lemma 10.3. (1) *The realization functor*

$$\mathcal{D}^b(\mathrm{ExCoh}([\hat{G}\backslash\hat{\mathcal{N}}_{\mathrm{Spr}}])) \to \mathrm{Perf}([\hat{G}\backslash\hat{\mathcal{N}}_{\mathrm{Spr}}])$$

is an equivalence of ∞-categories.
(2) *For any $\nu \in \mathbb{X}^+_\bullet$, there are isomorphisms*

$$\nabla^{\mathrm{ex}}_\nu \simeq \mathcal{O}(\nu), \tag{10.6}$$

$$\Delta^{\mathrm{ex}}_{-\nu} \simeq \mathcal{O}(-\nu) \tag{10.7}$$

(3) *For any $\nu \in \mathbb{X}_\bullet$, there are isomorphisms*

$$\nabla^{\mathrm{ex}}_\nu|_{\tilde{\mathcal{O}}_{\mathrm{r}}} \simeq \mathcal{O}(\nu^+)|_{\tilde{\mathcal{O}}_{\mathrm{r}}}, \tag{10.8}$$

$$\Delta^{\mathrm{ex}}_\nu|_{\mathcal{O}_{\mathrm{r}}} \simeq \mathcal{O}(\nu^-)|_{\mathcal{O}_{\mathrm{r}}}, \tag{10.9}$$

where $\mathcal{O}_{\mathrm{r}} \subset \hat{\mathcal{N}}$ is the regular orbit, and ν^+ (resp. ν^-) is the dominant (resp. anti-dominant) W_{fin}-conjugate of ν.

(4) *Assume G has enough minuscules. Then, for any $\nu \in \mathbb{X}_\bullet$, there is an isomorphism*

$$\mathcal{F}_{\mathcal{IW}}(L^{\mathrm{ex}}_\nu) \cong \mathrm{IC}^{\mathcal{IW}}_\nu. \tag{10.10}$$

Proof. The first three properties appear in [AR, Corollary 7.1.6, Lemmas 7.2.1 and 7.2.2]. We will explain how they follow from Proposition 10.1 for groups with enough minuscules and prove the last claim. It is well-known that $\mathcal{D}^b(\mathcal{P}(\mathrm{Hk}_{\mathcal{IW}})) \cong \mathcal{D}_{\mathrm{cons}}(\mathrm{Hk}_{\mathcal{IW}})$. Assertion (1) then follows from the equivalence (9.1). The first isomorphism in assertion (2) can be easily deduced from (6.7) and Proposition 10.1, and the second isomorphism follows analogously. The statement (3) follows from a standard induction argument on the length of the minimal element $w \in W_{\mathrm{fin}}$ such that $\nu = w\nu^+$(resp. $\nu = w\nu^-$) using (2) and [AR, Proposition 7.1.4]. Assertion (4) follows directly from Proposition 10.1 and Corollary 10.2. □

Corollary 10.4. *For any $A \in \mathrm{ExCoh}(\hat{\mathcal{N}}_{\mathrm{Spr}})$, $A|_{\mathcal{O}_{\mathrm{r}}}$ is a $\hat{G}$-equivariant vector bundle on $\mathcal{O}_{\mathrm{r}}$.*

Proof. The result follows from $\hat{G}/Z_{\hat{G}}(n_0) \simeq \mathcal{O}_{\mathrm{r}}$ and Lemma 10.3. □

Proposition 10.5. *For $\nu \in \mathbb{X}_\bullet$, we have*

$$L^{\mathrm{ex}}_\nu|_{\mathcal{O}_{\mathrm{r}}} \simeq \begin{cases} \mathcal{O} & \text{if } \ell(w_\nu) = 0 \\ 0 & \text{otherwise} \end{cases} \tag{10.11}$$

Proof. This is [AR, Proposition 7.2.4] and we sketch it here for groups with enough minuscules. If $\ell(w_\nu) = 0$, then $\nabla^{\mathcal{IW}}_\nu \cong \mathrm{IC}^{\mathcal{IW}}_\nu$, and we conclude the proof by Lemma 10.3. In general, there exists a unique $\mu \in \mathbb{X}_\bullet$ such that $\ell(w_\mu) = 0$ and $\mathrm{Fl}_{\mathcal{I},w_\nu}$ and $\mathrm{Fl}_{\mathcal{I},w_\mu}$ belong to the same connected component. Then the proof of Lemma 6.5 and Proposition 6.6 imply that $\mathrm{IC}^{\mathcal{IW}}_{w_\mu}$ is a composition factor of $\nabla^{\mathcal{IW}}_\mu$. Then Lemma 10.3 yields that L^{ex}_μ is a composition factor of ∇^{ex}_ν. Combining Lemma 3.9, Lemma 10.3, Corollary 10.4 and the previous discussion, both L^{ex}_μ and ∇^{ex}_ν restricts to an equivariant $\hat{G}$-line bundle on $\tilde{\mathcal{O}}_{\mathrm{r}}$. In particular, as a composition factor of ∇^{ex}_ν, L^{ex}_ν restricts to 0. □

Denote by $\mathrm{Perf}([\hat{G}\backslash\hat{\mathcal{N}}_{\mathrm{Spr}}])_{\mathrm{nr}}$ the full subcategory of perfect complexes supported on the complement of $\mathcal{O}_{\mathrm{r}}$.

Lemma 10.6. *The category $\mathrm{Perf}([\hat{G}\backslash\hat{\mathcal{N}}_{\mathrm{Spr}}])_{\mathrm{nr}}$ is generated by $\{L^{\mathrm{ex}}_\nu | \nu \in \mathbb{X}_\bullet, \ell(w_\nu) > 0\}$ under cones and shifts.*

Proof. This is [AR, Lemma 7.2.7] and we could also prove it for groups with enough minuscules via Corollary 10.4 and Proposition 10.5. □

For the rest of this section, we apply the previous discussion to study the relation between $\mathcal{P}_0(\mathrm{Hk}_{\mathcal{I}})$ and its full subcategory $\mathcal{P}^c_0(\mathrm{Hk}_{\mathcal{I}})$, culminating in the proof that they coincide assuming the existence of enough minuscules and so do $H \subset Z_{\hat{G}}(n_0)$ as promised in Remark 8.4. Recall we define the functor

$$\Pi^0 : \mathcal{P}_0(\mathrm{Hk}_{\mathcal{I}}) \to \mathcal{P}^c_0(\mathrm{Hk}_{\mathcal{I}}) \tag{10.12}$$

in §8. By definition, it factors through the anti-spherical category and we will denote by Π^0_{as} the resulting functor $\mathcal{P}_{\mathrm{as}}(\mathrm{Hk}_{\mathcal{I}}) \to \mathcal{P}_0(\mathrm{Hk}_{\mathcal{I}})$. Theorem 9.1 and Corollary 9.2 show that

$$\mathcal{F}^{\mathrm{as}} := \mathcal{D}^b(\Pi^0_{\mathrm{as}}) \circ \mathcal{F} : \mathrm{Perf}([\hat{G}\backslash\hat{\mathcal{N}}_{\mathrm{Spr}}]) \to \mathcal{D}^b(\mathcal{P}_{\mathrm{as}}(\mathrm{Hk}_{\mathcal{I}})) \tag{10.13}$$

is an equivalence for groups G with enough minuscules.

Proposition 10.7. *Assume G has enough minuscules. Then, there exists a unique t-exact equivalence of ∞-categories*

$$\mathcal{F}^{\mathrm{r}} : \mathrm{Perf}([\hat{G}\backslash\mathcal{O}_{\mathrm{r}}]) \to \mathcal{D}^b(\mathcal{P}_0(\mathrm{Hk}_{\mathcal{I}})), \tag{10.14}$$

fitting into the commutative diagram

$$\begin{tikzcd} \mathrm{Perf}([\hat{G}\backslash\hat{\mathcal{N}}_{\mathrm{Spr}}]) \arrow[r, "\mathcal{F}^{\mathrm{as}}"] \arrow[d] & \mathcal{D}^b(\mathcal{P}_{\mathrm{as}}(\mathrm{Hk}_{\mathcal{I}})) \arrow[d, "\mathcal{D}^b(\Pi^0_{\mathrm{as}})"] \\ \mathrm{Perf}([\hat{G}\backslash\mathcal{O}_{\mathrm{r}}]) \arrow[r, "\mathcal{F}^{\mathrm{r}}"] & \mathcal{D}^b(\mathcal{P}_0(\mathrm{Hk}_{\mathcal{I}})), \end{tikzcd} \tag{10.15}$$

where the left vertical arrow is induced by restriction.

Proof. The proof follows from the idea of [AR, Proposition 7.2.6]. We sketch the proof here and refer to *loc.cit* for details. We first observe that $\mathcal{D}^b(\Pi^0_{\mathrm{as}}) \circ \mathcal{F}^{\mathrm{as}}(L^{\mathrm{ex}}_{\nu}) = \Pi^0(\mathrm{IC}_{w_\nu})$ by Lemma 10.3 and Corollary 9.2. Then it follows from Lemma 10.6 that $\mathcal{D}^b(\Pi^0_{\mathrm{as}}) \circ \mathcal{F}^{\mathrm{as}}$ restricts to zero on $\mathrm{Perf}(\hat{G}\backslash\hat{\mathcal{N}}_{\mathrm{Spr}})_{\mathrm{nr}}$. Note that $\mathrm{Perf}(\hat{G}\backslash\mathcal{O}_{\mathrm{r}})$ is the quotient in Cat_∞ of $\mathrm{Perf}(\hat{\mathcal{N}}_{\mathrm{Spr}})$ by the non-regular full subcategory (idempotent completions are not necessary as the Springer variety and the regular orbit are smooth). On the other hand, as noticed before this proposition, the functor

$$\mathcal{D}^b(\Pi^0_{\mathrm{as}}) : \mathcal{D}^b(\mathcal{P}_{\mathrm{as}}(\mathrm{Hk}_{\mathcal{I}})) \to \mathcal{D}^b(\mathcal{P}_0(\mathrm{Hk}_{\mathcal{I}})) \tag{10.16}$$

is a quotient map in Cat_∞ with kernel given by the full subcategory generated by $\Pi^{\mathrm{as}}(\mathrm{IC}_w)$ with $\ell(w) > 0$. Thus $\mathcal{F}^{\mathrm{r}}$ is an equivalence.

By Corollary 10.2, the restriction functor $\mathcal{F}^{\mathrm{as}} : \mathrm{Perf}([\hat{G}\backslash\hat{\mathcal{N}}_{\mathrm{Spr}}]) \to \mathcal{D}^b(\mathcal{P}_{\mathrm{as}}(\mathrm{Hk}_{\mathcal{I}}))$ is t-exact with respect to the exotic t-structure on the source and the tautological t-structure on the target. Then, to prove $\mathcal{F}^{\mathrm{r}}$ is t-exact, it suffices to show that every simple object in $\mathrm{Coh}([\hat{G}\backslash\mathcal{O}_{\mathrm{r}}])$ is the restriction of an exotic coherent sheaf. The verification of the later assertion can be argued entirely on the coherent side as in [AR, Proposition 7.2.6]. □

Proposition 10.8. *With notations in Proposition 8.3, we have $\mathcal{P}^c_0(\mathrm{Hk}_{\mathcal{I}}) = \mathcal{P}_0(\mathrm{Hk}_{\mathcal{I}})$, and $H = Z_{\hat{G}}(n_0)$.*

Proof. The proof is completely analogous to [AR, Proposition 7.2.8] by our previous preparations. By construction, the projective objects in $\mathrm{Coh}([\hat{G}\backslash\hat{\mathcal{N}}_{\mathrm{Spr}}])$ map to $\mathcal{P}^c_0(\mathrm{Hk}_{\mathcal{I}})$ under $\Pi^0_{\mathrm{as}} \circ \mathcal{F}^{\mathrm{as}}$. Also every coherent sheaf on the regular orbit is a quotient of a projective object in $\mathrm{Coh}([\hat{G}\backslash\hat{\mathcal{N}}_{\mathrm{Spr}}])$ by [AR, Lemma 7.2.9]. Then it follows from Proposition 10.7 that the first assertion holds. Recall the equivalence Φ^0 from Proposition 8.3 between $\mathrm{Rep}H$ and $\mathcal{P}_0(\mathrm{Hk}_{\mathcal{I}})$. On the coherent side, we have an equivalence $\Psi : \mathrm{Coh}(\hat{G}\backslash\mathcal{O}_{\mathrm{r}}) \simeq \mathrm{Rep}(Z_{\hat{G}}(n_0))$ induced by the isomorphism $\mathcal{O}_{\mathrm{r}} \simeq \hat{G}/Z_{\hat{G}}(n_0)$ by the definition of the regular orbit itself. The second statement follows by showing that $\mathrm{For}^{Z_{\hat{G}}(n_0)}_H \circ \Psi$ is equivalent to $\Phi^0 \circ F^{\mathrm{r}}$ and we refer to the end of the proof of [AR, Proposition 7.2.8] for details. □

11. Equivariant coherent sheaves on the nilpotent cone

Recall the Springer resolution

$$p_{\mathrm{Spr}} : \hat{\mathcal{N}}_{\mathrm{Spr}} = \hat{G} \times^{\hat{B}} \mathrm{Lie}(\hat{U}) \to \hat{\mathcal{N}} \tag{11.1}$$

of the nilpotent cone of the dual group $\hat{G}$, defined over the coefficient field $\Lambda = \bar{\mathbb{Q}}_\ell$. In this section, we study the category $\mathrm{Coh}([\hat{G}\backslash\hat{\mathcal{N}}])$ by establishing a connection with a certain quotient of $\mathcal{P}(\mathrm{Hk}_{\mathcal{I}})$ and proving main results of [Bez09] in the mixed-characteristic setting for groups with enough minuscules.

Let $\mathcal{P}_{\mathrm{bas}}(\mathrm{Hk}_{\mathcal{I}})$ denote the quotient of $\mathcal{P}(\mathrm{Hk}_{\mathcal{I}})$ by the Serre subcategory spanned by the IC sheaves of $\mathrm{Fl}_{\mathcal{I},\leq w}$ for non-minimal w in its W_{fin}-double coset. Recall the anti-spherical category $\mathcal{P}_{\mathrm{as}}(\mathrm{Hk}_{\mathcal{I}})$ in Theorem 6.7. The natural functor $\mathcal{P}(\mathrm{Hk}_{\mathcal{I}}) \to \mathcal{P}_{\mathrm{bas}}(\mathrm{Hk}_{\mathcal{I}})$ factors through the quotient

$$\Pi^{\mathrm{as}}_{\mathrm{bas}} : \mathcal{P}_{\mathrm{as}}(\mathrm{Hk}_{\mathcal{I}}) \to \mathcal{P}_{\mathrm{bas}}(\mathrm{Hk}_{\mathcal{I}}). \tag{11.2}$$

In the sequel, we will relate this category to equivariant coherent sheaves on the nilpotent cone.

Theorem 11.1. *Assume G has enough minuscules. Then, there exists a unique equivalence of ∞-categories:*

$$\mathcal{F}_{\mathrm{bas}} \colon \mathcal{D}^b_{\mathrm{coh}}([\hat{G}\backslash\hat{\mathcal{N}}]) \to \mathcal{D}^b(\mathcal{P}_{\mathrm{bas}}(\mathrm{Hk}_{\mathcal{I}})), \tag{11.3}$$

making the following diagram commutes

$$\begin{tikzcd}
\mathcal{D}^b_{\mathrm{coh}}([\hat{G}\backslash\hat{\mathcal{N}}_{\mathrm{Spr}}]) \arrow[r, "\mathcal{F}_{\mathrm{as}}"] \arrow[d, "Rp_{\mathrm{Spr}*}"] & \mathcal{D}^b(\mathcal{P}_{\mathrm{as}}(\mathrm{Hk}_{\mathcal{I}})) \arrow[d, "\mathcal{D}^b(\Pi^{\mathrm{as}}_{\mathrm{bas}})"] \\
\mathcal{D}^b_{\mathrm{coh}}([\hat{G}\backslash\hat{\mathcal{N}}]) \arrow[r, "\mathcal{F}_{\mathrm{bas}}"] & \mathcal{D}^b(\mathcal{P}_{\mathrm{bas}}(\mathrm{Hk}_{\mathcal{I}})),
\end{tikzcd} \tag{11.4}$$

where $\mathcal{F}_{\mathrm{as}}$ is the composition of $\mathcal{F}$ with the functor $\mathcal{D}^b(\mathcal{P}(\mathrm{Hk}_{\mathcal{I}})) \to \mathcal{D}^b(\mathcal{P}_{\mathrm{as}}(\mathrm{Hk}_{\mathcal{I}}))$ induced by the quotient functor $\mathcal{P}(\mathrm{Hk}_{\mathcal{I}}) \to \mathcal{P}_{\mathrm{as}}(\mathrm{Hk}_{\mathcal{I}})$.

Proof. The proof follows the idea of [Bez09, Theorem 1] and [AR, Theorem 7.3.1] in equicharacteristic and we sketch it here. Recall that $\mathcal{D}^b_{\mathrm{coh}}([\hat{G}\backslash\hat{\mathcal{N}}_{\mathrm{Spr}}])$ is the bounded derived category of its abelian heart for the exotic t-structure, see Lemma 10.3. Let $\mathcal{D}$ be the Verdier quotient of $\mathcal{D}^b_{\mathrm{coh}}([\hat{G}\backslash\hat{\mathcal{N}}_{\mathrm{Spr}}])$ by the full subcategory spanned by the L^{ex}_ν with $\nu \notin \mathbb{X}^-_\bullet$ under cones and extensions. Then $Rp_{\mathrm{Spr}*}$ factors as the composition of the quotient $\Pi : \mathcal{D}^b_{\mathrm{coh}}([\hat{G}\backslash\hat{\mathcal{N}}_{\mathrm{Spr}}]) \to \mathcal{D}$ and a functor $\alpha : \mathcal{D} \to \mathcal{D}^b_{\mathrm{coh}}([\hat{G}\backslash\mathcal{N}])$ since $Rp_{\mathrm{Spr}*}(L^{\mathrm{ex}}_\nu) = 0$ for any $\nu \notin \mathbb{X}^-_\bullet$, compare with [Bez09, Lemma 1] and [AR, Lemma 7.3.3].

Similarly, let $\mathcal{D}'$ be the Verdier quotient of $\mathcal{D}^b(\mathcal{P}_{\mathrm{as}}(\mathrm{Hk}_{\mathcal{I}}))$ by the full subcategory spanned by the IC sheaves of the form IC_{w_ν} with $\nu \notin \mathbb{X}^-_\bullet$. Then $\mathcal{D}^b(\Pi^{\mathrm{as}}_{\mathrm{bas}})$ factors through $\mathcal{D}'$ via a functor $\alpha' \colon \mathcal{D}' \to \mathcal{D}^b(\mathcal{P}_{\mathrm{bas}}(\mathrm{Hk}_{\mathcal{I}}))$. We know by [Miy91, Theorem 3.2] that α' is an equivalence. The equivalence $\mathcal{F}_{\mathrm{as}}$ induces an equivalence $\mathcal{D} \simeq \mathcal{D}'$ by Corollary 9.2 and Lemma 10.3. Hence, it suffices to show α is an equivalence. The essential surjectivity follows from [Bez03, Lemma 7] and full faithfulness follows from the abstract [AR, Lemma 7.3.13] together with a few input calculations. ☐

In unpublished notes, Deligne introduces an analogue of the perverse t-structure [BBDG18] on the derived category of coherent sheaves on a Noetherian scheme with a dualizing complex. This t-structure has been studied and extended by Arinkin–Bezrukavnikov [AB10]. In this subsection, we compare the perverse t-structure on $\mathcal{D}^b_{\mathrm{coh}}([\hat{G}\backslash\mathcal{N}])$ with the exotic t-structure transported from the equivalence of Theorem 11.1 and prove [Bez09, Theorem 2, Corollary 1] in our setting.

The following lemma is due to Bezrukavnikov [Bez03].

Lemma 11.2. *The perverse coherent t-structure corresponding to the perversity function* $p(O) = \mathrm{codim}(O)/2$ *is the unique t-structure which has all* $p_{\mathrm{Spr}*}(\mathcal{O}(\nu))$ *lie in its heart.*

Proof. This is [Bez03, Corollary 3]. □

Note that we have an exotic t-structure on $\mathcal{D}^b_{\mathrm{coh}}([\hat{G}\backslash\hat{\mathcal{N}}])$ inherited from the derived category of the Springer variety in virtue of our realization of the former as a Verdier quotient of the latter in Corollary 11.3.

Corollary 11.3. *The exotic t-structure on* $\mathcal{D}^b_{\mathrm{coh}}([\hat{G}\backslash\hat{\mathcal{N}}])$ *identifies with the perverse coherent t-structure with perversity function* $p(O) = \mathrm{codim}(O)/2$.

Proof. The statement follows directly from Theorem 11.1 in light of Lemma 11.2. □

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DISTRIBUTIONS AND NORMALITY THEOREMS

JOÃO LOURENÇO

Abstract. We derive a Serre presentation of distribution algebras of loop groups in characteristic p and apply it to give a new proof of the normality of Schubert varieties inside parahoric affine Grassmannians, for all connected reductive groups whose fundamental group is p-torsion free.

Contents

1. Introduction

Let F be a field of characteristic p and G be a connected reductive group over F. If we want to understand the infinitesimal behavior of G near the identity, it is well known from modular representation theory that $\mathrm{Lie}(G)$ is insufficient. Indeed, the category of $\mathrm{Lie}(G)$-modules barely captures information on the category of G-representations. Instead, what one ought to consider is the F-algebra $\mathscr{D}(G)$ of distributions of G at the origin consisting of higher differential operators. This object is a sort of twisted divided power algebra and an explicit presentation in terms of generators and relations was given by Takeuchi [Tak83a, Tak83b] for split G. The idea for producing these generators and relations is to take the distributions of tori and root groups and evaluating on the rules of multiplications between those subgroups of G. In particular, the list of relations is not finite, and is best expressed in terms of generating series.

The point of $\mathscr{D}(G)$ is that it carries the same information as the formal group $\hat{G}$ given as the completion of G at the origin but in such a way that we get a covariant algebra object instead of a covariant geometric space (or contravariant algebra object by passing to formal sections). We became interested in it because of how it naturally fits into the study of loop groups. Assume from now on that $F = k(\!(t)\!)$ is a local field with finite residue field k. The loop group of G is the group object in ind-schemes $\mathrm{R}_{F/k}G$ whose functor of points is given by $R \mapsto G(R(\!(t)\!))$. Similarly, if we let $\mathcal{G}$ be a parahoric model of G over the ring of integers $O = k[\![t]\!]$ in the sense of Bruhat–Tits [BT72, BT84], then we have the group object $\mathrm{R}_{O/k}\mathcal{G}$ in schemes given by $R \mapsto \mathcal{G}(R[\![t]\!])$. The affine flag variety $\mathrm{Gr}_{\mathcal{G}}$ is the étale quotient $\mathrm{R}_{F/k}G/\mathrm{R}_{O/k}\mathcal{G}$ in the category of ind-schemes and classifies modifications of $\mathcal{G}$-torsors over O away from the residue field k. The Bruhat stratification yields Schubert varieties $\mathrm{Gr}_{\mathcal{G},l,\leq w}$ defined over the reflex field l of the element w in the absolute Iwahori–Weyl group.

Theorem 1.1. *If $p \nmid \#\pi_1(G_{\mathrm{der}})$, then $\mathrm{Gr}_{\mathcal{G},l,\leq w}$ is normal, Cohen–Macaulay, rational and globally $+$-regular.*

This result appears already in [Fal03, PR08, FHLR22] for every group except odd unitary ones if $p = 2$. The point of this paper is to give a new uniform proof, so first we have to review the history behind it. Faltings [Fal03] proved the theorem for split groups and his proof had two steps: (i) applying the Mehta–Ramanathan criterion [MR85] on φ-split varieties, where φ denotes the Frobenius map, to show that the transition maps of normalized Schubert varieties are closed immersions; (ii) use integral Lie-theoretic arguments to prove that the embedded Schubert varieties are already normal. Then, Pappas–Rapoport [PR08] applied this strategy to the case of tame groups. For non-tame G, we face the following obstacles: (i) demands a divisor on $\mathrm{Gr}_{\mathcal{G}}$ with specific degrees, usually defined via negative loop groups, which do not seem to exist beyond the tame case; (ii) requires an integral lift of the group theoretic data to a two dimensional ring such as $W(k)[\![t]\!]$, which does not seem to exist for odd unitary groups if $p = 2$.

A non-negligible portion of our research was dedicated to overcome some of the above obstacles. First, we show in [HLR24] that most Schubert varieties are not normal if $p \mid \#\pi_1(G_{\mathrm{der}})$, see [BR23] for the full classification at hyperspecial level. As for tameness, we lifted it almost completely in [Lou23, FHLR22] as follows. For part (i), we retrieve the critical divisors for all groups by a reduction to the split case, still relying on a case-by-case analysis. For part (ii), we constructed $W(k)$-lifts for all groups except odd unitary ones when $p = 2$, and could finish Faltings' proof of embedded normality.

Since then, we found an alternative approach to (i). In [CL25] we replace the Mehta–Ramanathan criterion [MR85] for φ-splitness by the Bhatt criterion [Bha12] for splinters, also known as globally $+$-regular varieties in [BMP+23]. This criterion involves choosing auxilliary $\mathbb{Q}$-Cartier boundaries inside Demazure varieties but does not require defining Cartier divisors in $\mathrm{Gr}_{\mathcal{G}}$. This gives a uniform proof of part (i) for all G. Improving the strategy for part (ii) is the goal of this paper: we want to explain a new method that uniformly proves embedded normality for all groups.

At this point, the reader probably already guessed how distributions fit into the normality picture. The main idea is that the integral Lie algebra methods should be replaced by dealing directly with the associative k-algebra $\mathscr{D}(\mathrm{R}_{F/k}G)$ of loop distributions. This algebra carries a natural topological structure, so we use the formalism of condensed mathematics of Clausen–Scholze [CS19]. We define a Serre presentation for $\mathscr{D}(\mathrm{R}_{F/k}G)$ following [Tak83a, Tak83b] to reduce to rank 1 groups. If G is a restriction of scalars of SL_2, we can make the calculations explicitly, as its Schubert varieties are lci. If G is a restriction of scalars of SU_3, we couldn't perform the calculations effectively, so instead more effort is required via the theory of local models, which we explain next.

Beilinson–Drinfeld [BD91] attach a more general loop group $\mathrm{R}_{O^2_\circ/O}\mathcal{G}$ to the parahoric model $\mathcal{G}$, where the restriction of scalars is along the inclusion of the second factor in the complement of the diagonal of O, and the $\mathcal{G}$-structure is along the first factor. It carries the jet subgroup $\mathrm{R}_{O^2/O}\mathcal{G}$ and the étale quotient $\mathrm{Gr}_{\mathcal{G},O}$ is called the Beilinson–Drinfeld Grassmannian. Its generic fiber is isomorphic to the usual affine Grassmannian $\mathrm{Gr}_{G,F}$ of the F-group G, whereas the special fiber equals $\mathrm{Gr}_{\mathcal{G},k}$. Let $\mathrm{Gr}_{\mathcal{G},O_E,\leq\mu}$ be the scheme-theoretic image of $\mathrm{Gr}_{G,E,\leq\mu} \to \mathrm{Gr}_{\mathcal{G},O_E}$, where μ is a conjugacy class of absolute coweights of G and E is its reflex field.

Theorem 1.2. *If $p \nmid \#\pi_1(G_{\mathrm{der}})$, then $\mathrm{Gr}_{\mathcal{G},O_E,\leq\mu}$ is normal and Cohen–Macaulay with φ-split special fiber.*

Again the proof of Theorem 1.2 can be found in [Zhu14, FHLR22] if $p > 2$ or G is SU_3-free. Zhu [Zhu14] handles tamely ramified groups via a global φ-splitting, which we cannot generalize to all G. Instead, we prove it via the unibranch theorem of [GL24] together with Theorem 1.1. We also use $\mathrm{Gr}_{\mathcal{G},O}$ and a weaker variant of Theorem 1.2 due to [CL25] to finish the proof of Theorem 1.1 when $G = \mathrm{SU}_3$. If $\mathcal{G}$ is a special parahoric with simply connected reductive quotient, then $\mathscr{D}(\mathrm{R}_{O^2_\circ/O}\mathcal{G})$ is generated by its unipotent part. Using the case of SL_3 in the generic fiber, this implies semi-normality of $\mathrm{Gr}_{\mathcal{G},O}$ and hence also of its special fiber $\mathrm{Gr}_{\mathcal{G},k}$ by a computation with minuscule coweights.

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2. Affine Grassmannians

2.1. **Ind-schemes.** In this paper, an ind-scheme will always mean a colimit of qcqs schemes along closed immersions. The qcqs hypothesis facilitates handling morphisms of ind-schemes, as they necessarily respect scheme presentations. Qcqs formal schemes obviously embed fully faithfully into the category of ind-schemes. Indeed, it is important for us to regard formal schemes as ind-schemes, and thus their underlying reduced subscheme equals the reduction in the category of ind-schemes.

Let k be a finite field and consider the category of pointed ind-k-schemes (X, x), where X is an ind-scheme over k and x is a k-valued point of X. We say that a pointed k-scheme (Z, z) is nilpotent if its reduction equals z (and thus Z is affine) and its radical ideal I_z is nilpotent, i.e., a power I_z^n of it vanishes for some $n \gg 0$. It is decisive to focus our attention on nilpotent rather than nil-ideals, i.e., those whose elements are nilpotent, because we will encounter many ideals which are not finitely generated.

Definition 2.1. The formal completion $\widehat{X}_x$ of a pointed ind-k-scheme (X, x) is the filtered colimit of all closed nilpotent pointed k-subschemes $(Z, x) \subset (X, x)$.

Similarly, we define the ring of formal sections $\Gamma(\widehat{X}_x, \mathcal{O})$ of (X, x) to be the limit of the rings $\Gamma(Z, \mathcal{O})$. This ring admits the structure of a solid commutative k-algebra in the sense of Clausen–Scholze [CS19, Proposition 7.5, Theorem 8.1], as $\Gamma(Z, \mathcal{O})$ is a colimit of finite k-modules and k is a finitely generated $\mathbb{Z}$-algebra. Moreover, this induces a contravariant functor from pointed ind-k-schemes to solid commutative k-algebras.

We shall also apply the notion of formally unramified, formally étale, and formally smooth maps $f\colon (X, x) \to (Y, y)$. For us, this means that

$$\mathrm{Hom}((Z, z), (X, x)) \to \mathrm{Hom}((W, w), (X, x)) \times_{\mathrm{Hom}((W,w),(Y,y))} \mathrm{Hom}((Z, z), (Y, y)) \qquad (2.1)$$

is either injective, bijective, or surjective, for every closed embedding $(W, w) \subset (Z, z)$ of nilpotent pointed k-schemes. During the main part of the paper, we will encounter formally étale maps of ind-schemes which are far from being representable, so the following assertion will be key.

Lemma 2.2. *A formally étale map $f\colon (X, x) \to (Y, y)$ of pointed ind-k-schemes induces an isomorphism on formal completions.*

Proof. Without loss of generality, we may and do assume that Y is a nilpotent scheme and X is its own formal completion at x. By formal étaleness, we get a unique section $s\colon Y \to X$ of f. This factors through a nilpotent subscheme X' by quasi-compactness and the resulting map $f'\colon X' \to Y$ is necessarily formally étale. In particular, we may now assume that $X = X'$ is a nilpotent k-scheme. Finally, we claim that also $s \circ f$ is the identity map of X. By formal étaleness, we can check this after post-composing with f, and it is then obvious. $\square$

Similarly, formal étaleness can be detected in terms of formal sections.

Lemma 2.3. *A map $f\colon (X,x) \to (Y,y)$ of pointed ind-schemes is formally étale if and only if it induces an isomorphism of their formal sections as solid commutative k-algebras.*

Proof. We may and do assume that Y is a nilpotent scheme and X is a nilpotent ind-scheme. Then, we have that $\Gamma(Y,\mathcal{O}) = \Gamma(X,\mathcal{O})$ where the left side is discrete (i.e., a colimit of finite k-modules) and the right side is a limit of discrete solid k-modules. In particular, the map factors through some nilpotent subscheme $X' \subset X$. We deduce that $\Gamma(X',\mathcal{O}) = \Gamma(X,\mathcal{O})$ and hence $X' = X$. □

We are also interested in understanding absolute properties for ind-schemes. We will say that an ind-scheme has a certain property (P) if it admits a presentation all of whose constituents satisfy the property (P), compare with [Ric20, Definition 1.15]. If (P) is preserved under closed immersions, then it does not depend on the choice of a presentation: this holds for many of the adjectives that we will employ for ind-schemes such as affine, separated, proper, projective. Then, there are other properties that one can also define via a universal property, such as reduced and semi-normal. It turns out that there always exist a universal reduced sub-ind-scheme $X^{\mathrm{red}} \to X$, see [Ric20, Lemma 1.17].

Let us review the notion of semi-normal schemes. A scheme X is semi-normal if every universal homeomorphism $Y \to X$ with trivial residue field extensions is an isomorphism. For any scheme X, there exists a initial morphism $X^{\mathrm{sn}} \to X$ with X^{sn} semi-normal: we call it the semi-normalization of X. The assignment $X \to X^{\mathrm{sn}}$ defines a functor, i.e., morphisms lift to their semi-normalizations. We refer to [Sta23, Tag 0EUS] for the proof of the previous assertions. Taking colimits, we see that $X \mapsto X^{\mathrm{sn}}$ extends to a functor from ind-schemes to sheaves on k-algebras. The resulting sheaf will be an ind-scheme too if the transition maps are closed immersions. Fortunately, this will happen for loop groups of reductive and unipotent groups.

2.2. **Loop groups.** From now on, let k be a finite field, $F = k(\!(t)\!)$ the local field of Laurent series with coefficients in k, and $O = k[\![t]\!]$ be the power series ring over k. In this paper, we are interested in the loop space $\mathrm{R}_{F/k}X$ of a finite type affine F-scheme X. This is the functor on k-algebras given by $A \mapsto X(A(\!(t)\!))$, and it is representable by an ind-k-scheme, compare with [PR08, 1.a]. If $\mathcal{X}$ is an affine O-model of X, we can define a jet space $\mathrm{R}_{O/k}\mathcal{X}$ as the functor given by $A \mapsto X(A[\![t]\!])$. It turns out that the natural map $\mathrm{R}_{O/k}\mathcal{X} \to \mathrm{R}_{F/k}X$ is a closed immersion and it realises the left side as an affine scheme that is almost never of finite type, compare again with [PR08, 1a]. Let us mention some of their basic properties, starting with jet spaces.

Lemma 2.4. *The functor $\mathcal{X} \mapsto \mathrm{R}_{O/k}\mathcal{X}$ from finite type affine O-schemes to affine k-schemes is a limit of the functors $\mathrm{R}_{O_n/k}$, where $O_n = O/t^nO$. In particular, $\mathrm{R}_{O/k}$ preserves immersions and carries smooth schemes to pro-smooth schemes.*

Proof. This is standard, compare with [PR08, 1a], [Zhu17b, Proposition 1.3.2] and [Ric20, Lemma 3.17]. The first part follows by definition itself of the jet group and affineness of $\mathcal{X}$ for commuting with limits. It is clear that it preserves closed immersions by general properties of limits along affine morphisms, and also open immersions because k is the reduction of O_n, which implies that $\mathrm{R}_{O_n/k}\mathcal{U} \subset \mathrm{R}_{O_n/k}\mathcal{X}$ is the pullback of $\mathcal{U}_k \subset \mathcal{X}_k$. The final assertion on smooth schemes is obvious by [CGP15, Proposition A.5.11]. □

For the loop group functor $\mathrm{R}_{F/k}$, it is no longer true that it respects open immersions. Indeed, we will see that $\mathrm{R}_{F/k}\mathbb{G}_{\mathrm{m},F}$ is not reduced in Proposition 2.8, whereas $\mathrm{R}_{F/k}\mathbb{G}_{\mathrm{a},F}$ clearly is. However, this still has a remedy at the formal level.

Lemma 2.5. *The functor $X \mapsto \mathrm{R}_{F/k}X$ from finite type affine F-schemes to affine ind-k-schemes preserves closed immersions and formally étale maps.*

Proof. The assertion for closed immersions follows easily by reduction to $X = \mathbb{A}^n_F$. As for formal étaleness, let (Z, z) be any nilpotent scheme. Showing that a map $\mathrm{R}_{F/k} f \colon \mathrm{R}_{F/k} X \to \mathrm{R}_{F/k} Y$ lifts uniquely against the inclusion $z \subset Z$ amounts to showing that $f \colon X \to Y$ lifts uniquely against $z(\!(t)\!) \subset Z(\!(t)\!)$, which holds by definition. Here, $(Z(\!(t)\!), z(\!(t)\!))$ indicates the nilpotent pointed F-scheme such that $\Gamma(Z(\!(t)\!), \mathcal{O}_{Z(\!(t)\!)}) = \Gamma(Z, \mathcal{O}_Z)(\!(t)\!)$. □

Let G be a connected reductive F-group and $\mathcal{G}$ be a parahoric O-model of G in the sense of [BT84, 5.1.9, Définition 5.2.6]. Note that the loop and jet spaces $\mathrm{R}_{F/k} G$ and $\mathrm{R}_{O/k} \mathcal{G}$ are now group objects in the category of ind-schemes. Now, we turn to the flag variety arising as the quotient

$$\mathrm{Gr}_{\mathcal{G}} = (\mathrm{R}_{F/k} G)/(\mathrm{R}_{O/k} \mathcal{G}) \tag{2.2}$$

for the étale topology. This is representable by an ind-scheme by [PR08, Theorem 1.4]. The Bruhat decomposition for parahoric models of reductive groups over non-archimedean fields assumes the form

$$G(\breve{F}) = \bigsqcup_{w \in W_{\mathcal{G}} \backslash W / W_{\mathcal{G}}} \mathcal{G}(\breve{O}) \dot{w} \mathcal{G}(\breve{O}). \tag{2.3}$$

Here, we let $T \subset G$ denote a maximally $\breve{F}$-split maximal F-torus, see [BT84, Corollaire 5.1.12], $\mathcal{T}$ be the connected Néron O-model of T, N the normalizer of T inside G, $W := N(\breve{F})/\mathcal{T}(\breve{O})$ the Iwahori–Weyl group, and $W_{\mathcal{G}} \subset W$ the subgroup of elements whose lifts $\dot{w} \in N(\breve{F})$ are contained in $\mathcal{G}(\breve{O})$, see [KP23, Theorem 7.5.3, Proposition 7.5.5]. For every $w \in W$, we define $\mathrm{Gr}_{\mathcal{G},l,w}$ as the étale descent to the reflex field l of w of the smooth locally closed orbit of w inside $\mathrm{Gr}_{\mathcal{G},\bar{k}}$ under the jet $\bar{k}$-group of $\mathcal{G}$. Its scheme-theoretic closure $\mathrm{Gr}_{\mathcal{G},l,\leq w}$ inside $\mathrm{Gr}_{\mathcal{G},l}$ is a reduced integral projective scheme, compare with [PR08, Definition 8.3]. This is called the Schubert variety associated with w and our goal in this paper is to study its geometry.

We denote by $\mathrm{Gr}^{\mathrm{sn}}_{\mathcal{G},l,\leq w}$ the semi-normalizations of the Schubert varieties embedded in $\mathrm{Gr}_{\mathcal{G},l}$. Note that, while we have canonical transition morphisms $\mathrm{Gr}^{\mathrm{sn}}_{\mathcal{G},l,\leq v} \to \mathrm{Gr}^{\mathrm{sn}}_{\mathcal{G},l,\leq w}$ over a common reflex field, it is not clear whether they are closed immersions. In particular, it is not a priori clear that the sheaf $\mathrm{Gr}^{\mathrm{sn}}_{\mathcal{G}}$ is an ind-scheme in our strict sense. Fortunately, these problems dissipate thanks to the following key theorem on semi-normalized Schubert varieties.

Theorem 2.6 ([FHLR22, CL25])**.** *The l-varieties $\mathrm{Gr}^{\mathrm{sn}}_{\mathcal{G},l,\leq w}$ are normal, Cohen–Macaulay, rational, compatibly φ-split and globally $+$-regular. In particular, the semi-normalization $\mathrm{Gr}^{\mathrm{sn}}_{\mathcal{G}}$ is an ind-k-scheme.*

As explained in the introduction, this result goes back to [Fal03, Theorem 8] for split G, [PR08, Theorem 8.4] for tame G, and [FHLR22, Theorem 4.1] for all G. A new uniform proof was given in [CL25, Theorem 4.8, Proposition 4.11]. Before concluding this section, we want to explain the several notions appearing in the statement and the proof strategies.

A noetherian scheme X is Cohen–Macaulay if the depth of its local rings (i.e., the maximal length of its regular sequences) equals their Krull dimensions, see [Sta23, Tag 00N7]. Equivalently, one can define Cohen–Macaulayness by demanding that the dualizing complex $\omega^\bullet_X$ is concentrated in a single degree or that the lower (i.e., below the Krull dimension) local cohomology groups $H^i_x(\mathcal{O}_{X,x})$ all vanish, compare with [Bha20, Definition 2.1, Example 2.5]. We say that a normal Cohen–Macaulay k-variety X is rational if for any proper birational map $f \colon Y \to X$ of k-schemes with Y smooth, the higher direct image sheaves $R^i f_* \mathcal{O}_Y$ vanish for all $i > 0$. It is enough to verify this condition for only one resolution by [CR11, Theorem 1] and it implies vanishing of $R^i f_* \omega_Y$ for all $i > 0$ by Cohen–Macaulayness and Grothendieck–Serre duality. Finally, we say that X is φ-split if the Frobenius $\varphi \colon \mathcal{O}_X \to \varphi_* \mathcal{O}_X$ splits as a map of $\mathcal{O}_X$-modules, and compatibility carries the obvious meaning, compare with [BS13, Definitions 5.0.1 and 5.1.4]. More generally, X is a splinter in the sense of [Bha12, Definition 0.1] or globally

$+$-regular in the sense of [BMP+23, Definition 6.1] if $\mathcal{O}_X \to \mathcal{O}_Y$ splits as $\mathcal{O}_X$-modules for any finite cover $Y \to X$. It is proved in [Bha12, Corollary 5.3] that splinters are automatically normal and Cohen–Macaulay. There is also a notion of strong φ-regularity generalizing φ-splitness and which is only slightly stronger than global $+$-regularity.

Let us now sketch the proof of Theorem 2.6. Let $\mathcal{I}$ be a Iwahori O-model of G obtained from $\mathcal{G}$ by dilatation. The transition map $\mathrm{Gr}_{\mathcal{I}} \to \mathrm{Gr}_{\mathcal{G}}$ is proper smooth, so it suffices to handle the Iwahori case. Up to translation into the neutral component and after enlarging the finite field k, any Schubert variety $\mathrm{Gr}_{\mathcal{I},\leq w}$ is resolved by the convolution Schubert variety $\mathrm{Gr}_{\mathcal{I},\leq s_\bullet}$ where $s_\bullet$ is a sequence of positive simple reflections in the absolute Iwahori–Weyl group. This variety is an iterated $\mathbb{P}^1_k$-bundle, so it is smooth in particular, and it is usually called the BSDH variety after work of Bott–Samelson [BS58], Demazure [Dem74], and Hansen [Han73]. Prior to [CL25], the existence of a φ-splitting of $\mathrm{Gr}_{\mathcal{I},\leq s_\bullet}$ was always deduced via the Mehta–Ramanathan criterion [MR85]. This requires producing a Cartier divisor on $\mathrm{Gr}_{\mathcal{I}}$ with degree 1 on each $\mathrm{Gr}_{\mathcal{I},\leq s_i}$, which is difficult to define homogeneously, see [FHLR22, Section 4] for the full proof using some case division. This upgrades to a splitting of the absolute integral closure by the proof of [Cas22, Theorem 1.4], whose method goes back to [LRPT06] for classical flag varieties. In [CL25], we approach the problem instead via inversion of adjunction following [Bha12, Proposition 7.2] refined as in [BMP+23, Theorem 7.2], i.e. we split absolute integral closure by induction on the length of $s_\bullet$. We have to perform a few calculations with auxilliary boundaries, but the proof remains uniform for all G throughout.

2.3. **A rank 1 example.** In this subsection, we calculate Schubert varieties for rank 1 split groups, and in particular verify their normality as in Theorem 2.9. We will do this by exploiting certain presentations of Schubert varieties at special level, which appear in [Zhu17a, Subsection 1.2.2] and also [SW20, Lemmas 19.3.5 to 19.3.7]. Let $G = \mathrm{GL}_{2,F}$, $\mathcal{G} = \mathrm{GL}_{2,O}$, and $\mu = (1,0)$ be its only dominant minuscule coweight. Up to translation, the Schubert varieties inside $\mathrm{Gr}_{\mathrm{GL}_{2,O}}$ are of the form $\mathrm{Gr}_{\mathrm{GL}_{2,O},\leq n\mu}$.

Proposition 2.7. *The variety* $\mathrm{Gr}_{\mathrm{GL}_{2,O},\leq n\mu}$ *is normal and lci.*

Proof. We follow [Zhu17a, Lemma B.4]. Note that every $\mathrm{R}_{O/k}\mathrm{GL}_{2,O}$-orbit has codimension at least 2 in a larger one, so we at least know that every Schubert variety is regular in codimension 1. We just have to verify that it is lci.

Let $\mathrm{Mat}_{2,O}$ be the O-scheme of 2-by-2 matrices and consider the closed k-subscheme $X_n \subset \mathrm{R}_{O/k}\mathrm{Mat}_{2,O} \cap \mathrm{R}_{F/k}\mathrm{GL}_{2,F}$ given by the preimage of t^n along the determinant map $\det\colon \mathrm{R}_{F/k}\mathrm{GL}_{2,F} \to \mathrm{Gr}_{\mathbb{G}_{\mathrm{m},O}}$. We claim that the map $X_n \to \mathrm{Gr}_{\mathrm{GL}_{2,O}}$ equals the natural $\mathrm{R}_{O/k}\mathrm{GL}_{2,O}$-torsor over $\mathrm{Gr}_{\mathrm{GL}_{2,O},\leq n\mu}$.

At the level of geometric points, this equality of closed subfunctors is an immediate consequence of the elementary divisor theorem. Therefore, it suffices to verify that X_n is reduced. For this, we note that X_n is a torsor over a finite type k-scheme Y_n under the congruence subgroup of $\mathrm{R}_{O/k}\mathrm{Mat}_{2,O}$ arising as the kernel of the map towards $\mathrm{R}_{O_{n+1}/k}\mathrm{Mat}_{2,O_{n+1}}$, where $O_{n+1} = O/t^{n+1}O$. We start by proving that Y_n is generically reduced. For this, consider its tangent space at the matrix $(t^n,0,0,1)$. It equals the k-submodule of $\mathrm{Mat}_2(O_{n+1})$ whose $(1,1)$-entry is divisible by t^n, and thus has dimension equal to $4(n+1)-n = 3n+4$. If we consider the stabilizer of the matrix $(t^n,0,0,1)$ under left and right multiplication by $\mathrm{R}_{O_{n+1}/k}\mathrm{GL}_{2,O_{n+1}}$, we get the matrix equality $(t^n a, b, t^n c, d) = (t^n e, t^n f, g, h)$, and hence the topologically dense smooth orbit has dimension equal to $8n+8-4n-4-n = 3n+4$, and this implies generic reducedness. Finally, if we write down the determinant as a power series modulo t^{n+1}, we conclude that Y_n sits inside $\mathbb{A}^{4n+5}_k$ and is defined by $n+1$ equations. This implies that Y_n is indeed a normal complete intersection. $\square$

2.4. **Normality.** In this subsection, we discuss reducedness and semi-normality of loop groups $\mathrm{R}_{F/k}G$ attached to connected reductive groups. For simplicity, we use the shorthand notation $\mathrm{R}^{\mathrm{red}}_{F/k}G := (\mathrm{R}_{F/k}G)^{\mathrm{red}}$, resp. $\mathrm{R}^{\mathrm{sn}}_{F/k}G := (\mathrm{R}_{F/k}G)^{\mathrm{sn}}$ to denoted the reduction and semi-normalization of our ind-schemes. Let us start by the problem of reducedness, since it is the most simple.

Proposition 2.8. *The ind-scheme $\mathrm{R}_{F/k}G$ is reduced if and only if G is semi-simple and $p \nmid \#\pi_1(G)$.*

Proof. The non-reducedness for non-semisimple G was proved in [PR08, Proposition 6.5]. The obstruction for tori T lies in the fact that the reduction of Gr_T is a zero-dimensional scheme locally of finite type, and hence $\mathrm{Lie}(\mathrm{R}^{\mathrm{red}}_{F/k}T) = \mathrm{Lie}(\mathcal{T})$ is not equal to $\mathrm{Lie}(T)$. Let $A = G/G_{\mathrm{der}}$ denote the abelian quotient of G. If $\mathrm{R}_{F/k}G$ were reduced, then $\mathrm{Lie}(G)$ would map onto $\mathrm{Lie}(\mathcal{A}) \subset \mathrm{Lie}(A)$, contradicting smoothness of the map of F-schemes $G \to A$.

Assume that G is semi-simple but p divides the order of $\pi_1(G_{\mathrm{der}})$. In [HLR24, Proposition 7.10], we proved that $\mathrm{R}_{F/k}G$ is non-reduced when G is tame. First, note that $G_{\mathrm{sc}} \to G$ is a central isogeny whose kernel μ is not étale. In particular, we know by [CGP15, Examples 1.3.2 and A.7.9] that $\mathrm{R}_{F/F^p}\mu$ is positive-dimensional and hence $\mathrm{R}_{F/F^p}G_{\mathrm{sc}}$ does not surject onto $\mathrm{R}_{F/F^p}G$. Since Schubert varieties of isogenous group have isomorphic semi-normalizations, we deduce that if $\mathrm{R}_{F/k}G$ were reduced, then we would have an equality

$$\mathrm{Lie}(\mathrm{R}_{F/F^p}G_{\mathrm{sc}}/\mathrm{R}_{F/F^p}\mu_p) + \mathrm{Lie}(\mathcal{G}) = \mathrm{Lie}(G) \tag{2.4}$$

of Lie algebras. But the left side is the sum of a proper F^p-subspace and an O^p-lattice, so this equality can never hold.

Next, we consider the case when G is simply connected, that was handled by [Fal03, Corollary 11] for split G and [PR08, Proposition 9.9] for tame G. It will follow from our calculations of distribution algebras later on that $\mathrm{R}^{\mathrm{red}}_{F/k}G \to \mathrm{R}_{F/k}G$ is formally étale, e.g., see Lemma 3.2 and Proposition 3.5. Modding out by the pro-smooth group $\mathrm{R}_{O/k}\mathcal{G}$, we deduce that the reduction map of $\mathrm{Gr}_{\mathcal{G}}$ is formally étale at every point by homogeneity. Now, [HLR24, Lemma 8.6] implies that it is an isomorphism.

Finally, we treat the case when G is semisimple and p does not divide the order of $\pi_1(G_{\mathrm{der}})$, due to [PR08, Theorem 6.1] for tame G. Notice that the kernel μ of $G_{\mathrm{sc}} \to G$ is étale. Consequently, $1_k = \mathrm{R}_{F/k}1_F \to \mathrm{R}_{F/k}\mu$ is formally étale, and the right side has finitely many points, so we conclude that it is also an étale k-scheme. In particular, $\mathrm{R}_{F/k}G_{\mathrm{sc}} \to \mathrm{R}_{F/k}G$ is an étale cover on neutral components and we deduce that $\mathrm{R}_{F/k}G$ is also reduced. □

Next, we move to the problem of semi-normality. In this paper, we work mostly with the full loop group $\mathrm{R}_{F/k}G$ during the proofs to fully exploit its multiplication law. However, for classical reasons, we state the result below for Schubert varieties in $\mathrm{Gr}_{\mathcal{G}}$.

Theorem 2.9. *If $p \nmid \#\pi_1(G_{\mathrm{der}})$, then $\mathrm{Gr}_{\mathcal{G},l,\leq w}$ is normal.*

In particular, we deduce by Theorem 2.6 that the Schubert varieties $\mathrm{Gr}_{\mathcal{G},\leq w}$ are normal, Cohen–Macaulay, rational, and globally $+$-regular. This result is found in [Fal03, Theorem 8] for split G, [PR08, Theorem 8.4] for tame G, and [FHLR22, Theorem 4.23] if $p > 2$ or G is SU_3-free. As mentioned in the introduction, those papers employ Lie-theoretic considerations to ad hoc $W(k)$-lifts of $\mathrm{R}_{F/k}G$. In this paper, we argue via distributions and therefore our proof must be postponed until the very end, see Corollaries 3.11 and 3.13. Before moving on, let us perform a helpful reduction step.

Lemma 2.10. *Theorem 2.9 holds if $\mathrm{R}^{\mathrm{sn}}_{F/k}G_{\mathrm{sc}} \to \mathrm{R}_{F/k}G_{\mathrm{sc}}$ is formally étale.*

Proof. First of all, note that, if p does not divide the order of $\pi_1(G_{\mathrm{der}})$, the neutral component of $\mathrm{R}^{\mathrm{red}}_{F/k}G$ admits $\mathrm{R}_{F/k}G_{\mathrm{sc}}$ as an étale cover by the proof of Proposition 2.8, so this reduces our problem to simply connected G. Suppose that $\mathrm{R}^{\mathrm{sn}}_{F/k}G \to \mathrm{R}_{F/k}G$ is formally étale. Quotienting out the right action of the pro-smooth jet group $\mathrm{R}_{O/k}\mathcal{G}$, we deduce that $\mathrm{Gr}^{\mathrm{sn}}_{\mathcal{G}} \to \mathrm{Gr}_{\mathcal{G}}$ is formally étale. Restricting to Schubert varieties, we also see that $\mathrm{Gr}^{\mathrm{sn}}_{\mathcal{G},\leq w} \to \mathrm{Gr}_{\mathcal{G},\leq w}$ is formally unramified. This implies by [Sta23, Tag 04XV] that the semi-normalization map is a closed immersion, and thus an isomorphism, around the identity. By $\mathrm{R}_{O/k}\mathcal{G}$-equivariance, this propagates to the entire variety, which is normal by Theorem 2.6. □

2.5. **Beilinson–Drinfeld deformation.** In this subsection, we discuss the deformation of the k-ind-scheme $\mathrm{Gr}_{\mathcal{G}}$ to the ring of integers O as defined by Beilinson–Drinfeld [BD91]. This means we have to consider the category of qcqs pointed ind-schemes (X,x) over O, where x stands for a section of the structure map $X \to \mathrm{Spec}(O)$. We can still define corresponding O-relative versions of the formal completion and the ring of formal sections, where the latter carries the structure of a solid commutative O-algebra for the t-adic topology, compare with [CS19, Proposition 7.9]. Note that Lemmas 2.2 and 2.3 hold also in the O-relative setting with the same proofs. Similarly, we can define the various properties such as separated, proper, affine, reduced, and semi-normal.

A novelty here is the notion of flatness over O (this is automatic over a field), which gives rise to a functor $X \mapsto X^{\mathrm{fl}}$ given as the scheme-theoretic image of the generic fiber and called the flat closure, see [HLR24, Definition 8.3]. We say moreover that (X,x) is formally flat if the formal completion is flat as an ind-scheme. Note that a normal flat O-scheme of finite type is formally flat because localizations of formal schemes are flat and normality is preserved under completion for excellent rings.

Let $\mathcal{X}$ be a finite type affine O-scheme. We define the loop space $\mathrm{R}_{O^2_\circ/O}\mathcal{X}$ as the affine ind-scheme representing the functor $A \mapsto \mathcal{X}(A(\!(t-a)\!))$ on O-algebras. Here, a denotes the image of $t \in O$ in A via the structure map and we regard $A(\!(t-a)\!)$ as an O-algebra instead via the formal variable t. Similarly, we have a closed affine subscheme $\mathrm{R}_{O^2/O}\mathcal{X} \subset \mathrm{R}_{O^2_\circ/O}\mathcal{X}$ called the jet space and representing the functor $A \mapsto \mathcal{X}(A[\![t-a]\!])$ on O-algebras. Lemmas 2.4 and 2.5 admit counterparts in the O-relative setting and we also have the following flatness result.

Lemma 2.11. *If G is quasi-split with induced maximal torus T, then the ind-scheme $\mathrm{R}_{O^2_\circ/O}\mathcal{G}$ is formally flat.*

Proof. This is proved in [HLR24, Corollary 8.5, Proposition 8.8] for tame G and without assuming T to be induced. Note that $\mathcal{G}$ has a big cell $\mathcal{C} := \mathcal{U}^- \times \mathcal{T} \times \mathcal{U}^+$ as an open neighborhood of the identity by [BT84, Théorème 3.8.1], where products are fibered over O. The big cell itself can be identified with an open neighborhood of the zero section in $\mathbb{A}^n_O$, where n equals the dimension of G. In particular, by the O-relative variant of Lemma 2.5, the loop spaces $\mathrm{R}_{O^2_\circ/O}\mathcal{G}$ and $\mathrm{R}_{O^2_\circ/O}\mathbb{A}^n_O$, have isomorphic formal completions, and one checks easily that the latter is formally flat. □

For a parahoric O-model $\mathcal{G}$ of a given connected reductive F-group G, we can define the affine Grassmannian

$$\mathrm{Gr}_{\mathcal{G},O} := \mathrm{R}_{O^2_\circ/O}\mathcal{G}/\mathrm{R}_{O^2/O}\mathcal{G} \tag{2.5}$$

as the quotient for the étale topology and it is representable by a projective ind-scheme by [PZ13, Proposition 5.5]. Its generic fiber $\mathrm{Gr}_{G,F}$ is the usual affine Grassmannian attached to the F-group G, whereas the special fiber $\mathrm{Gr}_{\mathcal{G},k}$ is the affine flag variety we've studied so far. Let μ be a conjugacy class of geometric coweights of G with reflex field E, and consider the associated Schubert variety $\mathrm{Gr}_{G,E,\leq\mu}$. We define the local model $\mathrm{Gr}_{\mathcal{G},O_E,\leq\mu}$ as being the scheme-theoretic image of the former in $\mathrm{Gr}_{\mathcal{G},O_E}$.

Theorem 2.12. *The semi-normal flat O-scheme $\mathrm{Gr}^{\mathrm{sn}}_{\mathcal{G},O_E,\leq\mu}$ has φ-split special fiber. In particular, it is normal, Cohen–Macaulay and φ-rational.*

Proof. This result is proved in [Zhu14, Theorems 3.8 and 3.9] and [HR22, Theorem 2.1] for tame G, in [FHLR22, Theorem 5.4] when $p > 2$ or SU_3-free G and in [CL25, Theorem 4.15, Corollary 4.16] for general G. Let us give an explanation of the proof.

In [GL24, Theorem 1.3], we saw that $\mathrm{Gr}_{\mathcal{G},O_E,\leq\mu}$ is unibranch, i.e., its normalization is a universal homeomorphism, by a nearby cycle calculation relying on the Wakimoto filtration of [AB09, Theorem 4], compare also with [ALWY23, Theorem 4.17]. One sees that the inclusion in $\mathrm{Gr}_{\mathcal{G},O_E,\leq\mu}$ of the jet group orbit of any representative $\nu \in X_*(T)$ of μ is a universal homeomorphism onto its image by Zariski's main theorem and unibranchness of local models. Now, proper monomorphisms are closed immersions, but both schemes above are flat and reduced, so we conclude that the orbit map is an open immersion, compare with [Ric16, Corollary 2.14]. Joint with [HR21, Theorem 6.12], compare also with [AGLR22, Theorem 6.16], this implies that the special fiber of $\mathrm{Gr}_{\mathcal{G},O_E,\leq\mu}$ is generically reduced. By flatness, the special fiber of the normalization of the local model is also S_1, hence itself reduced by Serre's criterion. In particular, the special fiber of the normalization is covered by the φ-split, hence weakly normal, variety $\mathrm{Gr}^{\mathrm{sn}}_{\mathcal{G},k,\leq\mu}$ in light of Theorem 2.6, so this normalization equals $\mathrm{Gr}^{\mathrm{sn}}_{\mathcal{G},O_E,\leq\mu}$.

To finish the proof, we have two options. If we invoke Theorem 2.9 and cover G by a z-extension and invoking Theorem 2.9, we can show that the special fiber also maps to $\mathrm{Gr}^{\mathrm{sn}}_{\mathcal{G},k,\leq\mu}$. Since it is reduced, we deduce that this map is an isomorphism. As for the remaining assertions, we can derive them formally from the φ-splitness of the special fiber just like in [FHLR22, Theorem 5.4]. Alternatively, one can compute the dimension of global sections of ample line bundles in $\mathrm{Gr}^{\mathrm{sn}}_{\mathcal{G},k,\leq\mu}$, and then show that it coincides with that of $\mathrm{Gr}_{G,E,\leq\mu}$: this computation is performed in [CL25] via Demazure resolutions and a combinatorial argument. Then, we deduce that the special fiber is already semi-normal. □

Note that during the above proof, we came really close to showing the embedded version of the previous normality theorem.

Theorem 2.13. *If $p \nmid \#\pi_1(G_{\mathrm{der}})$, then $\mathrm{Gr}_{\mathcal{G},O_E,\leq\mu}$ is normal.*

Proof. This follows from Theorem 2.9 and Theorem 2.12 as follows. The special fiber of $\mathrm{Gr}^{\mathrm{sn}}_{\mathcal{G},O_E,\leq\mu}$ is semi-normal, hence it embeds into the ind-scheme $\mathrm{Gr}^{\mathrm{sn}}_{\mathcal{G},k}$. Hence, the semi-normalization of the local model is an isomorphism in the generic fiber and a closed immersion in the special fiber, so it must be an isomorphism by flatness and Nakayama's lemma. □

Funnily enough, to prove Theorem 2.9 beyond SU_3-free groups, we will need to use Theorem 2.12 in rank 1, see Corollary 3.13. This is allowed, because there is an independent proof due to [CL25]. Before concluding this section, we state a helpful version of Theorem 2.12 at the ind-scheme level.

Corollary 2.14. *Assume G is simply connected. Then, $\mathrm{Gr}^{\mathrm{sn}}_{\mathcal{G},O}$ is an ind-scheme and its special fiber identifies with $\mathrm{Gr}^{\mathrm{sn}}_{\mathcal{G},k}$.*

Proof. By étale descent, we may enlarge k so that G is quasi-split by Steinberg's theorem. Let $\rho^\vee$ be the half-sum of all coroots. It is clear that for simply connected G, we have $\mu \leq N\rho^\vee$ for $N \gg 0$. Also the definition field of $\rho^\vee$ is F. In particular, we see that $\mathrm{Gr}^{\mathrm{sn}}_{\mathcal{G},O}$ equals the colimit of the local models $\mathrm{Gr}^{\mathrm{sn}}_{\mathcal{G},O,\leq N\rho^\vee}$ in the category of O-sheaves. We must show that the transition morphisms of our presentation are closed immersions. This is true on geometric fibers by Theorems 2.6 and 2.12, so we conclude by flatness and Nakayama's lemma. □

3. Distributions

In this section, we study the notion of distributions of ind-schemes, which arise as non-linear differential operators near a given point. They capture essentially the same information as formal completions at a point, but with the added bonus that group distributions form a solid associative Hopf k-algebra. We give a Serre presentation of $\mathscr{D}(\mathrm{R}_{F/k}G)$ in terms of its rank 1 and unipotent parts, and use this to prove the normality theorem.

3.1. **Preparations.** During this section, we work again in the category of pointed ind-schemes (X, x) over a finite field k, i.e., X is an arbitrary ind-scheme and x is a k-valued point of X. Our notion of distributions is similar to [HLR24, Definition 7.1], except we drop the Artinian condition and take topological information into account.

Definition 3.1. The solid k-module $\mathscr{D}(X, x)$ is the filtered colimit of the solid k-modules $\mathrm{Hom}_k(\Gamma(Z, \mathcal{O}_Z), k)$, as (Z, x) runs over all closed nilpotent pointed k-subschemes of (X, x).

The k-module $\mathscr{D}(X, x)$ has a natural solid structure in the sense of Clausen–Scholze [CS19, Proposition 7.5, Theorem 8.1], because k is a finitely generated $\mathbb{Z}$-algebra, and we can write the global sections of Z as the filtered colimit of its finitely generated k-submodules, so its k-module dual equals the cofiltered limit of its finitely generated k-quotients. Note that $\mathscr{D}(X, x)$ only depends on the formal completion of (X, x): in fact, it equals the solid k-dual of the ring of formal sections. Later on, whenever x is understood, we will just omit it from the notation.

Given a morphism $(X, x) \to (Y, y)$ of pointed ind-k-schemes, there is a natural map $\mathscr{D}(X, x) \to \mathscr{D}(Y, y)$. Indeed, any closed nilpotent subscheme $Z_X \subset X$ maps to Y via a closed nilpotent subscheme $Z_Y \subset Y$, and this induces a morphism of solid k-modules $\mathscr{D}(Z_X, x) \to \mathscr{D}(Z_Y, y)$ that is natural in the various closed nilpotent subschemes. Note that closed immersions induce monomorphisms at the level of distributions. In particular, we see that $\mathscr{D}(Z, z)$ embeds into $\mathscr{D}(X, x)$ for any closed nilpotent subscheme $Z \subset X$ supported at x. In the next lemma, we impose no representability nor finiteness condition on the given morphism.

Lemma 3.2. *Let $f\colon (X, x) \to (Y, y)$ be a map of pointed ind-schemes. Then, $\mathscr{D}(f)$ is an isomorphism if and only if f is formally étale at x.*

Proof. First, we handle the if direction. By Lemma 2.2, formally étale maps induce an isomorphism between formal completions. The definition of $\mathscr{D}$ only depends on formal completions, so the result is clear. Now, we handle the only if direction. Note that the condensed k-dual of $\mathscr{D}(Z, z)$ equals its ring $\Gamma(Z, \mathcal{O}_Z)$ of global sections. In particular, the ring of formal sections of (X, x) equals the condensed k-dual of $\mathscr{D}(X, x)$. Since $\mathscr{D}(f)$ is an isomorphism, we conclude that f induces an isomorphism of formal rings. Hence, our claim follows from Lemma 2.3. □

We want to understand when maps of distributions are surjections in the category of solid k-modules. For this, we say that a map $f\colon (X, x) \to (Y, y)$ of pointed ind-schemes over k is formally dominant if the scheme-theoretic image functor along f preserves formal completions.

Lemma 3.3. *Let $(X, x) \to (Y, y)$ be a formally dominant map of pointed ind-k-schemes. Then, $\mathscr{D}(f)$ is a surjective map of solid k-modules.*

Proof. We may assume that X is nilpotent and Y is the scheme-theoretic image of X along f (and thus also nilpotent). At the level of formal sections, we have an inclusion $\Gamma(Y, \mathcal{O}) \to \Gamma(X, \mathcal{O})$ of discrete solid k-modules. Upon taking k-duals, this turns into a surjection of solid k-modules. □

If f is a scheme-theoretic dominant map of finite type k-schemes, then it is also formally dominant by Chevalley's lemma, compare with [HLR24, Lemma 7.3]. However, beware that this is false as soon as we drop finiteness, as revealed by the endomorphism of the scheme $\mathrm{R}_{O/k}\mathbb{A}^1_k$ given by $\sum t^i z_i \mapsto \sum t^i z_i^i$. Indeed, the scheme-theoretic images of nilpotent schemes along that endomorphism are always of finite type.

Next, we show that distribution modules are factorizable in products of ind-schemes.

Lemma 3.4. *Let (X, x) and (Y, y) be pointed k-schemes. The canonical map of solid k-modules*

$$\mathscr{D}(X, x) \otimes_k^{\blacksquare} \mathscr{D}(Y, y) \to \mathscr{D}(X \times Y, (x, y)) \tag{3.1}$$

is an isomorphism.

Proof. Notice that if we let Z_X, resp. Z_Y, run over all nilpotent thickenings of x at X, resp. y at Y, then the colimit of the closed nilpotent subschemes $Z_X \times_k Z_Y$ recovers the formal completion of $X \times_k Y$ at (x, y). In particular, we may assume that X and Y are themselves nilpotent, as the solid tensor product preserves colimits. But since k is a field, taking k-module duals commutes with tensor products, and it is easy to check that the isomorphism respects the solid structure. □

Note that $\mathscr{D}$ is a covariant functor, so the diagonal $\Delta\colon X \to X \times X$ yields a comultiplication map

$$\mu := \Delta_*\colon \mathscr{D}(X, x) \to \mathscr{D}(X \times X, (x, x)) \simeq \mathscr{D}(X, x) \otimes_k^{\blacksquare} \mathscr{D}(X, x) \tag{3.2}$$

making the distribution k-module into a cocommutative solid k-coalgebra, with counit $\epsilon\colon k = \mathscr{D}(*) \to \mathscr{D}(X, x)$ induced by x. Assume $X = G$ is a k-group ind-scheme. We define its distribution k-algebra $\mathscr{D}(G) := \mathscr{D}(G, 1)$ as the solid k-module of distributions based at the origin. The multiplication map $m\colon G \times G \to G$ induces a multiplication map

$$m := m_*\colon \mathscr{D}(G \times G) \simeq \mathscr{D}(G) \otimes_k^{\blacksquare} \mathscr{D}(G) \to \mathscr{D}(G) \tag{3.3}$$

making the distribution k-module into a cocommutative associative solid Hopf k-algebra, with antipode induced by the inverse map of G. Beware that this associative algebra ought to be as commutative as G itself, and hence it is very rarely so.

3.2. **A Serre presentation.** Let G be a connected reductive F-group and assume it is residually split in the sense of [KP23, Definition 9.10.11] throughout this subsection. By [KP23, Proposition 9.10.12], G is also quasi-split. In particular, we can fix a pinning in the sense of [BT84, Section 4.1], i.e., the data consisting of a maximally split maximal F-torus T of G, a Borel F-subgroup $B \subset G$, and certain isomorphisms x_α^{-1} between U_α and certain explicit groups that we describe below.

Our results in this section consist in giving a solid associative k-algebra presentation for $\mathscr{D}(\mathrm{R}_{F/k}G)$ in terms of $\mathscr{D}(\mathrm{R}_{F/k}U^{\pm})$. We have a notion of coproduct in the category of solid associative k-algebras by taking the usual construction in condensed associative k-algebras and then solidifying it.

Proposition 3.5. *If G is simply connected and residually split, the natural map*

$$*^{\blacksquare}_{\alpha \in \pm\Delta} \mathscr{D}(\mathrm{R}_{F/k}U_\alpha) \to \mathscr{D}(\mathrm{R}_{F/k}G) \tag{3.4}$$

of solid associative k-algebras is an epimorphism.

Proof. This will be proved along several computational lemmas that appear below. First, we note that we have a decomposition

$$\mathscr{D}(\mathrm{R}_{F/k}U^-) \otimes_k^{\blacksquare} \mathscr{D}(\mathrm{R}_{F/k}T) \otimes_k^{\blacksquare} \mathscr{D}(\mathrm{R}_{F/k}U^+) = \mathscr{D}(\mathrm{R}_{F/k}G) \tag{3.5}$$

by combining formal étaleness with Lemma 3.4. In order to handle the unipotent parts of the distribution algebra, it is enough to show that

$$*^{\blacksquare}_{\alpha \in \Delta} \mathscr{D}(\mathrm{R}_{F/k}U_\alpha) \to \mathscr{D}(\mathrm{R}_{F/k}U^+) \tag{3.6}$$

is a surjection of solid associative k-algebras, which will be done in Lemma 3.6. Finally, to handle the torus we observe that there is also a decomposition

$$\mathscr{D}(\mathrm{R}_{F/k}T) = \otimes^{\blacksquare}_{\alpha \in \Delta} \mathscr{D}(\mathrm{R}_{F/k}T_\alpha) \tag{3.7}$$

in the category of solid k-modules by Lemma 3.4. Here, T_α denotes the intersection of T with the subgroup G_α generated by $U_{\pm\alpha}$. In particular, this reduces our statement to the rank 1 case, which is handled in Lemma 3.7. □

Below, we perform the explicit computations required to verify the claims used in Proposition 3.5. For this, let us recall that are fixing the data of certain compatible isomorphisms as follows. Assume first that 2α is not a root, and the pinning is of the form $x_\alpha \colon \mathrm{R}_{F_\alpha/F}\mathbb{G}_{\mathrm{a},F_\alpha} \to U_\alpha$, where F_α/F is a separable field extension that is unique up to conjugation under Gal_F. If 2α is a root, then the pinning takes the form $x_\alpha \colon \mathrm{R}_{F_{2\alpha}/F}\mathbb{G}_{\mathrm{p},F_\alpha/F_{2\alpha}}$, where F_α/F is a separable field extension, and $F_\alpha/F_{2\alpha}$ is a quadratic field extension. Here, the group $\mathbb{G}_{\mathrm{p},F_\alpha/F_{2\alpha}}$ is a three-dimensional $F_{2\alpha}$-group described explicitly in [BT84, 4.1.15] whose rational points are pairs in $(F_\alpha)^2$ with trace-zero second coordinate.

Lemma 3.6. *The map (3.6) is a surjection of solid associative k-algebras.*

Proof. Observe that $\mathscr{D}(U^+)$ is the product of the $\mathscr{D}(U_\alpha)$ as α runs over all positive roots by Lemma 3.4. This statement is proved by induction on the height of a root, by exploiting the commutator relations inside U^+ explicitly written down in [BT84, Addendum]. It is then clear that we are reduced to proving the statement for almost simple simply connected groups of rank 2 with $\Delta = \{\alpha, \beta\}$, an exhaustive list being given by restrictions of scalars of groups of type A_2, C_2, ${}^2A_3$, ${}^2A_4$, ${}^{3,6}D_4$, and G_2.

First, if G is of type A_2, then Φ^+ contains exactly three roots $\alpha, \beta, \alpha+\beta$, then we get the commutator formula

$$[x_\alpha(z_1), x_\beta(z_2)] = x_{\alpha+\beta}(z_1 z_2). \tag{3.8}$$

It is clear that this map is formally dominant, because it has bounded coefficients and hits a topological k-basis of F. In particular, $\mathscr{D}(\mathrm{R}_{F/k}U_{\alpha+\beta})$ is contained in the image of (3.6) by Lemma 3.3.

If instead G is of type C_2 or ${}^2A_3$, then Φ^+ contains exactly four roots of the form $\alpha, \beta, \alpha+\beta, 2\alpha+\beta$, and we get

$$[x_\alpha(z_1), x_\beta(z_2)] = x_{\alpha+\beta}(z_1 z_2) x_{2\alpha+\beta}(N(z_1) z_2) \tag{3.9}$$

where $N(z_1) = z_1^2$ or the norm of the quadratic extension in the non-split case, and the variables are understood to be formal. Again, one can check easily by this expression that the commutator map is formally dominant. In particular, we deduce by Lemma 3.3 that $\mathscr{D}(\mathrm{R}_{F/k}U_{\alpha+\beta})$ and $\mathscr{D}(\mathrm{R}_{F/k}U_{2\alpha+\beta})$ both lie in the image of (3.6).

If G is of type ${}^{3,6}D_4$ or G_2, then $\Phi^+ = \{\alpha, \beta, \alpha+\beta, 2\alpha+\beta, 3\alpha+\beta, 3\alpha+2\beta\}$, and we get

$$[x_\alpha(z_1), x_\beta(z_2)] = x_{\alpha+\beta}(z_1 z_2) x_{2\alpha+\beta}(\theta(z_1) z_2) x_{3\alpha+\beta}(N(z_1) z_2) x_{3\alpha+2\beta}(N(z_1) z_2^2), \tag{3.10}$$

where $N(z_1) = z_1^3$ in the split case and is the usual norm of the fixed cubic extension in the non-split case, $\theta(z_1) z_1 = N(z_1)$, and the variables are understood to be formal. In this case, the commutator is not dominant for dimension reasons. However, one can observe that the formal completion of $\mathrm{R}_{F/k}U_{\alpha+\beta}$ is contained in the scheme-theoretic image of the above map, because $z_1 z_2$ is algebraically idenpendent from the remaining polynomials. In particular, $\mathscr{D}(\mathrm{R}_{F/k}U_{\alpha+\beta})$ is contained in the image of (3.6). This means we can transport $x_\alpha(z_1 z_2)$ to the left side of the equation, and we are reduced to the remaining three root groups. We continue this procedure first by showing containment of $\mathscr{D}(\mathrm{R}_{F/k}U_{3\alpha+2\beta})$, then of $\mathscr{D}(\mathrm{R}_{F/k}U_{2\alpha+\beta})$, and finally of $\mathscr{D}(\mathrm{R}_{F/k}U_{3\alpha+\beta})$.

Finally, we consider the case where G is of type ${}^2A_4$. It follows that $\Phi^+ = \{\alpha, \beta, \alpha+\beta, 2\alpha, 2\alpha+\beta, 2\alpha+2\beta\}$. Therefore, we get

$$[x_\alpha(z_1, z_2), x_\beta(z_3)] = x_{\alpha+\beta}(\sigma(z_1 z_3), N(z_1) z_2) x_{2\alpha+\beta}(z_3 N(z_1) + z_3 z_2) \tag{3.11}$$

where N denotes the norm of $F_\alpha/F_{2\alpha}$ and σ the non-trivial involution. We can see that the formal completion of $\mathrm{R}_{F/k}U_{\alpha+\beta}\ \mathscr{D}(U_\Delta)$ is contained in the scheme-theoretic image of the above map. As above, this is enough to show that $\mathscr{D}(\mathrm{R}_{F/k}U_{\alpha+\beta})$ and $\mathscr{D}(\mathrm{R}_{F/k}U_{2\alpha+\beta})$ both lie in the image of (3.6). □

In the next lemma, we handle the rank 1 case. For this, we recall that there is a natural isomorphism $\alpha^\vee : \mathrm{R}_{F_\alpha/F}\mathbb{G}_{\mathrm{m},F_\alpha} \to T_\alpha$, where the right side equals the intersection of T with the subgroup G_α generated by $U_{\pm\alpha}$.

Lemma 3.7. *Assume G is simply connected, residually split and has rank 1. Then, (3.4) is a surjection of solid associative k-algebras.*

Proof. Let α be the positive simple root of G. First, we assume that $G = \mathrm{R}_{F_\alpha/F}\mathrm{SL}_{2,F_\alpha}$. Inside $\mathrm{R}_{F/k}G$, we have the following equation

$$x_\alpha(z_1)x_{-\alpha}(z_2) = x_{-\alpha}\big(\frac{z_2}{1+z_1z_2}\big)\alpha^\vee(1+z_1z_2)x_\alpha\big(\frac{z_1}{1+z_1z_2}\big) \tag{3.12}$$

where the variables are understood to be formal. If we isolate the term $\alpha^\vee(1+z_1z_2)$ on the right side, we see that the map is formally dominant. This implies by Lemma 3.3 that $\mathscr{D}(\mathrm{R}_{F/k}T)$ is contained in the image of (3.4). By the big cell factorization, this implies the desired surjectivity in the split case.

Next, we handle the rank 1 quasi-split group $G = \mathrm{R}_{F_{2\alpha}/F}\mathrm{SU}_{3,F_\alpha/F_{2\alpha}}$ where F_α/F is a separable field extension and $F_\alpha/F_{2\alpha}$ is quadratic. In $\mathrm{R}_{F/k}G$, we have the following equality

$$x_\alpha(z_1,z_2)x_{-\alpha}(z_3,z_4) = x_{-\alpha}(f_1,f_2)\alpha^\vee(1+g)x_\alpha(f_3,f_4) \tag{3.13}$$

where the f_i are explicit rational functions on the z_i involving the quadratic involution σ, which we omit for simplicity, and

$$g = -\sigma(z_1)z_3 + (z_2+\lambda N(z_1))(z_4+\lambda N(z_3)), \tag{3.14}$$

compare with [BT84, 4.1.12]. The map resulting from isolating $\alpha^\vee(g)$ in the right side can be checked to be formally dominant, so we see by Lemma 3.3 that $\mathscr{D}(\mathrm{R}_{F/k}T)$ is contained in the image of (3.4), implying that this is a surjection. □

Now, we have all the necessary tools at our disposal to establish a Serre presentation for the loop distribution algebra $\mathscr{D}(\mathrm{R}_{F/k}G)$ of a simply connected F-group G, inspired by [Tak83a, Proposition 3.6] and [Tak83b, Theorem 5.1]. For this, we need to introduce the following notation: given $\alpha \in \Delta$, a positive simple root, we let $G_{\pm\alpha}$ be the derived subgroup generated by $U_{\pm\alpha}$, then we let $V_{\pm\alpha} \subset U^\pm$ be the unique smooth connected unipotent subgroup such that $U^\pm = U_{\pm\alpha} \times V_{\pm\alpha}$, and finally we denote by $Q_{\pm\alpha}$ the semi-direct product $G_{\pm\alpha} \ltimes U_{\pm\alpha}$. Let us emphasize that $Q_{\pm\alpha}$ is a kind of derived parabolic subgroup. Indeed, it equals the extension of the derived subgroup $G_{\pm\alpha}$ of the standard Levi of the minimal parahoric $P_{\pm\alpha}$ by its unipotent radical $U_{\pm\alpha}$.

Theorem 3.8. *If G is simply connected and residually split, the kernel of the surjection*

$$\mathscr{D}(\mathrm{R}_{F/k}U^+) *^{\blacksquare} \mathscr{D}(\mathrm{R}_{F/k}U^-) \to \mathscr{D}(\mathrm{R}_{F/k}G) \tag{3.15}$$

of solid associative k-algebras is the solid ideal generated by the kernels of

$$\mathscr{D}(\mathrm{R}_{F/k}U^\pm) *^{\blacksquare} \mathscr{D}(\mathrm{R}_{F/k}U_{\mp\alpha}) \to \mathscr{D}(\mathrm{R}_{F/k}Q_{\pm\alpha}), \tag{3.16}$$

as $\alpha \in \Delta$ runs through all positive simple roots.

Proof. Let $\mathscr{U}$ be the solid associative k-algebra given by the generators and relations described in the statement of the theorem. We have a surjection $\mathscr{U} \to \mathscr{D}(\mathrm{R}_{F/k}G)$ of solid associative k-algebras by Proposition 3.5. Our first goal is to define a section s of the previous surjection in the category of solid k-modules. By definition, the algebra $\mathscr{U}$ contains $\mathscr{D}(U^\pm)$ and $\mathscr{D}(T_\alpha)$

for any $\alpha \in \Delta$ as solid associative k-subalgebras. This yields the desired section by using the factorizations (3.5) and (3.7).

To finish the proof, it is enough by Lemma 3.6 to prove stability of the corresponding solid k-submodule $\mathrm{im}(s)$ under left multiplication by every $\mathscr{D}(\mathrm{R}_{F/k}U_\alpha)$ for any root $\alpha \in \Delta$. Before doing this however, we prove that the solid module given as the product of the $\mathscr{D}(\mathrm{R}_{F/k}T_\alpha)$ gives rise to a solid commutative k-subalgebra of $\mathscr{U}$. Let $\alpha \neq \beta \in \Delta$ be distinct positive simple roots. We claim that the natural conjugation map

$$\mathscr{D}(\mathrm{R}_{F/k}T_\alpha) \otimes^{\blacksquare} \mathscr{D}(\mathrm{R}_{F/k}G_\beta) \to \mathscr{D}(\mathrm{R}_{F/k}G_\beta) \tag{3.17}$$

lifts to $\mathscr{U}$, which clearly implies the desired commutativity. Indeed, $\mathscr{D}(\mathrm{R}_{F/k}G_\beta)$ is surjected upon by the algebra coproduct of $\mathscr{D}(\mathrm{R}_{F/k}U_{\pm\beta})$ according to Lemma 3.7. The latter solid algebra carries a conjugation action by $\mathscr{D}(\mathrm{R}_{F/k}T_\alpha)$ and this action is compatible with the maps to $\mathscr{U}$, because our universal solid algebra contains $\mathscr{D}(\mathrm{R}_{F/k}Q_{\pm\alpha})$ as a solid subalgebra and $U_{\pm\beta}, T_\alpha \subset Q_{\pm\alpha}$.

Finally, we check stability of the section s to the natural surjection $\mathscr{U} \to \mathscr{D}(\mathrm{R}_{F/k}G)$ under left multiplication by $\mathscr{D}(\mathrm{R}_{F/k}U_\alpha)$ for every $\alpha \in \Delta$. Writing $U^- = V_{-\alpha} \times U_{-\alpha}$, we deduce from Lemma 3.4 a decomposition

$$\mathscr{D}(\mathrm{R}_{F/k}U^-) = \mathscr{D}(\mathrm{R}_{F/k}V_{-\alpha}) \otimes^{\blacksquare} \mathscr{D}(\mathrm{R}_{F/k}U_{-\alpha}). \tag{3.18}$$

On the other hand, we know that $\mathscr{D}(\mathrm{R}_{F/k}U_\alpha)$ normalizes $\mathscr{D}(\mathrm{R}_{F/k}V_{-\alpha})$ inside $\mathscr{U}$ due to our imposed relations coming from $\mathscr{D}(\mathrm{R}_{F/k}Q_\alpha)$. In other words, we can switch the order in which $\mathscr{D}(\mathrm{R}_{F/k}V_{-\alpha})$ and $\mathscr{D}(\mathrm{R}_{F/k}U_\alpha)$ are multiplied inside $\mathscr{U}$. Finally, we can assemble the product

$$\mathscr{D}(\mathrm{R}_{F/k}U_\alpha) \otimes^{\blacksquare} \mathscr{D}(\mathrm{R}_{F/k}U_{-\alpha}) \subset \mathscr{D}(\mathrm{R}_{F/k}G_\alpha) \tag{3.19}$$

inside $\mathscr{U}$, because Q_α contains G_α, and then use the factorization (3.5) for G_α to switch the order of the loop distributions of $\pm\alpha$. Since the solid commutative k-subalgebra $\mathscr{D}(\mathrm{R}_{F/k}T) \subset \mathscr{U}$ normalizes $\mathscr{D}(\mathrm{R}_{F/k}U_\alpha)$, we can finally pull this factor across and absorb it into $\mathscr{D}(\mathrm{R}_{F/k}U^+)$, concluding our claim. □

3.3. **Proof of normality.** In this section, we finally prove Theorem 2.9 (which also implies Theorem 2.13) using our newly acquired distribution skills. The first result needed is a surjectivity one, giving us some control on what happens with the distributions of the semi-normal loop group.

Lemma 3.9. *If G is residually split, the natural map of solid associative k-algebras*

$$*^{\blacksquare}_{\alpha\in\Delta}\, \mathscr{D}(\mathrm{R}_{O/k}\mathcal{G}_{s_\alpha}) \to \mathscr{D}(\mathrm{R}^{\mathrm{sn}}_{F/k}G) \tag{3.20}$$

is a surjection. Here, $\mathcal{G}_{s_\alpha}$ denotes the parahoric O-model attached to a wall of a fixed alcove with associated Iwahori O-model $\mathcal{I}$.

Proof. Consider the Demazure resolution $\mathrm{Gr}_{\mathcal{I},\leq s_\bullet} \to \mathrm{Gr}^{\mathrm{sn}}_{\mathcal{I},\leq w}$ which is a proper birational cover. In particular, it surjects at the level of distributions supported at the identity by Lemma 3.3 and Chevalley's lemma, compare with [HLR24, Lemma 7.3]. This map lifts to the natural $\mathrm{R}_{O/k}\mathcal{I}$-torsors on the right as follows

$$\mathrm{R}_{O/k}\mathcal{G}_{s_1} \times^{\mathrm{R}_{O/k}\mathcal{I}} \times \cdots \times^{\mathrm{R}_{O/k}\mathcal{I}} \mathrm{R}_{O/k}\mathcal{G}_{s_n} \to (\mathrm{R}^{\mathrm{sn}}_{F/k}G)_{\leq w} \tag{3.21}$$

by multiplying the subgroups on the left inside $\mathrm{R}^{\mathrm{sn}}_{F/k}G$. Note that the left side is built out of parahoric jet groups and it surjects again by formal dominance after taking $\mathscr{D}$, see Lemma 3.3. Passing to colimits, we get our claim. □

Our next step consists of a reduction to rank 1 groups.

Proposition 3.10. *Assume G is simply connected and residually split. If Theorem 2.9 holds for every rank 1 subgroup G_α, then it also does for G.*

Proof. It is enough by Lemma 2.10 to show that the semi-normalization map for $\mathrm{R}_{F/k}G$ is formally étale, and this can be verified at the distribution level, i.e., if we show that

$$\mathscr{D}(\mathrm{R}^{\mathrm{sn}}_{F/k}G) \to \mathscr{D}(\mathrm{R}_{F/k}G) \tag{3.22}$$

is an isomorphism, thanks to Lemma 3.2.

By Theorem 3.8, we are reduced to showing that $\mathscr{D}(\mathrm{R}^{\mathrm{sn}}_{F/k}G)$ satisfies the generators and relations described in that statement. First of all, note that the semi-normal ind-group scheme $\mathrm{R}_{F/k}U^{\pm}$ naturally sits inside $\mathrm{R}^{\mathrm{sn}}_{F/k}G$ as a closed subgroup by naturality of the semi-normalization functor. Next, we note that by Lemma 3.4 applied to the big cell of a parahoric $\mathcal{G}$ fixing an alcove of the standard appartment, we have a factorization

$$\mathscr{D}(\mathrm{R}_{O/k}\mathcal{G}) = \mathscr{D}(\mathrm{R}_{O/k}\mathcal{U}^-) \otimes^{\blacksquare} \mathscr{D}(\mathrm{R}_{O/k}\mathcal{T}) \otimes^{\blacksquare} \mathscr{D}(\mathrm{R}_{O/k}\mathcal{U}^+), \tag{3.23}$$

where $\mathcal{U}^{\pm}$ and $\mathcal{T}$ denote the corresponding O-models of $U^{\pm}$ and T sitting inside $\mathcal{G}$ as smooth closed subgroups. Since the jet group $\mathrm{R}_{O/k}\mathcal{T}$ is also semi-normal by Lemma 2.4, it naturally lifts to $\mathrm{R}^{\mathrm{sn}}_{F/k}G$. Moreover, it decomposes as a product of the $\mathrm{R}_{O/k}\mathcal{T}_\alpha$ for $\alpha \in \Delta$, because G is simply connected. By assumption, we also know that $\mathrm{R}_{F/k}G_\alpha$ is semi-normal, so it lifts canonically to $\mathrm{R}^{\mathrm{sn}}_{F/k}G$ compatibly with $\mathrm{R}_{F/k}U_{\pm\alpha}$ and $\mathrm{R}_{O/k}\mathcal{T}_\alpha$. In particular, we deduce that

$$\mathscr{D}(\mathrm{R}_{F/k}U^+) *^{\blacksquare} \mathscr{D}(\mathrm{R}_{F/k}U^-) \to \mathscr{D}(\mathrm{R}^{\mathrm{sn}}_{F/k}G) \tag{3.24}$$

is a surjection of solid associative k-algebras, by combining Lemmas 3.7 and 3.9.

Seeing as $\mathscr{D}(\mathrm{R}^{\mathrm{sn}}_{F/k}G) \to \mathscr{D}(\mathrm{R}_{F/k}G)$ is a surjection of solid associative k-algebras, it is enough to prove that the kernel of (3.24) contains that of (3.15). Due to Theorem 3.8, this follows by observing that the semi-normal ind-scheme $\mathrm{R}_{F/k}Q_{\pm\alpha}$ lifts canonically to $\mathrm{R}^{\mathrm{sn}}_{F/k}G$ by naturality of the semi-normalization functor and our assumed rank 1 case. □

In the next corollary, we say that a simply connected group is SU_3-free if all of its rank 1 subgroups G_α over any unramified extension of F are inner forms of restrictions of scalars of SL_2.

Corollary 3.11. *Theorem 2.9 holds for all* SU_3*-free groups.*

Proof. By Lemma 2.10, we may assume that G is simply connected. After enlarging k, we may also assume by étale descent that G is residually split. If G is SU_3-free, then all its rank 1 simply connected subgroups G_α are isomorphic to a restriction of scalars of SL_2. By Proposition 2.7, we know that the Schubert varieties of SL_2 are normal, so we conclude by the previous Proposition 3.10. □

For any G, we know by Corollary 2.14 that the semi-normal O-loop group $\mathrm{R}^{\mathrm{sn}}_{O^2_\circ/O}\mathcal{G}$ is an ind-scheme and its special fiber is semi-normal, i.e., it identifies with our original loop group $\mathrm{R}^{\mathrm{sn}}_{F/k}G$ over k. Once again, we must show that the natural morphism

$$\mathrm{R}^{\mathrm{sn}}_{O^2_\circ/O}\mathcal{G} \to \mathrm{R}_{O^2_\circ/O}\mathcal{G} \tag{3.25}$$

is a formally étale map of formally O-flat ind-schemes, by the O-flat version of Lemma 2.2. This has to be true in light of Theorem 2.13, but if we prove it independently for some parahoric model $\mathcal{G}$, then it would also yield Theorem 2.9 for its generic fiber G.

In order to do this, we define again $\mathscr{D}(X, x)$ for a pointed ind-scheme over O as the solid O-dual of the ring of formal sections, where we regard O with its t-adic analytic structure. Note that $\mathscr{D}(X, x)$ only depends on the flat closure of the formal completion. We can see that most of our general lemmas on distributions continue to be true over O provided that we restrict to maps of formally flat pointed ind-schemes over O, most notably Lemmas 3.2 and 3.4. However, there is no analogue of Lemma 3.3, because injections of solid O-modules do not necessarily dualize to surjections, e.g., if the cokernel has torsion. In our concrete situation, we can remedy this failure thanks to the following explicit calculation with root groups.

Proposition 3.12. *Assume G is residually split and that the reductive quotient of $\mathcal{G}_k$ is simply connected. Then, the natural map*

$$\mathscr{D}(\mathrm{R}_{O^2_\circ/O}\mathcal{U}^+) *^{\blacksquare}_O \mathscr{D}(\mathrm{R}_{O^2_\circ/O}\mathcal{U}^-) \to \mathscr{D}(\mathrm{R}_{O^2_\circ/O}\mathcal{G}) \tag{3.26}$$

is a surjection of solid associative O-algebras.

Proof. Note that the big cell induces a decomposition

$$\mathscr{D}(\mathrm{R}_{O^2_\circ/O}\mathcal{G}) = \mathscr{D}(\mathrm{R}_{O^2_\circ/O}\mathcal{U}^-) \otimes^{\blacksquare}_O \mathscr{D}(\mathrm{R}_{O^2_\circ/O}\mathcal{T}) \otimes^{\blacksquare}_O \mathscr{D}(\mathrm{R}_{O^2_\circ/O}\mathcal{U}^+) \tag{3.27}$$

of solid O-modules by the O-flat version of Lemma 3.4 and formal flatness of each of the terms, compare with Lemma 2.11. Similarly, we also have a factorization

$$\mathscr{D}(\mathrm{R}_{O^2_\circ/O}\mathcal{T}) = \otimes^{\blacksquare}_{O,\alpha\in\Delta}\mathscr{D}(\mathrm{R}_{O^2_\circ/O}\mathcal{T}_\alpha) \tag{3.28}$$

as solid O-modules, owing to the simply connectedness of G, and a combination again of Lemmas 2.11 and 3.4. In other words, it is clear that we can reduce to rank 1 groups G. After enlarging k, these are either isomorphic to a restriction of scalars of SL_2 or SU_3. In the first case, $\mathcal{G} = \mathrm{R}_{O_\alpha/O}\mathrm{SL}_2$ and it is easy to check (and not really necessary for our proof of Theorem 2.9 anyway) the surjectivity, so we leave this task to the reader.

If $G = \mathrm{R}_{F_{2\alpha}/F}\mathrm{SU}_{3,F_\alpha/F_{2\alpha}}$, then we can describe the special parahoric model $\mathcal{G}$ with simply connected reductive quotient as follows. Let $\lambda \in F^1_\alpha$ be a trace 1 element of maximal valuation, $\mu \in F^0_\alpha$ be an arbitrary element of trace 0. Then, by [BT84, 4.3.5] the closed subgroups $\mathcal{U}_{\pm\alpha} \subset \mathcal{G}$ are the unique smooth connected O-models of $U_{\pm\alpha}$ whose integral points are given by the subset $\nu_\pm O_\alpha \times \mu^{\pm 1} O_{2\alpha} \subset F_\alpha \times F^0_\alpha$, where $\nu_\pm$ is any element such that $\lambda N(\nu_\pm)$ equals $\mu^{\pm 1} t_\alpha$ up to a unit, where t_α is a uniformizer for O_α.

Write the base change of the punctured disk $O^2_\circ$ along $O \to O_\alpha$ (resp. $O_{2\alpha}$) as R_α (resp. $R_{2\alpha}$). Note that we have an exchange equation (3.13) at the formal completions, which expresses the middle term $\alpha^\vee(1+g)$ as a product of points of $\mathcal{U}_{\pm\alpha}$. If we set $z_1 = z_3 = 0$, then g simplifies to $z_2 z_4$. Here, $z_2 \in \mu R_{2\alpha}, z_4 \in \mu^{-1} R_{2\alpha}$, so we have obtained the distributions with coefficients in $R_{2\alpha}$. Similarly, if we set $z_1 = z_4 = 0$, then g simplifies to $\lambda z_2 N(z_3)$ with $z_2 \in \mu R_{2\alpha}$ and $z_3 \in \nu_- R_\alpha$, so we get the distributions with coefficients in $t_\alpha R_{2\alpha}$. Clearly, R_α is a free $R_{2\alpha}$-module with basis $\{1, t_\alpha\}$, so we deduce the desired surjectivity. □

Corollary 3.13. *Theorem 2.9 holds for all groups.*

Proof. We may and do assume that G is simply connected by Lemma 2.10. By étale descent, we may also assume that G is residually split. Furthermore, by Proposition 3.10 and Corollary 3.11, it suffices to treat the case where G is a restriction of scalars of SU_3. Let $\mathcal{G}$ be a special parahoric model of G such that $\mathcal{G}_k$ has simply connected reductive quotient, whose existence is ensured by [FHLR22, Lemma 4.11]. Notice that (3.25) is a map of formally flat ind-schemes by Corollary 2.14 and Lemma 2.11. Indeed, for the semi-normal loop group, it is already flat from the beginning and this property passes to the formal completion using that excellent normal local rings are analytically irreducible.

We are going to show that the semi-normalization map over O is formally étale. It is an isomorphism over F by étale descent applied to Corollary 3.11 for the split form SL_3 of G. In particular, the map

$$\mathscr{D}(\mathrm{R}^{\mathrm{sn}}_{O^2_\circ/O}\mathcal{G}) \to \mathscr{D}(\mathrm{R}_{O^2_\circ/O}\mathcal{G}) \tag{3.29}$$

of torsion-free solid O-modules is a monomorphism. But Proposition 3.12 tells us that it is also surjective, as unipotent loop groups lift to the semi-normalization. We deduce formal étaleness by the O-flat variant of Lemma 3.2. Passing to the special fibers, we deduce that $\mathrm{R}^{\mathrm{sn}}_{F/k}G \to \mathrm{R}_{F/k}G$ is formally étale, thanks to Corollary 3.13. □

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A MODULAR RAMIFIED GEOMETRIC SATAKE EQUIVALENCE

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Abstract. We extend the ramified geometric Satake equivalence due to Zhu (for tamely ramified groups) and the third named author (in full generality) from rational coefficients to include modular and integral coefficients.

Contents

1. Introduction

1.1. **Ramified geometric Satake equivalence.** The geometric Satake equivalence, first fully established by Mirković–Vilonen [MV07] after important contributions of Lusztig [Lus83], Ginzburg [Gin00] and Beĭlinson–Drinfeld [BD00], has now become a cornerstone of geometric approaches to a variety of subjects in representation theory and number theory. It consists of an equivalence of monoidal categories relating the category of perverse sheaves on the affine Grassmannian of a reductive algebraic group to representations of the Langlands dual reductive group. This construction has many variants, in which one e.g. changes the coefficients of the sheaf theory, or the field of definition of the geometric objects. See, for instance, Cass–van den Hove–Scholbach [CvdHS23] and the references cited therein, as well as Table 1.

The present paper is a contribution to another variant of this story, initially developed by Zhu [Zhu15], where one replaces the (split) reductive algebraic group over the base field $\mathbb{F}$ (of characteristic p) of which one takes the affine Grassmannian by a *possibly nonsplit* reductive

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Name	*Input group*	*Coefficients*	*Tannakian side*	*Reference*
absolute	G split / $\mathbb{C}$	comm. ring $\Bbbk$	$\mathsf{Rep}(G^\vee_\Bbbk)$	[MV07]
motivic	G split / scheme S	motives	$\mathsf{Rep}_{G^\vee}(\mathrm{MTM}(S))$	[CvdHS23]
tamely ramified	$\mathcal{G}$ spec. parahoric / $\mathbb{F}[\![t]\!]$	$\mathbb{Q}_\ell$	$\mathsf{Rep}((G^\vee_{\mathbb{Q}_\ell})^I)$	[Zhu15]
ramified	$\mathcal{G}$ spec. parahoric / $\mathbb{F}[\![t]\!]$	$\mathbb{Q}_\ell$	$\mathsf{Rep}((G^\vee_{\mathbb{Q}_\ell})^I)$	[Ric16]
mod. ramified	$\mathcal{G}$ spec. parahoric / $\mathbb{F}[\![t]\!]$	$\Lambda = \mathbb{Q}_\ell, \mathbb{Z}_\ell, \mathbb{F}_\ell$	$\mathsf{Rep}((G^\vee_\Lambda)^I)$	this paper

TABLE 1. Some variants of the geometric Satake equivalence

group over $\mathbb{F}(\!(t)\!)$ and a parahoric model over $\mathbb{F}[\![t]\!]$ attached to a special facet. For ℓ-adic coefficients with $\ell \neq p$, a version of the Satake equivalence in this setting was obtained in [Zhu15] assuming the reductive group splits over a tame extension, and then by the third author [Ric16] in full generality. Here we extend these constructions to *modular and integral coefficients*, i.e. to categories of perverse sheaves with coefficients in a finite field of characteristic ℓ or the ring of integers of a finite extension of $\mathbb{Q}_\ell$ with $\ell \neq p$. (See also Remark 1.3 regarding the analytic setting and Remark 1.4 for an extension to the motivic setting.)

1.2. **Ramified affine Grassmannians.** Quite generally, one can associate a positive loop group $\mathrm{L}^+\mathcal{H}$ and an affine Grassmannian (sometimes called an *affine flag variety*) to any smooth affine group scheme $\mathcal{H}$ over $\mathbb{F}[\![t]\!]$. (Here $\mathbb{F}$ is a separably closed field; for this part of the discussion it could be an arbitrary field.) Of particular interest is the case when $\mathcal{H}$ is a parahoric integral model of a connected reductive algebraic group over $\mathbb{F}(\!(t)\!)$ attached to a facet in the associated Bruhat–Tits building, see in particular Pappas–Rapoport [PR08]. This class is however too vast to expect a geometric Satake equivalence (e.g. since it contains *all* partial affine flag varieties attached to split groups). A geometric study of the cases where some of the important properties of the affine Grassmannians arising in [MV07] (in particular, regarding parity of dimensions of orbits) was undertaken in [Ric16]; it turns out that a nice class of cases is that when the facet involved is *special*. This case covers the setting considered in [MV07] (corresponding to hyperspecial facets of split groups), and also those arising in [Zhu15]. More geometric properties of such *twisted* affine Grassmannians (in particular, regarding semi-infinite orbits) were later established in Anschütz–Gleason–Lourenço–Richarz [AGLR22].

For twisted affine Grassmannians, and for ℓ-adic coefficients, the third author established an equivalence generalizing those of [MV07] and [Zhu15], relating the category of equivariant perverse sheaves on the affine Grassmannian, equipped with the convolution monoidal structure, to the category of representations of a certain algebraic group, described as follows. The reductive group G over $F = \mathbb{F}(\!(t)\!)$ involved in the constructions splits over the separable closure F^{s} of F. One can then consider the pinned group $G^\vee_{\mathbb{Q}_\ell}$ over $\mathbb{Q}_\ell$ which is Langlands dual to this split group. Since the dual group arises from a group over $\mathbb{F}(\!(t)\!)$, it is equipped with an action of the Galois group I of F^{s} over F preserving the pinning. The Tannakian group considered above is then identified with the fixed point subgroup $(G^\vee_{\mathbb{Q}_\ell})^I$. This group is a possibly disconnected algebraic group over $\mathbb{Q}_\ell$, whose neutral component is reductive.

Remark 1.1. It is quite remarkable that the Tannakian group above only depends on G, and not on the choice of special facet. In fact there are reductive groups that admit essentially different special facets; the associated affine Grassmannians are not isomorphic (and, in fact, have quite different geometric properties), yet the associated categories of perverse sheaves are equivalent. See [Zhu15, p. 411] for more detailed comments on this phenomenon. The same phenomenon occurs for the categories with integral or modular coefficients considered below.

1.3. **Fixed points of groups of pinning-preserving automorphisms of reductive group schemes.** The equivalence of [MV07] (considered here in the setting of étale sheaves) has a version where the field of coefficients is replaced by a ring Λ which is either a finite field of characteristic $\ell \neq p$ or the ring of integers of a finite extension of $\mathbb{Q}_\ell$. It therefore seemed reasonable to expect a generalization of the equivalences of [Zhu15, Ric16] for such coefficients, involving the group scheme $(G^\vee_\Lambda)^I$ of fixed points of the action as above on the dual split reductive group scheme $G^\vee_\Lambda$ over the given ring of coefficients Λ.

As a first step towards this goal, in the companion paper [ALRR23] we study the group schemes that arise in this way, exploiting crucially the fact that the action of I stabilizes the pinning of $G^\vee_\Lambda$. We show in particular that these group schemes are always flat over Λ (so, their categories of representations are abelian), but not necessarily smooth; in particular, for an appropriate action of $\mathbb{Z}/2\mathbb{Z}$ on $\mathrm{GL}_{2n+1,\mathbb{Z}_2}$ the group of fixed points turns out to be an example of a non-reductive quasi-reductive group scheme in the sense of Prasad–Yu [PY06].

1.4. **Nearby cycles and the relation with the dual group.** The reason why the dual group $G^\vee_\Lambda$ of $F^{\mathrm{s}} \otimes_F G$ occurs in this story is the following. Consider our integral model $\mathcal{G}$ over $\mathbb{F}[\![t]\!]$ and the associated affine Grassmannian $\mathrm{Gr}_{\mathcal{G}}$, but also the group scheme over the ring $F^{\mathrm{s}}[\![z]\!]$ (where z is another formal variable) obtained by base change from the split group $F^{\mathrm{s}} \otimes_F G$. Then $\mathrm{Gr}_{\mathcal{G}}$ can be described as a degeneration of the affine Grassmannian $\mathrm{Gr}_{F^{\mathrm{s}}[\![z]\!] \otimes_F G}$ (an ind-scheme over F^{s}): there exists an ind-scheme $\mathrm{Gr}_{\mathcal{G},\overline{S}}$ over the spectrum $\overline{S}$ of the integral closure of $\mathbb{F}[\![t]\!]$ in F^{s} and a diagram with cartesian squares

$$\begin{tikzcd} \mathrm{Gr}_{F^{\mathrm{s}}[\![z]\!] \otimes_F G} \arrow[r, hook] \arrow[d] & \mathrm{Gr}_{\mathcal{G},\overline{S}} \arrow[d] & \mathrm{Gr}_{\mathcal{G}} \arrow[l, hook'] \arrow[d] \\ \mathrm{Spec}(F^{\mathrm{s}}) \arrow[r, hook] & \overline{S} & \mathrm{Spec}(\mathbb{F}). \arrow[l, hook'] \end{tikzcd} \tag{1.1}$$

Associated with this diagram we have a nearby cycles functor sending perverse sheaves on $\mathrm{Gr}_{F^{\mathrm{s}}[\![z]\!] \otimes_F G}$ (which can be described as representations of $G^\vee_\Lambda$ via the "usual" geometric Satake equivalence) to perverse sheaves on $\mathrm{Gr}_{\mathcal{G}}$. The main result of the paper is an equivalence of monoidal categories

$$\mathsf{Perv}_{\mathrm{L}^+\mathcal{G}}(\mathrm{Gr}_{\mathcal{G}}, \Lambda) \cong \mathsf{Rep}((G^\vee_\Lambda)^I), \tag{1.2}$$

between the categories of $\mathrm{L}^+\mathcal{G}$-equivariant Λ-perverse sheaves on $\mathrm{Gr}_{\mathcal{G}}$ and of representations of $(G^\vee_\Lambda)^I$ on finitely generated Λ-modules, under which the nearby cycles functor considered above corresponds to the functor of restriction along the natural embedding $(G^\vee_\Lambda)^I \subset G^\vee_\Lambda$. This is summarized in the following commutative diagram:

$$\begin{tikzcd} \mathsf{Perv}_{\mathrm{L}^+G}(\mathrm{Gr}_{F^{\mathrm{s}}[\![z]\!] \otimes_F G}, \Lambda) \arrow[r, "\text{“usual” geometric Satake}", "\sim"'] \arrow[d, "\text{nearby cycles}"'] & \mathsf{Rep}(G^\vee_\Lambda) \arrow[d, "\text{restriction}"] \\ \mathsf{Perv}_{\mathrm{L}^+\mathcal{G}}(\mathrm{Gr}_{\mathcal{G}}, \Lambda) \arrow[r, "(1.2)", "\sim"'] & \mathsf{Rep}((G^\vee_\Lambda)^I). \end{tikzcd}$$

As for the usual geometric Satake equivalence, for a perverse sheaf $\mathscr{F}$, the underlying Λ-module of the representation associated with $\mathscr{F}$ is the total cohomology $\mathsf{H}^\bullet(\mathrm{Gr}_{\mathcal{G}}, \mathscr{F})$.

1.5. **Comments on the proof.** The proof of the equivalence (1.2) in the ℓ-adic case is based on a general result of Bezrukavnikov regarding central functors with domain the category of representations of an algebraic group. We do have such a functor for arbitrary coefficients thanks to the constructions explained in §1.4, but Bezrukavnikov's result has no counterpart for integral coefficients, and its direct application for positive-characteristic coefficients in our setting presents some difficulties. We therefore follow a different, and in some sense more explicit, route and treat in parallel the cases of a finite extension $\mathbb{K}$ of $\mathbb{Q}_\ell$ (for which we provide a proof which is different from those in [Zhu15, Ric16]), its ring of integers $\mathbb{O}$, and the residue field $\Bbbk$ of $\mathbb{O}$, using in a crucial way some "change of scalars" arguments to transfer information from one case to another.

Part of our constructions are parallel to those used in [MV07]: we consider some "weight functors" constructed using semi-infinite orbits (building on geometric facts established in [AGLR22]), and show representability of these functors to construct a flat Λ-bialgebra $\mathrm{B}_{\mathcal{G}}(\Lambda)$ and an equivalence of monoidal categories between $\mathsf{Perv}_{\mathrm{L}^+\mathcal{G}}(\mathrm{Gr}_{\mathcal{G}}, \Lambda)$ and the category of $\mathrm{B}_{\mathcal{G}}(\Lambda)$-comodules which are finitely generated over Λ. Further, we show that we have canonical isomorphisms

$$\mathrm{B}_{\mathcal{G}}(\mathbb{K}) \cong \mathbb{K} \otimes_{\mathbb{O}} \mathrm{B}_{\mathcal{G}}(\mathbb{O}), \quad \mathrm{B}_{\mathcal{G}}(\Bbbk) \cong \Bbbk \otimes_{\mathbb{O}} \mathrm{B}_{\mathcal{G}}(\mathbb{O}). \tag{1.3}$$

The nearby cycles functor of §1.4 and the Tannakian formalism provide a morphism of Λ-bialgebras $\mathscr{O}(G^\vee_\Lambda) \to \mathrm{B}_{\mathcal{G}}(\Lambda)$, which is easily seen to factor through a morphism

$$\mathscr{O}((G^\vee_\Lambda)^I) \to \mathrm{B}_{\mathcal{G}}(\Lambda), \tag{1.4}$$

and what remains to be seen is that this morphism is an isomorphism.

Here we note a first important difference with the constructions in [MV07]: in the present setting, we do not know any geometric construction of the commutativity constraint for convolution in $\mathsf{Perv}_{\mathrm{L}^+\mathcal{G}}(\mathrm{Gr}_{\mathcal{G}}, \Lambda)$ that corresponds under (1.2) to the obvious commutativity constraint for the tensor product of representations. (This is due to the absence, in this setting, of the Beĭlinson–Drinfeld deformations of the affine Grassmannian over products of curves.) To bypass this difficulty, we prove surjectivity of the morphism (1.4) in case $\Lambda = \mathbb{K}$; this implies in particular that $\mathrm{B}_{\mathcal{G}}(\mathbb{K})$ is commutative. Using (1.3) we deduce that $\mathrm{B}_{\mathcal{G}}(\mathbb{O})$ is commutative, and then that $\mathrm{B}_{\mathcal{G}}(\Bbbk)$ is commutative too.

The other argument for which the Beĭlinson–Drinfeld deformations are crucially used in [MV07] is in the construction of the monoidal structure on the "fiber functor" given by total cohomology (which induces the comultiplication in $\mathrm{B}_{\mathcal{G}}(\Lambda)$). Here we bypass this difficulty in a different way, constructing this monoidal structure by a method which is new even in the setting of [MV07], and which corrects a pervasive error in the literature on this topic (see Remark 5.6).

Once these properties are known it is not difficult to check that

$$\mathcal{G}^\vee_\Lambda := \mathrm{Spec}(\mathrm{B}_{\mathcal{G}}(\Lambda))$$

is a group scheme over Λ, and that (1.4) provides a morphism of group schemes $\mathcal{G}^\vee_\Lambda \to (G^\vee_\Lambda)^I$. To prove that this morphism is an isomorphism we first treat the case of relative rank 1, and exploit once again the possibility of transferring information between coefficients $\mathbb{K}$, $\mathbb{O}$ and $\Bbbk$. We also use a number of properties of the groups $(G^\vee_\Lambda)^I$ proved in [ALRR23], and specific arguments to treat the case when the quasi-reductive group over $\mathbb{Z}_2$ mentioned in §1.3 appears.

Remark 1.2. We emphasize that $(G^\vee_\Lambda)^I$ is not a reductive group scheme over Λ in general. We find it remarkable that such groups can arise as a Tannakian group for a category of perverse sheaves.

Remark 1.3. In the case where $\mathbb{F} = \mathbb{C}$, one can also consider $\mathrm{Gr}_{\mathcal{G}}(\mathbb{C})$ with the *analytic* topology, and work with sheaves of Λ-modules where Λ is any unital, commutative, noetherian ring of finite global dimension as in [MV07]. In particular, in the analytic setting, one can take $\Lambda = \mathbb{Z}$. The main results of this paper should remain valid in the analytic setting, with minor modifications to the proofs, but with one caveat: In this paper, we work with étale sheaves on algebraic stacks, and we are not aware of a reference that treats analytic sheaves on stacks in the appropriate generality. However, this issue can likely be circumvented by working with "equivariant derived categories in families" as in [AR, Chapter 10].

Remark 1.4. Thibaud van den Hove has announced a motivic refinement of the ramified Satake equivalence with coefficients in $\mathbb{Z}[\frac{1}{p}]$, $\mathbb{Q}$, or $\mathbb{F}_\ell$ for $\ell \neq p$.

1.6. **Motivation.** The main motivation for the constructions in [Zhu15, Ric16] was the application to some properties of Shimura varieties. At this point it does not seem that our integral and modular versions lead to any specific new application in this direction; in fact our desire to establish the equivalence (1.2) rather came from representation theory. Namely, in many cases the group $(G^\vee_\Bbbk)^I$ is still a reductive group. It was conjectured by Brundan [Bru98], and proved by him in most cases, that the restriction to $(G^\vee_\Bbbk)^I$ of any tilting $G^\vee_\Bbbk$-module remains tilting. (The remaining cases were later treated by van der Kallen [vdK01]). This proof is based on case-by-case considerations, which in our opinion does not explain the true meaning of this property. We hope that the geometric description of the restriction functor in terms of nearby cycles will lead to an alternative (and more satisfactory) proof of this property.

Let us also point out that in the recent update of Fargues–Scholze [FS21, §VIII.5] a uniform proof of this result, not relying on case-by-case considerations, is given under the assumption that I acts on $G^\vee_\Bbbk$ through of finite quotient of order prime to the characteristic of $\Bbbk$, see [FS21, Theorem VIII.5.15]. It would be interesting to see whether the coprimality assumption can be removed in the present setting.

1.7. **Contents.** In Section 2 we recall (and sometimes complete) some results on the geometry of affine Grassmannians associated with special parahoric models of reductive groups, following [Zhu15, Ric16]. In Section 3 we recall (and, again, sometimes complete) results from [AGLR22] and Haines–Richarz [HR21] which form the basis for the construction of the weight (or "constant term") functors. In Section 4 we introduce the category of equivariant sheaves on our affine Grassmannian, and some of the structures we have on this category. In Sections 5 we construct a monoidal structure on the total cohomology functor, and in Section 6 we construct the bialgebra $\mathrm{B}_{\mathcal{G}}(\Lambda)$ from §1.5, and establish some of its basic properties. Section 7 links our constructions to the "absolute" case studied in [MV07]. In Section 8 we explain the construction of the nearby cycles functor from §1.4, and how to obtain from it the morphism (1.4). Finally, in Section 9 we show that this morphism is an isomorphism, which completes the proof of the equivalence (1.2).

The paper finishes with two appendices: in Appendix A we prove that $\mathrm{L}^+\mathcal{G}$-equivariance is automatic for perverse sheaves on $\mathrm{Gr}_{\mathcal{G}}$ which are constant along $\mathrm{L}^+\mathcal{G}$-orbits, and in Appendix B we collect the results on étale sheaves on stacks that are used in the main text.

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2. Affine Grassmannians associated with special facets

2.1. **Loop groups and affine Grassmannians.** Let k be a field and x be an indeterminate. Given a k-algebra R we consider the rings $R[\![x]\!]$ and $R(\!(x)\!)$ of power and Laurent power series in x with coefficients in R respectively. Given an affine group scheme $\mathcal{H}$ over $k[\![x]\!]$, resp. an affine group scheme H over $k(\!(x)\!)$, we can define the positive loop group $\mathrm{L}^+\mathcal{H}$, resp. the loop group $\mathrm{L}H$, as the functor from k-algebras to groups defined by

$$\mathrm{L}^+\mathcal{H}(R) = \mathcal{H}(R[\![x]\!]), \quad \text{resp.} \quad \mathrm{L}H(R) = H(R(\!(x)\!)).$$

It is well known (see e.g. [Ric20, Lemma 3.17]) that the functor $\mathrm{L}\mathcal{H}$ is represented by an ind-affine group ind-scheme over k, and that $\mathrm{L}^+\mathcal{H}$ is represented by an affine group scheme over k.

Regarding $\mathrm{L}^+\mathcal{H}$, more specifically, for any $i \geq 0$ we can consider the functor $\mathrm{L}_i^+\mathcal{H}$ defined by $\mathrm{L}_i^+\mathcal{H}(R) = \mathcal{H}(R[x]/(x^{i+1}))$. Then we have

$$\mathrm{L}^+\mathcal{H} = \lim_{i \geq 0} \mathrm{L}_i^+\mathcal{H},$$

and for any $i \geq 0$ the functor $\mathrm{L}_i^+\mathcal{H}$ is an affine scheme, which is smooth over k if $\mathcal{H}$ is smooth over $k[\![x]\!]$. Note also that we have an obvious morphism of group ind-schemes

$$\mathrm{L}^+\mathcal{H} \to \mathrm{L}(\mathcal{H} \otimes_{k[\![x]\!]} k(\!(x)\!)),$$

which is representable by a closed immersion if $\mathcal{H}$ is of finite type.

Let $\mathcal{H}$ be an affine group scheme over $k[\![x]\!]$, and set $H := \mathcal{H} \otimes_{k[\![x]\!]} k(\!(x)\!)$. The *affine Grassmannian* associated with $\mathcal{H}$ is the fppf quotient $[\mathrm{L}H/\mathrm{L}^+\mathcal{H}]_{\mathrm{fppf}}$, i.e. the fppf sheaf associated with the presheaf

$$R \mapsto \mathrm{L}H(R)/\mathrm{L}^+\mathcal{H}(R). \tag{2.1}$$

It is a standard fact (see e.g. [Ric20, Proposition 3.18]) that if $\mathcal{H}$ is smooth over $k[\![x]\!]$ this sheaf identifies with the functor $\mathrm{Gr}_{\mathcal{H}}$ from k-algebras to sets sending R to the set of isomorphism classes of pairs $(\mathcal{E}, \alpha)$ where $\mathcal{E} \to \mathrm{Spec}(R[\![x]\!])$ is an fppf[1] $\mathcal{H}$-torsor and α is a section of $\mathcal{E}$ over $\mathrm{Spec}(R(\!(x)\!))$. In particular, this functor is represented by a separated ind-scheme of ind-finite type over k; see [Ric20, Theorem 3.4]. In fact, [Ric20, Proposition 3.18] also contains the following claim, which will be useful in the present paper.

Lemma 2.1. *If $\mathcal{H}$ is a smooth affine group scheme over $k[\![x]\!]$, then we have $\mathrm{Gr}_{\mathcal{H}} = [\mathrm{L}H/\mathrm{L}^+\mathcal{H}]_{\text{ét}}$, i.e. the functor $\mathrm{Gr}_{\mathcal{H}}$ is also the* étale *sheafification of the presheaf* (2.1).

Remark 2.2.
1. What we call the *affine Grassmannian* of $\mathcal{H}$ is sometimes called the *partial affine flag variety* of $\mathcal{H}$, at least when $\mathcal{H}$ is a parahoric group scheme in the sense of Bruhat–Tits (see e.g. [PR08]). This is justified by the fact that partial affine flag varieties attached to split reductive groups over k (as e.g. in Görtz [Gör10, Definition 2.6]) are special cases of this construction. The cases that we will consider below give rise to ind-schemes whose properties are very close to those of the "usual" affine Grassmannians as considered in [MV07, Gör10]. To simplify terminology and notation, and following the conventions in [Zhu17, Ric20], we will call all of these ind-schemes affine Grassmannians.
2. It is proved in Česnavičius [Čes24, Theorem 2.5] that if $\mathcal{H}$ is the base change of a reductive group scheme over k, then $\mathrm{Gr}_{\mathcal{H}}$ is the *Zariski* sheafification of the presheaf quotient $\mathrm{L}H/\mathrm{L}^+\mathcal{H}$, and moreover that no sheafification is needed under additional assumptions, see [Čes24, Theorem 3.4]. These results do not apply in the general setting considered below, and we do not know if these properties are satisfied in our case.

The following fact is clear from the definitions as functors.

Lemma 2.3. *Let $\mathcal{H}$ be a smooth affine group scheme over $k[\![x]\!]$. If k' is an extension of k, there exist canonical isomorphisms of k'-(ind-)schemes*

$$\mathrm{L}(k'(\!(x)\!) \otimes_{k(\!(x)\!)} H) \xrightarrow{\sim} k' \otimes_k (\mathrm{L}H), \qquad \mathrm{L}^+(k'[\![x]\!] \otimes_{k[\![x]\!]} \mathcal{H}) \xrightarrow{\sim} k' \otimes_k (\mathrm{L}^+\mathcal{H}),$$
$$\mathrm{Gr}_{k'[\![x]\!] \otimes_{k[\![x]\!]} \mathcal{H}} \xrightarrow{\sim} k' \otimes_k \mathrm{Gr}_{\mathcal{H}}.$$

For any k-scheme Y and any $y \in Y(k)$ we have the corresponding tangent space T_yY, see [Sta22, Tag 0B2C], which identifies with the subset of $Y(k[\varepsilon]/\varepsilon^2)$ consisting of points whose

[1] Since $\mathcal{H}$ is assumed to be smooth here, the notions of fpqc, fppf or étale torsors coincide. Moreover, any such torsor is representable by an $\mathcal{H}$-principal bundle, that is, a (necessarily smooth affine) scheme which is étale locally on the base isomorphic to $\mathcal{H}$.

image under $\epsilon \mapsto 0$ in $Y(k)$ is y. This definition extends in the obvious way to ind-schemes. In particular, if $\mathcal{H}$ is a smooth affine group scheme over $k[\![x]\!]$, and setting as above $H := \mathcal{H} \otimes_{k[\![x]\!]} k(\!(x)\!)$, we can consider the Lie algebras $\mathscr{L}ie(\mathrm{L}H)$ and $\mathscr{L}ie(\mathrm{L}^+\mathcal{H})$ of the group ind-scheme $\mathrm{L}H$ and of the group scheme $\mathrm{L}^+\mathcal{H}$, defined as the tangent spaces at their unit point. We can also consider the base point e of $\mathrm{Gr}_{\mathcal{H}}$, and the associated tangent space $T_e\mathrm{Gr}_{\mathcal{H}}$.

Lemma 2.4. *Let $\mathcal{H}$ be a smooth affine group scheme over $k[\![x]\!]$. There exists a canonical identification*

$$T_e\mathrm{Gr}_{\mathcal{H}} \cong \mathscr{L}ie(\mathrm{L}H)/\mathscr{L}ie(\mathrm{L}^+\mathcal{H}).$$

Proof. The exact sequence of pointed sheaves $1 \to \mathrm{L}^+\mathcal{H} \to \mathrm{L}H \to \mathrm{Gr}_{\mathcal{H}} \to 1$ induces an exact sequence of k-vector spaces

$$0 \to \mathscr{L}ie(\mathrm{L}^+\mathcal{H}) \to \mathscr{L}ie(\mathrm{L}H) \to T_e\mathrm{Gr}_{\mathcal{H}}.$$

In fact this sequence is also exact on the right: by Lemma 2.3 and compatibility of tangent spaces with extension of the base field, without loss of generality we may and do assume that k is algebraically closed. Then this easily follows from the fact that

$$\mathrm{Gr}_{\mathcal{H}}(k[\varepsilon]/\varepsilon^2) = \mathrm{L}H(k[\varepsilon]/\varepsilon^2)/\mathrm{L}^+\mathcal{H}(k[\varepsilon]/\varepsilon^2)$$

by Lemma 2.1, since $k[\varepsilon]/\varepsilon^2$ is a strictly henselian ring. □

In this paper, we will apply the affine Grassmannian construction in two different settings, that we now explain. Let $\mathbb{F}$ be an algebraically closed field, and set

$$F := \mathbb{F}(\!(t)\!), \quad O_F := \mathbb{F}[\![t]\!]$$

where t is an indeterminate. We denote by F^s a separable closure of F, and set

$$I := \mathrm{Gal}(F^s/F),$$

which is the inertia group of F as the residue field $\mathbb{F}$ is algebraically closed. For any F-scheme X we will write X_{F^s} for $F^s \otimes_F X$.

Given a smooth affine group scheme $\mathcal{H}$ over O_F, we consider the associated affine Grassmannian $\mathrm{Gr}_{\mathcal{H}}$ over $\mathbb{F}$ (where the indeterminate x is equal to t). On the other hand, if K is either F or F^s and P is a smooth affine group scheme over K, and if z is an indeterminate, we also consider the above construction for the smooth group scheme $K[\![z]\!] \otimes_K P$ over $K[\![z]\!]$ (and the indeterminate $x = z$) to obtain the K-ind-scheme $\mathrm{Gr}_{K[\![z]\!] \otimes_K P}$. This ind-scheme is the object considered in [MV07] or [Gör10, Definition 2.5]. Lemma 2.3 implies that if P is a smooth affine group scheme over F we have an identification

$$\mathrm{Gr}_{F^s[\![z]\!] \otimes_{F^s} (F^s \otimes_F P)} \xrightarrow{\sim} F^s \otimes_F \mathrm{Gr}_{F[\![z]\!] \otimes_F P}. \tag{2.2}$$

2.2. **Reductive groups and special facets.** From now on we fix a smooth affine group scheme $\mathcal{G}$ over O_F, and set

$$G := F \otimes_{O_F} \mathcal{G}.$$

It is important to know exactly for which $\mathcal{G}$ the associated ind-scheme $\mathrm{Gr}_{\mathcal{G}}$ is ind-proper (equivalently, ind-projective, as it is always ind-quasi-projective). A full criterion can be found in [Lou22, Théorème 5.2], and relates to (a generalization of) Bruhat–Tits theory. For our purposes, we will assume that G is a connected reductive group over F and that the special fiber $\mathbb{F} \otimes_{O_F} \mathcal{G}$ is connected; in this case it was shown (earlier) in [Ric16, Theorem A] that $\mathrm{Gr}_{\mathcal{G}}$ is ind-proper if and only if $\mathcal{G}$ is a parahoric group scheme in the sense of Bruhat–Tits [BT84].

The case that will be relevant in this paper is the following. Consider the (extended) Bruhat–Tits building $\mathscr{B}(G, F)$ associated with the reductive group G over the local field F. The parahoric group schemes are attached to the facets in $\mathscr{B}(G, F)$. We henceforth impose the following two assumptions:

- $\mathcal{G}$ is the parahoric group scheme attached to some facet $\mathfrak{a} \subset \mathscr{B}(G, F)$.
- The facet $\mathfrak{a}$ is *special*.

The latter assumption is equivalent to a certain geometric condition on $\mathrm{Gr}_{\mathcal{G}}$: see [Ric16, Theorem B]. For an explicit example of this setting with G nonsplit, see [Ric13, §4].

2.3. **Tori, (co)weights, and (co)roots.** By Steinberg's theorem,[2] G is quasi-split, i.e. there exist Borel subgroups $B \subset G$ defined over F. By [BT65, Corollaire 4.16], any such B contains a maximal F-split torus $A \subset G$ in its radical, and by [BT65, Théorème 4.15] the centralizer $T := Z_G(A)$ of A is then a Levi subgroup of B, and thus a maximal torus of G. The facet $\mathfrak{a}$ belongs to the apartment $\mathscr{A}(G, A', F)$ associated with some maximal F-split torus A'. By conjugacy of maximal F-split tori in G (see [BT65, Théorème 4.21]), we can and will assume that $A = A'$.

Let us denote by $\mathcal{A}$, resp. $\mathcal{T}$, the scheme-theoretic closure of A, resp. T, in $\mathcal{G}$. By [Ric16, Appendix 1], $\mathcal{T}$ is the unique parahoric group scheme of T. As explained in [PR08, §3.b], $\mathcal{T}$ therefore identifies with the connected Néron model of T. The group scheme $\mathcal{A}$ is also the scheme-theoretic closure of A in $\mathcal{T}$; in view of [BT84, §4.4], $\mathcal{A}$ is therefore the natural split torus extending A, which coincides with its connected Néron model.

Recall that if H is an F-torus, the F^s-torus H_{F^s} is split, see [BT65, Proposition 1.5]; we will denote by $\mathbb{X}^*(H)$, resp. $\mathbb{X}_*(H)$, its character, resp. cocharacter, lattice. (We emphasize that, contrary to what the notation might suggest, in general $\mathbb{X}^*(H)$ and $\mathbb{X}_*(H)$ consist of characters and cocharacters of H_{F^s} and not of H. If H is already split however, they can be seen as characters and cocharacters of H.) Given a group C and a $\mathbb{Z}$-module M endowed with a linear action of C, we will denote by $M_C := \mathbb{Z} \otimes_{\mathbb{Z}[C]} M$ the module of coinvariants. (Here $\mathbb{Z}[C]$ is the group algebra of C over $\mathbb{Z}$, and $\mathbb{Z}$ is endowed with the trivial action of C.)

In our present setting, we will in particular consider the torus T, its base change T_{F^s}, and the lattices $\mathbb{X}^*(T)$ and $\mathbb{X}_*(T)$. The group I acts on $\mathbb{X}_*(T)$, and this action factors through a finite quotient since T in fact splits over a finite separable extension; we will consider the associated coinvariants $\mathbb{X}_*(T)_I$. Recall the Kottwitz homomorphism $T(F) \to \mathbb{X}_*(T)_I$, see e.g. [PR08, §2.a.2]; this morphism is functorial on the category of F-tori, compatible with products and is given for $T = \mathbb{G}_{\mathrm{m}}$ by the natural map $F^\times \to F^\times/O_F^\times = \mathbb{X}_*(\mathbb{G}_{\mathrm{m}})$, where the identification is induced by $\mu \mapsto \mu(t)$. As explained in [PR08, §5.a], this morphism factors through an isomorphism

$$T(F)/\mathcal{T}(O_F) \xrightarrow{\sim} \mathbb{X}_*(T)_I. \tag{2.3}$$

The reductive group G_{F^s} over F^s is split since it admits the split maximal torus T_{F^s}, see [DG11, Exp. XXII, Proposition 2.2]. We can therefore consider its roots and coroots with respect to T_{F^s}, which will be denoted

$$\Phi_{\mathrm{abs}} \subset \mathbb{X}^*(T) \quad \text{and} \quad \Phi^\vee_{\mathrm{abs}} \subset \mathbb{X}_*(T)$$

respectively. (The subscript "abs" stands for "absolute.") These subsets are stable under the I-actions on $\mathbb{X}^*(T)$ and $\mathbb{X}_*(T)$. The nonzero weights of T_{F^s} in the Lie algebra of the Borel subgroup $B_{F^s} \subset G_{F^s}$ form a system of positive roots $\Phi^+_{\mathrm{abs}} \subset \Phi_{\mathrm{abs}}$ which is stable under the action of I. The subset of dominant coweights of T_{F^s} (with respect to this choice of positive roots) will be denoted $\mathbb{X}_*(T)^+$.

We will consider in particular the sublattice $\mathbb{Z}\Phi^\vee_{\mathrm{abs}} \subset \mathbb{X}_*(T)$ generated by the coroots. The quotient $\mathbb{X}_*(T)/\mathbb{Z}\Phi^\vee_{\mathrm{abs}}$ is called the *algebraic fundamental group* of G, and is denoted $\pi_1(G)$.

[2]Here "Steinberg's theorem" refers to [Ste65, Corollary 10.2(a)]. This statement has the assumption that the base field is perfect, but it is well known (and explicitly stated in [BS68, §8.6]) that this assumption can be removed if the group is assumed to be reductive.

Lemma 2.5. *The exact sequence* $\mathbb{Z}\Phi^\vee_{\mathrm{abs}} \hookrightarrow \mathbb{X}_*(T) \twoheadrightarrow \pi_1(G)$ *induces an exact sequence*

$$(\mathbb{Z}\Phi^\vee_{\mathrm{abs}})_I \hookrightarrow \mathbb{X}_*(T)_I \twoheadrightarrow \pi_1(G)_I.$$

Proof. By right exactness of the coinvariants functor, it suffices to prove that the morphism $(\mathbb{Z}\Phi^\vee_{\mathrm{abs}})_I \to \mathbb{X}_*(T)_I$ is injective. However, $\mathbb{Z}\Phi^\vee_{\mathrm{abs}}$ has a basis consisting of the simple coroots, which is permuted by I. As a consequence $(\mathbb{Z}\Phi^\vee_{\mathrm{abs}})_I$ is free, with a basis in bijection with I-orbits of simple coroots. To prove the claim, it therefore suffices to prove that the induced morphism

$$\mathbb{Q} \otimes_{\mathbb{Z}} \left((\mathbb{Z}\Phi^\vee_{\mathrm{abs}})_I\right) \to \mathbb{Q} \otimes_{\mathbb{Z}} \left(\mathbb{X}_*(T)_I\right) \tag{2.4}$$

is injective. Here the first term identifies with $(\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z}\Phi^\vee_{\mathrm{abs}})_I$, and the second one with $(\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{X}_*(T))_I$. Now the subspace $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z}\Phi^\vee_{\mathrm{abs}} \subset \mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{X}_*(T)$ has an I-stable complement, consisting of the elements orthogonal to all roots, which implies the injectivity of (2.4). □

The modules considered in Lemma 2.5 appear in the description of the set of connected components $\pi_0(\mathrm{Gr}_{\mathcal{G}})$ of $\mathrm{Gr}_{\mathcal{G}}$; namely, by [PR08, Theorem 0.1] the Kottwitz morphism induces a bijection

$$\pi_0(\mathrm{Gr}_{\mathcal{G}}) \xrightarrow{\sim} \pi_1(G)_I. \tag{2.5}$$

Now we consider the lattices $\mathbb{X}^*(A)$ and $\mathbb{X}_*(A)$ associated with the split torus A; the system of (relative) roots of (G, A) will be denoted

$$\Phi \subset \mathbb{X}^*(A).$$

Restriction along $A \subset T$ induces a canonical morphism $\mathbb{X}^*(T) \to \mathbb{X}^*(A)$, which is I-equivariant with respect to the trivial action on $\mathbb{X}^*(A)$. Since G is quasi-split, this morphism sends Φ_{abs} onto Φ. The image Φ^+ of Φ^+_{abs} is a system of positive roots for Φ; moreover, if we denote by $\Phi^{\mathrm{s}}_{\mathrm{abs}}$, resp. Φ^{s}, the associated basis of Φ_{abs}, resp. Φ, then $\Phi^{\mathrm{s}}_{\mathrm{abs}}$ is stable under the action of I, and our morphism $\mathbb{X}^*(T) \to \mathbb{X}^*(A)$ restricts to a surjection

$$\Phi^{\mathrm{s}}_{\mathrm{abs}} \twoheadrightarrow \Phi^{\mathrm{s}} \tag{2.6}$$

whose fibers are exactly the I-orbits in $\Phi^{\mathrm{s}}_{\mathrm{abs}}$. (For all of this, see [BT84, §4.1.2].)

2.4. **Iwahori–Weyl group and Schubert varieties.** The $\mathrm{L}^+\mathcal{G}$-orbit subschemes inside $\mathrm{Gr}_{\mathcal{G}}$ are smooth and locally closed, giving rise to a topological stratification. We will now describe these orbits together with their closures, following [Ric13, §§1-2] and [Ric16, §2.1 and §3].

The *Iwahori–Weyl group* of (G, A) is the quotient

$$W := \big(N_G(A)\big)(F)/\mathcal{T}(O_F),$$

where $N_G(A)$ is the normalizer of A in G. This group contains the quotient

$$T(F)/\mathcal{T}(O_F) \cong \mathbb{X}_*(T)_I$$

(see (2.3)) as a normal subgroup, and the quotient is the finite Weyl group

$$W_0 := \big(N_G(A)\big)(F)/T(F).$$

Setting

$$W_{\mathfrak{a}} := \big(N_G(A)(F) \cap \mathcal{G}(O_F)\big)/\mathcal{T}(O_F),$$

the composition

$$W_{\mathfrak{a}} \hookrightarrow W \to W_0$$

is an isomorphism (see [PR08, Appendix, Proposition 13] or [Ric13, Remark 1.4]). So we obtain an identification

$$W \cong W_{\mathfrak{a}} \ltimes \mathbb{X}_*(T)_I.$$

Consider the natural composition $\mathbb{X}_*(A) \to \mathbb{X}_*(T) \to \mathbb{X}_*(T)_I$; the induced morphism

$$a \colon \mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{X}_*(A) \to \mathbb{Q} \otimes_{\mathbb{Z}} (\mathbb{X}_*(T)_I) \tag{2.7}$$

is an isomorphism. We set

$$\mathbb{X}_*(T)_I^+ = \{\lambda \in \mathbb{X}_*(T)_I \mid \forall \alpha \in \Phi^+,\ \langle a^{-1}(\overline{\lambda}), \alpha\rangle \geq 0\},$$

where $\overline{\lambda}$ denotes the image of λ in $\mathbb{Q} \otimes_{\mathbb{Z}} (\mathbb{X}_*(T)_I)$.

Lemma 2.6. (1) *The composition*

$$\mathbb{X}_*(T)_I^+ \hookrightarrow \mathbb{X}_*(T)_I \twoheadrightarrow \mathbb{X}_*(T)_I/W_0$$

is a bijection; in other words, $\mathbb{X}_*(T)_I^+$ is a system of representatives for the W_0-orbits in $\mathbb{X}_*(T)_I$.

(2) *The composition*

$$\mathbb{X}_*(T)^+ \hookrightarrow \mathbb{X}_*(T) \twoheadrightarrow \mathbb{X}_*(T)_I$$

factors through a map $\mathbb{X}_*(T)^+ \to \mathbb{X}_*(T)_I^+$. This map is surjective if one of the following equivalent conditions holds, where Z is the (scheme-theoretic) center of G:

(a) *Z is a torus;*

(b) *the abelian group $\mathbb{X}^*(Z)$ of characters of $F^s \otimes_F Z$ is torsion free;*

(c) *the natural map $\mathbb{X}_*(T) \to \mathbb{X}_*(T/Z)$ is surjective.*

Proof. (1) Let Z be the center of G, and consider the adjoint group $G_{\mathrm{ad}} := G/Z$. Let also T_{ad}, resp. A_{ad}, be the image of T, resp. A, in G_{ad}; then T_{ad}, resp. A_{ad}, is a maximal, resp. maximal split, torus of G_{ad}, see [BT72, Théorème 2.20]. If we denote by Φ_{ad}, resp. $W_{\mathrm{ad},0}$, the (relative) root system, resp. Weyl group, of $(G_{\mathrm{ad}}, A_{\mathrm{ad}})$, then the quotient map $G \to G_{\mathrm{ad}}$ induces bijections $\Phi \cong \Phi_{\mathrm{ad}}$ and $W_0 \cong W_{\mathrm{ad},0}$, compatible with the action of the latter on the former. So the map $T \to T_{\mathrm{ad}}$ induces a commutative diagram of pointed sets

$$\begin{tikzcd}
\mathbb{X}_*(T)_I^+ \arrow[r] \arrow[d] & Q^+ \arrow[r, hook] \arrow[d] & \mathbb{X}_*(T_{\mathrm{ad}})_I^+ \arrow[d] \\
\mathbb{X}_*(T)_I/W_0 \arrow[r] & Q/W_0 \arrow[r, hook] & \mathbb{X}_*(T_{\mathrm{ad}})_I/W_0,
\end{tikzcd}$$

where Q is the image of $\mathbb{X}_*(T)_I \to \mathbb{X}_*(T_{\mathrm{ad}})_I$ and Q^+ is its intersection with $\mathbb{X}_*(T_{\mathrm{ad}})_I^+$. The left horizontal maps are surjective. Moreover, since the action of W_0 and the pairing with the relative roots is trivial on $\ker(\mathbb{X}_*(T)_I \to \mathbb{X}_*(T_{\mathrm{ad}})_I)$, their fibers are canonically bijective to this kernel, which is a subgroup of the monoid $\mathbb{X}_*(T)_I^+$. Hence, to prove our claim it suffices to prove bijectivity of the map $Q^+ \to Q/W_0$.

For this it is convenient to make the connection with the "échelonnage root system" as follows. Fixing a point in $\mathfrak{a}$ defines an identification of

$$V := \mathbb{X}_*(A_{\mathrm{ad}}) \otimes_{\mathbb{Z}} \mathbb{R} = \mathbb{X}_*(T_{\mathrm{ad}})_I \otimes_{\mathbb{Z}} \mathbb{R}$$

(see (2.7)) with the apartment $\mathscr{A}(G_{\mathrm{ad}}, A_{\mathrm{ad}}, F)$ on which the Iwahori–Weyl group $W_{\mathrm{ad}} = W(G_{\mathrm{ad}}, A_{\mathrm{ad}})$ acts by affine transformations. There exists a reduced root system $\Sigma \subset V^*$, called *échelonnage root system* by Bruhat–Tits, such that the associated affine Weyl group $W_{\mathrm{af}}(\Sigma)$ is isomorphic to the Iwahori–Weyl group $W_{\mathrm{sc}} = W(G_{\mathrm{sc}}, A_{\mathrm{sc}})$ of the simply connected cover $G_{\mathrm{sc}} \to G_{\mathrm{ad}}$ (see [BT72, Proposition 2.24]) with A_{sc} being the preimage of A_{ad} in G_{sc}, and such that the identification $W_{\mathrm{af}}(\Sigma) = W_{\mathrm{sc}}$ is compatible with the actions on V; see [Hai18, Section 6.1]. In particular, $W_0(\Sigma) = W_0$ for the finite Weyl groups. One necessarily has $\mathbb{Q} \cdot \Phi = \mathbb{Q} \cdot \Sigma$ (in fact, those are the same up to multiples of $2^{\pm 1}$), so the positive relative roots Φ^+ determine a system of positive roots $\Sigma^+ \subset \Sigma$. In V we have (in general strict) inclusions of $\mathbb{Z}$-sublattices

$$Q^\vee(\Sigma) = \mathrm{image}(\mathbb{X}_*(T_{\mathrm{sc}})_I \to \mathbb{X}_*(T_{\mathrm{ad}})_I) \subset Q \subset \mathbb{X}_*(T_{\mathrm{ad}})_I \subset P^\vee(\Sigma),$$

where $Q^\vee(\Sigma) \subset P^\vee(\Sigma)$ is the coroot (resp. coweight) lattice of Σ, and T_{sc} is the centralizer of A_{sc} in G_{sc} (which is the same as the preimage of T_{ad} in G_{sc}). Here we note that $\mathbb{X}_*(T_{\mathrm{ad}})_I$ is torsion free because $\mathbb{X}_*(T_{\mathrm{ad}})$ admits a basis permuted by I (namely, the fundamental coweights). Hence, the bijectivity of $Q^+ \to Q/W_0$ follows from standard facts on reduced root systems.

(2) Since the positive relative roots Φ^+ are restrictions of the positive absolute roots Φ^+_{abs}, the restriction of the quotient map $\mathbb{X}_*(T) \to \mathbb{X}_*(T)_I$ to $\mathbb{X}_*(T)^+$ factors through a map $\mathbb{X}_*(T)^+ \to \mathbb{X}_*(T)^+_I$.

To see that the conditions (2a), (2b) and (2c) are equivalent we note that the center Z of G is a F-group scheme of multiplicative type. So Z is a torus if and only if $\mathbb{X}^*(Z)$ is torsion free, which shows the equivalence of (2a) and (2b). Next, the exact sequence $1 \to Z \to T \to T_{\mathrm{ad}} \to 1$ (where we use the notation of the proof of (1)) induces a short exact sequence $0 \to \mathbb{X}^*(T_{\mathrm{ad}}) \to \mathbb{X}^*(T) \to \mathbb{X}^*(Z) \to 0$ of $\mathbb{Z}$-modules where $\mathbb{X}^*(T_{\mathrm{ad}})$ and $\mathbb{X}^*(T)$ are torsion free. Applying the functor $\mathrm{Hom}_{\mathbb{Z}\text{-Mod}}(\text{-}, \mathbb{Z})$ we obtain an exact sequence

$$0 \to \mathrm{Hom}_{\mathbb{Z}\text{-Mod}}(\mathbb{X}^*(Z), \mathbb{Z}) \to \mathbb{X}_*(T) \to \mathbb{X}_*(T_{\mathrm{ad}}) \to \mathrm{Ext}^1_{\mathbb{Z}\text{-Mod}}(\mathbb{X}^*(Z), \mathbb{Z}) \to 0.$$

Hence, $\mathbb{X}^*(Z)$ being torsion free is equivalent to the surjectivity of $\mathbb{X}_*(T) \to \mathbb{X}_*(T_{\mathrm{ad}})$, which shows the equivalence of (2b) and (2c).

Next, we show surjectivity of the map $\mathbb{X}_*(T)^+ \to \mathbb{X}_*(T)^+_I$ in case $G = G_{\mathrm{ad}}$ in the notation used above. In this case, the fundamental coweights $(\omega^\vee_\alpha : \alpha \in \Phi^{\mathrm{s}}_{\mathrm{abs}})$ form a $\mathbb{Z}$-basis of $\mathbb{X}_*(T)$, and a $\mathbb{Z}_{\geq 0}$-basis of $\mathbb{X}_*(T)^+$. For any $\alpha \in \Phi^{\mathrm{s}}_{\mathrm{abs}}$, the image $\bar{\omega}^\vee_\alpha$ of $\omega^\vee_\alpha$ in $\mathbb{X}_*(T)_I$ only depends on the orbit $I \cdot \alpha$, and the family $(\bar{\omega}^\vee_\alpha : \alpha \in \Phi^{\mathrm{s}}_{\mathrm{abs}}/I)$ forms a $\mathbb{Z}$-basis of $\mathbb{X}_*(T)_I$. We claim that this family is also a $\mathbb{Z}_{\geq 0}$-basis of $\mathbb{X}_*(T)^+_I$, which will imply the desired claim.

In fact, the map a in (2.7) is a composition of the natural maps

$$\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{X}_*(A) \xrightarrow{a_1} \mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{X}_*(T) \xrightarrow{a_2} \mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{X}_*(T)_I.$$

For $\alpha, \beta \in \Phi^{\mathrm{s}}_{\mathrm{abs}}$ one has

$$\langle \bar{\omega}^\vee_\alpha, \beta_{|A} \rangle = \langle a_1(\bar{\omega}^\vee_\alpha), \beta \rangle.$$

If we fix a finite quotient $\bar{I}$ of I through which the action on $\mathbb{X}_*(T)$ factors, then the element $a_1(\bar{\omega}^\vee_\alpha)$ is $\bar{I}$-invariant, so

$$\langle a_1(\bar{\omega}^\vee_\alpha), \beta \rangle = \tfrac{1}{|\bar{I}|} \langle \textstyle\sum_{i \in \bar{I}} i \cdot a_1(\bar{\omega}^\vee_\alpha), \beta \rangle = \tfrac{1}{|\bar{I}|} \langle a_1(\bar{\omega}^\vee_\alpha), \textstyle\sum_{i \in \bar{I}} i \cdot \beta \rangle.$$

Now $\sum_{i \in \bar{I}} i \cdot \beta$ is I-invariant, and the elements $a_1(\bar{\omega}^\vee_\alpha)$ and $\omega^\vee_\alpha$ have the same image in $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{X}_*(T)_I$, hence we have

$$\tfrac{1}{|\bar{I}|} \langle a_1(\bar{\omega}^\vee_\alpha), \textstyle\sum_{i \in \bar{I}} i \cdot \beta \rangle = \tfrac{1}{|\bar{I}|} \langle \omega^\vee_\alpha, \textstyle\sum_{i \in \bar{I}} i \cdot \beta \rangle = \begin{cases} \frac{1}{|I \cdot \alpha|} & \text{if } \beta \in I\alpha; \\ 0 & \text{otherwise.} \end{cases}$$

From this computation we deduce that a linear combination of the $\bar{\omega}^\vee_\alpha$ has nonnegative pairing with any simple (relative) root if and only if the coefficient of each $\bar{\omega}^\vee_\alpha$ is nonegative, which proves the desired claim.

Finally, we will prove that the map $\mathbb{X}_*(T)^+ \to \mathbb{X}_*(T)^+_I$ is surjective in case the morphism $\mathbb{X}_*(T) \to \mathbb{X}_*(T_{\mathrm{ad}})$ is surjective. Setting $K := \ker(\mathbb{X}_*(T) \to \mathbb{X}_*(T_{\mathrm{ad}}))$, we consider the diagram of natural maps

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r] \arrow[d, two heads] & \mathbb{X}_*(T) \arrow[r] \arrow[d, two heads] & \mathbb{X}_*(T_{\mathrm{ad}}) \arrow[r] \arrow[d, two heads] & 0 \\ & K_I \arrow[r] & \mathbb{X}_*(T)_I \arrow[r] & \mathbb{X}_*(T_{\mathrm{ad}})_I \arrow[r] & 0. \end{tikzcd}$$

If $\bar{\lambda} \in \mathbb{X}_*(T)^+_I$, then its image in $\mathbb{X}_*(T_{\mathrm{ad}})_I$ belongs to $\mathbb{X}_*(T_{\mathrm{ad}})^+_I$, hence is the image of some $\mu \in \mathbb{X}_*(T_{\mathrm{ad}})^+$ by the case treated above. If ν is any preimage of μ in $\mathbb{X}_*(T)$, then its image $\bar{\nu}$ in $\mathbb{X}_*(T)_I$ has the same image as $\bar{\lambda}$ in $\mathbb{X}_*(T_{\mathrm{ad}})_I$. Hence there exists $\eta \in K$ whose image in K_I has image $\bar{\lambda} - \bar{\nu}$ in $\mathbb{X}_*(T)_I$. Then $\lambda := \nu + \eta$ has image $\bar{\lambda}$ in $\mathbb{X}_*(T)_I$, and its image in $\mathbb{X}_*(T_{\mathrm{ad}})$

is μ. Since the pairing of any coweight of T with an absolute root only depends on its image in $\mathbb{X}_*(T_{\mathrm{ad}})$ we have $\lambda \in \mathbb{X}_*(T)^+$, which finishes the proof. □

Remark 2.7. Contrary to what is asserted in [Ric16, Remark 3.8] and [HR20, page 3227, lines 30–31], the surjectivity of $\mathbb{X}_*(T)^+ \to \mathbb{X}_*(T)_I^+$ fails in general. Both references use the false claim implicitly through [Zhu15, Proof of Corollary 2.8] where the same false claim appears. The proof can be fixed using Corollary A.4 below by passing to adjoint groups and referring to Lemma 2.6(2) in this case.

For an explicit example, let $G = \mathrm{SU}_3$ be the special unitary group on a 3-dimensional hermitian vector space defined by some separable quadratic extension F'/F. Then, $\mathbb{X}_*(T)$ can be identified with the group $\mathbb{Z}^3_{\Sigma=0}$ of elements $(a,b,c) \in \mathbb{Z}^3$ with $a+b+c=0$, with the non-trivial Galois involution in $\mathrm{Gal}(F'/F)$ acting by $(a,b,c) \mapsto (-c,-b,-a)$. The subgroup $(1-I)\mathbb{Z}^3_{\Sigma=0}$ identifies with the subset of vectors $(-b,2b,-b)$ with $b \in \mathbb{Z}$, and the map $(a,b,c) \mapsto a-c$ induces an isomorphism $\mathbb{Z}^3_{\Sigma=0}/(1-I)\mathbb{Z}^3_{\Sigma=0} \simeq \mathbb{Z}$. Hence, $\mathbb{X}_*(T) \to \mathbb{X}_*(T)_I$ identifies with $\mathbb{Z}^3_{\Sigma=0} \to \mathbb{Z}$, $(a,b,c) \mapsto a-c$. The dominant elements $(\mathbb{Z}^3_{\Sigma=0})^+$ with respect to the upper triangular Borel subgroup are given by the condition $a \geq b \geq c$ in $\mathbb{Z}^3_{\Sigma=0}$ and by $a \geq 0$ in $\mathbb{Z}$. An easy calculation shows that the image of $(\mathbb{Z}^3_{\Sigma=0})^+ \to \mathbb{Z}^+ = \mathbb{Z}_{\geq 0}$ is $2\mathbb{Z}_{\geq 0}$, so the map is not surjective.

The Iwahori–Weyl group W acts on the apartment $\mathscr{A}(G,A,F)$ by affine transformations. Any choice of a point in $\mathfrak{a}$ defines an identification of

$$\mathbb{X}_*(A) \otimes_{\mathbb{Z}} \mathbb{R} = \mathbb{X}_*(T)_I \otimes_{\mathbb{Z}} \mathbb{R}$$

(see (2.7)) with $\mathscr{A}(G,A,F)$ such that an element μ of the subgroup $T(F)/\mathcal{T}(O_F) \cong \mathbb{X}_*(T)_I$ (see (2.3)) of W acts by translation by $-\mu$. Let $\mathfrak{a}_0$ be the alcove in $\mathscr{A}(G,A,F)$ containing $\mathfrak{a}$ in its closure and which belongs to the chamber corresponding under this identification to the chamber in $X_*(A) \otimes_{\mathbb{Z}} \mathbb{R}$ determined by B. This choice determines a structure of a quasi-Coxeter group on W; see [PR08, Appendix, Lemma 14] and [Ric13, §1] for details. In terms of this structure, one can consider a subset

$${}_{\mathfrak{a}}W^{\mathfrak{a}} \subset W$$

characterized in [Ric13, (1.5)], which is a system of representatives for the double quotient

$$W_{\mathfrak{a}} \backslash W / W_{\mathfrak{a}} \cong \mathbb{X}_*(T)_I / W_0,$$

and is thus in canonical bijection with the double quotient

$$\mathcal{G}(O_F) \backslash G(F) / \mathcal{G}(O_F) = \mathrm{L}^+\mathcal{G}(\mathbb{F}) \backslash \mathrm{L}G(\mathbb{F}) / \mathrm{L}^+\mathcal{G}(\mathbb{F});$$

see [Ric13, Lemma 1.3]. Since $\mathfrak{a}$ is special, the subset ${}_{\mathfrak{a}}W^{\mathfrak{a}}$ has an explicit description, which follows from [Ric13, Corollary 1.8] or the more general claim [Ric16, Lemma 3.9]: it coincides with the subset

$$\mathbb{X}_*(T)_I^+ \subset \mathbb{X}_*(T)_I \subset W.$$

Any $\mu \in \mathbb{X}_*(T)_I$ determines a point in $T(F)/\mathcal{T}(O_F)$ (see (2.3)), and hence an $\mathbb{F}$-point of $\mathrm{Gr}_{\mathcal{G}}$, which we denote t^μ. We denote by $\mathrm{Gr}^\mu_{\mathcal{G}}$ the $\mathrm{L}^+\mathcal{G}$-orbit of t^μ, and by $\mathrm{Gr}^{\leq\mu}_{\mathcal{G}}$ the scheme-theoretic closure of $\mathrm{Gr}^\mu_{\mathcal{G}}$. We also denote by

$$j^\mu \colon \mathrm{Gr}^\mu_{\mathcal{G}} \to \mathrm{Gr}_{\mathcal{G}}, \qquad j^{\leq\mu} \colon \mathrm{Gr}^{\leq\mu}_{\mathcal{G}} \to \mathrm{Gr}_{\mathcal{G}} \tag{2.8}$$

the natural immersions. Then $\mathrm{Gr}^{\leq\mu}_{\mathcal{G}}$ is a projective variety over $\mathbb{F}$, and $\mathrm{Gr}^\mu_{\mathcal{G}}$ is a smooth open dense subscheme of $\mathrm{Gr}^{\leq\mu}_{\mathcal{G}}$, which admits a paving by affine spaces;[3] see [Ric16, §2.1] for details. This notation is justified by the following fact. Recall from Lemma 2.5 and its proof that the submodule $(\mathbb{Z}\Phi^\vee_{\mathrm{abs}})_I \subset \mathbb{X}_*(T)_I$ admits a basis in natural bijection with the I-orbits of simple

[3]The paving by affine spaces in this general setting can be deduced from [PR08, Proposition 8.7] using Demazure resolutions.

roots. We can therefore define an order $\leq$ on $\mathbb{X}_*(T)_I$ by declaring that $\lambda \leq \mu$ iff $\mu - \lambda$ belongs to the submonoid generated by this basis. Then on the underlying sets we have

$$|\mathrm{Gr}_{\mathcal{G}}^{\leq\mu}| = \bigsqcup_{\substack{\lambda \in \mathbb{X}_*(T)_I^+ \\ \lambda \leq \mu}} |\mathrm{Gr}_{\mathcal{G}}^{\lambda}|; \tag{2.9}$$

see [Ric13, Corollary 1.8 and Proposition 2.8].

Let $\rho \in \mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{X}^*(T)$ be one-half the sum of the roots in Φ_{abs}^+. Since this element is I-invariant, the pairing

$$\langle -, 2\rho \rangle \colon \mathbb{X}_*(T) \to \mathbb{Z}$$

factors through a map $\mathbb{X}_*(T)_I \to \mathbb{Z}$, which we denote similarly. With this notation, for any $\lambda \in \mathbb{X}_*(T)_I^+$ we have

$$\dim(\mathrm{Gr}_{\mathcal{G}}^{\lambda}) = \langle \lambda, 2\rho \rangle, \tag{2.10}$$

see again [Ric13, Corollary 1.8 and Proposition 2.8].

Remark 2.8. It is known that $\mathrm{Gr}_{\mathcal{G}}^{\leq\mu}$ is always geometrically unibranch (i.e. the normalization morphism is a universal homeomorphism, see [Kol16, Corollary 32]), see [HR23, Proposition 3.1]. Under some additional mild assumptions, $\mathrm{Gr}_{\mathcal{G}}^{\leq\mu}$ turns out to also be normal and Cohen–Macaulay, due to results of Faltings [Fal03], Pappas–Rapoport [PR08], Fakhruddin–Haines–Lourenço–Richarz [FHLR22] and the second named author [Lou23] (see also [HLR18, BR23] for failures of this property). We do not need these results below, since universal homeomorphisms induce equivalences on small étale topoi.

2.5. **Convolution schemes.** Given a smooth affine group scheme $\mathcal{H}$ over O_F, we can consider the "convolution functor" $\mathrm{Conv}_{\mathcal{H}}$ over $\mathbb{F}$ whose R-points consist of isomorphism classes of triples $(\mathcal{E}_1, \mathcal{E}_2, \alpha, \beta)$ where $\mathcal{E}_1, \mathcal{E}_2$ are $\mathcal{H}$-torsors over $\mathrm{Spec}(R[\![t]\!])$, α is a trivialization of $\mathcal{E}_1$ over $\mathrm{Spec}(R(\!(t)\!))$, and β is an isomorphism between the restrictions of $\mathcal{E}_1$ and $\mathcal{E}_2$ to $\mathrm{Spec}(R(\!(t)\!))$. We have a "multiplication" morphism

$$m \colon \mathrm{Conv}_{\mathcal{H}} \to \mathrm{Gr}_{\mathcal{H}}, \ (\mathcal{E}_1, \mathcal{E}_2, \alpha, \beta) \mapsto (\mathcal{E}_2, \beta \circ \alpha),$$

where we use the moduli description of $\mathrm{Gr}_{\mathcal{H}}$ recalled in §2.1, as well as the projection map

$$\mathrm{pr}_1 \colon \mathrm{Conv}_{\mathcal{H}} \to \mathrm{Gr}_{\mathcal{H}}, \ (\mathcal{E}_1, \mathcal{E}_2, \alpha, \beta) \mapsto (\mathcal{E}_1, \alpha).$$

It is clear that the product map

$$(\mathrm{pr}_1, m) \colon \mathrm{Conv}_{\mathcal{H}} \to \mathrm{Gr}_{\mathcal{H}} \times \mathrm{Gr}_{\mathcal{H}} \tag{2.11}$$

is an isomorphism; in particular, this shows that $\mathrm{Conv}_{\mathcal{H}}$ is representable by an ind-scheme, which is ind-projective if $\mathcal{H}$ is parahoric. In this case, m is ind-projective as well.

This ind-scheme can also be constructed as a twisted product

$$\mathrm{Conv}_{\mathcal{H}} \cong \mathrm{L}H \times^{\mathrm{L}^+\mathcal{H}} \mathrm{L}H/\mathrm{L}^+\mathcal{H} \tag{2.12}$$

(where, as usual, $H = F \otimes_{O_F} \mathcal{H}$, and we consider étale quotients). From this perspective, given any locally closed subschemes $X, Y \subset \mathrm{Gr}_{\mathcal{H}}$ where Y is $\mathrm{L}^+\mathcal{H}$-stable, we can define a locally closed subscheme

$$X \mathbin{\tilde{\times}} Y := (X \times_{\mathrm{Gr}_{\mathcal{H}}} \mathrm{L}H) \times^{\mathrm{L}^+\mathcal{H}} Y \qquad \subset \mathrm{Conv}_{\mathcal{H}}.$$

For instance, if $\mathcal{G}$ is as in §2.2, given $\lambda, \mu \in \mathbb{X}_*(T)_I^+$, we set

$$\mathrm{Conv}_{\mathcal{G}}^{(\lambda,\mu)} := \mathrm{Gr}_{\mathcal{G}}^{\lambda} \mathbin{\tilde{\times}} \mathrm{Gr}_{\mathcal{G}}^{\mu}.$$

The closures of these schemes are denoted by

$$\mathrm{Conv}_{\mathcal{G}}^{\leq(\lambda,\mu)} := \overline{\mathrm{Conv}_{\mathcal{G}}^{(\lambda,\mu)}} = \mathrm{Gr}_{\mathcal{G}}^{\leq\lambda} \mathbin{\tilde{\times}} \mathrm{Gr}_{\mathcal{G}}^{\leq\mu}.$$

In view of (2.9) we have

$$|\mathrm{Conv}_{\mathcal{G}}^{\leq(\lambda,\mu)}| = \bigsqcup_{\substack{\lambda',\mu' \in \mathbb{X}_*(T)_I^+ \\ \lambda' \leq \lambda,\, \mu' \leq \mu}} |\mathrm{Conv}_{\mathcal{G}}^{(\lambda',\mu')}|. \tag{2.13}$$

Note that in the special case where $\mathcal{H}$ is abelian, the left multiplication action of $\mathrm{L}^+\mathcal{H}$ on $\mathrm{Gr}_{\mathcal{H}}$ is trivial. We deduce an identification

$$\mathrm{Conv}_{\mathcal{H}} \cong \mathrm{L}\mathcal{H} \times^{\mathrm{L}^+\mathcal{H}} \mathrm{Gr}_{\mathcal{H}} \cong \mathrm{Gr}_{\mathcal{H}} \times \mathrm{Gr}_{\mathcal{H}} \qquad \text{if } \mathcal{H} \text{ is abelian.} \tag{2.14}$$

We emphasize that this isomorphism is *different* from the one in (2.11).

2.6. **Some quotient stacks.** In this subsection we introduce some stacks that will be used later in our study of sheaves on $\mathrm{Gr}_{\mathcal{H}}$.

Let $\mathrm{Hk}_{\mathcal{H}}$ be the Hecke stack (over $\mathbb{F}$) associated with $\mathrm{Gr}_{\mathcal{H}}$, such that $\mathrm{Hk}_{\mathcal{H}}(R)$ is the category of triples $(\mathcal{E}_1, \mathcal{E}_2, \alpha)$ where $\mathcal{E}_1, \mathcal{E}_2$ are $\mathcal{H}$-bundles on $\mathrm{Spec}(R[\![t]\!])$ and α is an isomorphism between their restrictions to $\mathrm{Spec}(R(\!(t)\!))$. We have a natural morphism

$$h\colon \mathrm{Gr}_{\mathcal{H}} \to \mathrm{Hk}_{\mathcal{H}}$$

sending $(\mathcal{E}, \alpha)$ to $(\mathcal{E}_0, \mathcal{E}, \alpha)$ where $\mathcal{E}_0$ is the trivial $\mathcal{H}$-bundle. This morphism is an $\mathrm{L}^+\mathcal{H}$-bundle; as in [FS21, Proposition 6.1.7] it factors through an isomorphism

$$[\mathrm{L}^+\mathcal{H}\backslash \mathrm{Gr}_{\mathcal{H}}]_{\text{ét}} \xrightarrow{\sim} \mathrm{Hk}_{\mathcal{H}} \tag{2.15}$$

where the left-hand side is the étale quotient stack of $\mathrm{Gr}_{\mathcal{H}}$ by the action of $\mathrm{L}^+\mathcal{H}$. (In particular, the left-hand side identifies with the fppf quotient stack of $\mathrm{Gr}_{\mathcal{H}}$ by the action of $\mathrm{L}^+\mathcal{H}$.)

By the proof of its representability (see e.g. [Ric20], or [RS20, Lemma A.5] for a more general result), the ind-scheme $\mathrm{Gr}_{\mathcal{H}}$ admits a presentation $\mathrm{Gr}_{\mathcal{H}} = \mathrm{colim}_{i \geq 0} X_i$ where each X_i is an $\mathbb{F}$-scheme of finite type such that the action of $\mathrm{L}^+\mathcal{H}$ on $\mathrm{Gr}_{\mathcal{H}}$ factors through an action on X_i, and such that the latter action factors though an action of $\mathrm{L}^+_{n_i}\mathcal{H}$ for some $n_i \geq 0$. One can also assume that if $i \leq j$ we have $n_i \leq n_j$, so that the composition $\mathrm{L}^+\mathcal{H} \to \mathrm{L}^+_{n_j}\mathcal{H}$ factors through the quotient morphism $\mathrm{L}^+_{n_i}\mathcal{H} \to \mathrm{L}^+_{n_j}\mathcal{H}$. Then we have

$$\mathrm{Hk}_{\mathcal{H}} = \mathrm{colim}_i [\mathrm{L}^+\mathcal{H}\backslash X_i]_{\text{ét}}, \quad [\mathrm{L}^+\mathcal{H}\backslash X_i]_{\text{ét}} = \lim_{n \geq n_i} [\mathrm{L}^+_n\mathcal{H}\backslash X_i]_{\text{ét}}. \tag{2.16}$$

(Here again, each étale quotient stack identifies with the corresponding fppf quotient stack. Note that each $[\mathrm{L}^+_n\mathcal{H}\backslash X_i]_{\text{ét}}$ is an algebraic stack over $\mathbb{F}$, see [Sta22, Tag 06FI], which is moreover of finite type.)

We will also consider the Hecke convolution stack $\mathrm{HkConv}_{\mathcal{H}}$ over $\mathbb{F}$, defined in such a way that $\mathrm{HkConv}_{\mathcal{H}}(R)$ is the category of tuples $(\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3, \alpha, \beta)$ where $\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3$ are $\mathcal{H}$-bundles over $\mathrm{Spec}(R[\![t]\!])$, α is an isomorphism between the restrictions of $\mathcal{E}_1$ and $\mathcal{E}_2$ to $\mathrm{Spec}(R(\!(t)\!))$, and β is an isomorphism between the restrictions of $\mathcal{E}_2$ and $\mathcal{E}_3$ to $\mathrm{Spec}(R(\!(t)\!))$. We have a canonical morphism

$$\tilde{h}\colon \mathrm{Conv}_{\mathcal{H}} \to \mathrm{HkConv}_{\mathcal{H}}$$

sending $(\mathcal{E}_1, \mathcal{E}_2, \alpha, \beta)$ to $(\mathcal{E}_0, \mathcal{E}_1, \mathcal{E}_2, \alpha, \beta)$ where $\mathcal{E}_0$ is the trivial $\mathcal{H}$-bundle. This morphism is an $\mathrm{L}^+\mathcal{H}$-bundle, and factors through an isomorphism

$$[\mathrm{L}^+\mathcal{H}\backslash \mathrm{Conv}_{\mathcal{H}}]_{\text{ét}} \xrightarrow{\sim} \mathrm{HkConv}_{\mathcal{H}}.$$

We also have descriptions of this stack similar to those in (2.16).

The morphisms $\mathrm{pr}_1\colon \mathrm{Conv}_{\mathcal{H}} \to \mathrm{Gr}_{\mathcal{H}}$ and $m\colon \mathrm{Conv}_{\mathcal{H}} \to \mathrm{Gr}_{\mathcal{H}}$ have analogues at the level of quotient stacks, which will also be denoted

$$\mathrm{pr}_1\colon \mathrm{HkConv}_{\mathcal{H}} \to \mathrm{Hk}_{\mathcal{H}}, \qquad m\colon \mathrm{HkConv}_{\mathcal{H}} \to \mathrm{Hk}_{\mathcal{H}},$$

and which send $(\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3, \alpha, \beta)$ to $(\mathcal{E}_1, \mathcal{E}_2, \alpha)$ and $(\mathcal{E}_1, \mathcal{E}_3, \beta \circ \alpha)$, respectively. We also have morphisms

$$p\colon \mathrm{HkConv}_{\mathcal{H}} \to \mathrm{Hk}_{\mathcal{H}} \times \mathrm{Hk}_{\mathcal{H}}, \qquad p\colon \mathrm{Conv}_{\mathcal{H}} \to \mathrm{Gr}_{\mathcal{H}} \times \mathrm{Hk}_{\mathcal{H}},$$

where the former sends $(\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3, \alpha, \beta)$ to $\big((\mathcal{E}_1, \mathcal{E}_2, \alpha), (\mathcal{E}_2, \mathcal{E}_3, \beta)\big)$, and the latter is defined similarly. These maps fit into cartesian squares

$$\begin{tikzcd} \mathrm{HkConv}_{\mathcal{H}} \arrow[r, "p"] \arrow[d, "\mathrm{pr}_1"'] & \mathrm{Hk}_{\mathcal{H}} \times \mathrm{Hk}_{\mathcal{H}} \arrow[d] & \mathrm{Conv}_{\mathcal{H}} \arrow[r, "p"] \arrow[d, "\mathrm{pr}_1"'] & \mathrm{Gr}_{\mathcal{H}} \times \mathrm{Hk}_{\mathcal{H}} \arrow[d] \\ \mathrm{Hk}_{\mathcal{H}} \arrow[r] & \mathrm{Hk}_{\mathcal{H}} \times [\mathrm{L}^+\mathcal{H}\backslash \mathrm{pt}]_{\text{ét}}, & \mathrm{Gr}_{\mathcal{H}} \arrow[r] & \mathrm{Gr}_{\mathcal{H}} \times [\mathrm{L}^+\mathcal{H}\backslash \mathrm{pt}]_{\text{ét}}, \end{tikzcd} \tag{2.17}$$

where the bottom horizontal arrows are induced by the obvious morphism $\mathrm{pt} \to [\mathrm{L}^+\mathcal{H}\backslash \mathrm{pt}]_{\text{ét}}$.

In the case where $\mathcal{H}$ is abelian, we have a counterpart to (2.14): an isomorphism

$$\mathrm{HkConv}_{\mathcal{H}} \cong \mathrm{Hk}_{\mathcal{H}} \times \mathrm{Gr}_{\mathcal{H}} \qquad \text{if } \mathcal{H} \text{ is abelian.} \tag{2.18}$$

In this case, the following diagram commutes:

$$\begin{tikzcd} & \mathrm{Gr}_{\mathcal{H}} \times \mathrm{Gr}_{\mathcal{H}} \arrow[dl, "h \times h"'] \arrow[r, "(2.14)", "\sim"'] \arrow[d, "h \times \mathrm{id}"] & \mathrm{Conv}_{\mathcal{H}} \arrow[r, "m"] \arrow[d, "\tilde{h}"] & \mathrm{Gr}_{\mathcal{H}} \arrow[d, "h"] \\ \mathrm{Hk}_{\mathcal{H}} \times \mathrm{Hk}_{\mathcal{H}} & \mathrm{Hk}_{\mathcal{H}} \times \mathrm{Gr}_{\mathcal{H}} \arrow[l, "\mathrm{id} \times h"] \arrow[r, "(2.18)", "\sim"'] & \mathrm{HkConv}_{\mathcal{H}} \arrow[r, "m"] \arrow[ll, bend left, "p"] & \mathrm{Hk}_{\mathcal{H}}. \end{tikzcd} \tag{2.19}$$

In later sections we will consider the morphisms m, p, h and $\tilde{h}$ for the group $\mathcal{G}$, but also for some subgroups. To avoid confusions, when necessary we will add a subscript to these notations to indicate which group is considered.

3. Semi-infinite orbits

3.1. **Attractors and fixed points in Bruhat–Tits theory.** Recall that for any base scheme S, given an S-scheme X endowed with an action of the multiplicative group $\mathbb{G}_{\mathrm{m},S}$ we have associated *attractor* and *fixed points* functors from S-schemes to sets, denoted X^+ and X^0; see e.g. [Ric19]. These functors are not always representable, but they are under reasonable assumptions that will be satisfied in all the examples we consider below. We also have natural morphisms of functors $X^0 \to X^+$ and $X^+ \to X$. The former admits a left inverse $X^+ \to X^0$ (the "limit" morphism). If X is separated over S then the morphism $X^+ \to X$ is a monomorphism, see [Ric19, Remark 1.19(ii)].

Let us now consider the setting of Section 2, fix a cocharacter

$$\lambda\colon \mathbb{G}_{\mathrm{m},F} \to A \subset G,$$

and then make $\mathbb{G}_{\mathrm{m},F}$ act on G by conjugation via this cocharacter. Denote the attractor and fixed-points functors by

$$P_\lambda := G^+ \qquad \text{and} \qquad M_\lambda := G^0,$$

respectively. These functors are represented by smooth subgroup schemes of G. Indeed, by [CGP15, Proposition 2.2.9], P_λ is a parabolic subgroup of G containing A, and M_λ is a Levi factor of P_λ. Moreover, every parabolic subgroup of G can be described in this way as an attractor.

Remark 3.1. (1) To amplify the preceding comment: if $P \subset G$ is a parabolic subgroup containing A, then $P = P_\lambda$ for some $\lambda \in \mathbb{X}_*(A)$. (This follows from the conjugacy of maximal F-split tori in P: see [BT65, Lemme 4.6 and Proposition 11.6].) In this case, M_λ is the unique Levi factor of P containing A. The parabolic subgroup P_λ is standard with respect to B (i.e., contains B) if and only if λ is dominant (with respect to $\Phi^+_{\mathrm{abs}} \subset \mathbb{X}^*(T)$).

(2) If one replaces λ by the opposite cocharacter $-\lambda$, then we obtain the same Levi subgroup: $M_{-\lambda} = M_\lambda$, and the parabolic subgroup $P_{-\lambda}$ is the opposite parabolic subgroup such that $P_\lambda \cap P_{-\lambda} = M_\lambda$.

Next, we observe that λ extends uniquely to a cocharacter

$$\lambda_{O_F} \colon \mathbb{G}_{\mathrm{m},O_F} \to \mathcal{A} \subset \mathcal{G}$$

of O_F-groups by the universal property of the Néron model $\mathcal{A}$ (or, in a more elementary way, by the equality $\mathrm{Hom}_{O_F\text{-gps}}(\mathbb{G}_{\mathrm{m},O_F}, \mathcal{A}) = \mathrm{Hom}_{F\text{-gps}}(\mathbb{G}_{\mathrm{m},F}, A)$, which follows from the facts that $\mathcal{A}$ is a split torus and $\mathrm{Spec}(O_F)$ is connected). We can therefore consider, as above, the action of $\mathbb{G}_{\mathrm{m},O_F}$ on $\mathcal{G}$ induced by λ_{O_F}, as well as the corresponding attractors and fixed points

$$\mathcal{P}_\lambda := \mathcal{G}^+ \qquad \text{and} \qquad \mathcal{M}_\lambda := \mathcal{G}^0$$

over O_F. According to [HR21, Lemma 4.5] (and its proof), $\mathcal{P}_\lambda$ and $\mathcal{M}_\lambda$ are both smooth affine group schemes over O_F, and they coincide with the scheme-theoretic closures of their counterparts in G:

$$\mathcal{P}_\lambda = \overline{P_\lambda} \qquad \text{and} \qquad \mathcal{M}_\lambda = \overline{M_\lambda}. \tag{3.1}$$

In particular, the group schemes $\mathcal{P}_\lambda$ and $\mathcal{M}_\lambda$ depend only on P_λ and M_λ, and not on the choice of the cocharacter λ. The natural limit morphism $\mathcal{P}_\lambda \to \mathcal{M}_\lambda$ is split by the embedding $\mathcal{M}_\lambda \to \mathcal{P}_\lambda$, and its kernel $\mathcal{U}_\lambda$ is a smooth affine O_F-group scheme with connected unipotent geometric fibers.

Since M_λ contains A, we can consider the apartment $\mathscr{A}(M_\lambda, A, F)$ in the building $\mathscr{B}(M_\lambda, F)$ of M_λ. This identifies with $\mathscr{A}(G, A, F)$, hence we can consider the unique facet $\mathfrak{a}_{M_\lambda}$ in $\mathscr{A}(M_\lambda, A, F)$ containing $\mathfrak{a}$. In view of [Ric16, Appendix 1], we know that:

- $\mathcal{M}_\lambda$ is the parahoric group scheme attached to the facet $\mathfrak{a}_{M_\lambda} \subset \mathscr{B}(M_\lambda, F)$;
- the facet $\mathfrak{a}_{M_\lambda}$ is special.

In other words, $\mathcal{M}_\lambda$ matches the set-up of §2.2. (As usual, these data depend only on M_λ, and not on the choice of λ.)

3.2. **Attractors and fixed points on the affine Grassmannian.** We continue with the setting of §3.1. The composition

$$\mathbb{G}_{\mathrm{m},\mathbb{F}} \subset \mathrm{L}^+\mathbb{G}_{\mathrm{m},O_F} \xrightarrow{\mathrm{L}^+\lambda_{O_F}} \mathrm{L}^+\mathcal{G} \tag{3.2}$$

and the $\mathrm{L}^+\mathcal{G}$-action on $\mathrm{Gr}_{\mathcal{G}}$ provide a $\mathbb{G}_{\mathrm{m},\mathbb{F}}$-action on the ind-scheme $\mathrm{Gr}_{\mathcal{G}}$. It is clear by construction that this action preserves Schubert varieties. Furthermore, by [HR21, Lemma 5.3] it is Zariski locally linearizable in the sense of [Ric19], and thus the fixed points $\mathrm{Gr}^0_{\mathcal{G}}$ and the attractor $\mathrm{Gr}^+_{\mathcal{G}}$ (defined in the obvious way, generalizing the definition for schemes) are representable by ind-schemes (see [HR21, Theorem 2.1]). Moreover the natural morphism $\mathrm{Gr}^0_{\mathcal{G}} \to \mathrm{Gr}_{\mathcal{G}}$ is representable by a closed immersion; the natural morphism $\mathrm{Gr}^+_{\mathcal{G}} \to \mathrm{Gr}_{\mathcal{G}}$ is bijective (but not a homeomorphism); and its restriction to each connected component of $\mathrm{Gr}^+_{\mathcal{G}}$ is representable by a locally closed immersion. Below we identify precisely this sub-ind-scheme, confirming the expectation in [HR21, Remark 4.8].

The morphisms of group schemes

$$\mathcal{M}_\lambda \to \mathcal{P}_\lambda \to \mathcal{G}$$

induce $\mathbb{G}_{\mathrm{m},\mathbb{F}}$-equivariant morphisms of the corresponding affine Grassmannians

$$\mathrm{Gr}_{\mathcal{M}_\lambda} \to \mathrm{Gr}_{\mathcal{G}} \quad \text{and} \quad \mathrm{Gr}_{\mathcal{P}_\lambda} \to \mathrm{Gr}_{\mathcal{G}}.$$

Since $\mathbb{G}_{\mathrm{m},\mathbb{F}}$ acts trivially on $\mathrm{Gr}_{\mathcal{M}_\lambda}$, the former factors through a morphism

$$\mathrm{Gr}_{\mathcal{M}_\lambda} \to \mathrm{Gr}^0_{\mathcal{G}}. \tag{3.3}$$

Similarly, since the action of $\mathbb{G}_{\mathrm{m},O_F}$ on $\mathcal{P}_\lambda$ extends to an action of the monoid $\mathbb{A}^1_{O_F}$, the action of $\mathbb{G}_{\mathrm{m},\mathbb{F}}$ on $\mathrm{Gr}_{\mathcal{P}_\lambda}$ extends to an action of $\mathbb{A}^1_{\mathbb{F}}$, which implies that the monomorphism $(\mathrm{Gr}_{\mathcal{P}_\lambda})^+ \to \mathrm{Gr}_{\mathcal{P}_\lambda}$ is an isomorphism. We deduce a morphism

$$\mathrm{Gr}_{\mathcal{P}_\lambda} = (\mathrm{Gr}_{\mathcal{P}_\lambda})^+ \to \mathrm{Gr}^+_{\mathcal{G}}. \tag{3.4}$$

Proposition 3.2. *The morphisms* (3.3) *and* (3.4) *are isomorphisms.*

Remark 3.3. In particular, Proposition 3.2 and (3.1) show that $\mathrm{Gr}^+_{\mathcal{G}}$, resp. $\mathrm{Gr}^0_{\mathcal{G}}$, only depends on $\mathcal{G}$ and P_λ, resp. M_λ, and not on the actual choice of λ.

For the proof of this proposition we will need a preliminary result on tangent spaces (see §2.1).

Lemma 3.4. *Let $e \in \mathrm{Gr}_{\mathcal{P}_\lambda}(\mathbb{F})$ be the base point, and denote similarly its image under* (3.4)*. Then* (3.4) *induces an isomorphism*

$$T_e \mathrm{Gr}_{\mathcal{P}_\lambda} \xrightarrow{\sim} T_e \mathrm{Gr}^+_{\mathcal{G}}.$$

Similarly, if we denote by $e' \in \mathrm{Gr}_{\mathcal{M}_\lambda}(\mathbb{F})$ the base point and its image under (3.3)*, then* (3.3) *induces an isomorphism*

$$T_{e'} \mathrm{Gr}_{\mathcal{M}_\lambda} \xrightarrow{\sim} T_{e'} \mathrm{Gr}^0_{\mathcal{G}}.$$

Proof. We give the proof of the first claim, and leave that of the second one to the reader. As the formation of attractors commutes with that of tangent spaces at fixed points (see [Dri13, Proposition 1.4.11(vi)]), we have

$$T_e \mathrm{Gr}^+_{\mathcal{G}} = (T_e \mathrm{Gr}_{\mathcal{G}})^+;$$

moreover the right-hand side identifies with the subspace in $T_e \mathrm{Gr}_{\mathcal{G}}$ spanned by weight vectors of nonnegative weight for the action of $\mathbb{G}_{\mathrm{m},\mathbb{F}}$ induced by the action on $\mathrm{Gr}_{\mathcal{G}}$. To prove the desired claim, we therefore have to show that (3.4) induces an isomorphism

$$(T_e \mathrm{Gr}_{\mathcal{G}})^+ = T_e \mathrm{Gr}_{\mathcal{P}_\lambda}. \tag{3.5}$$

By Lemma 2.4, we have isomorphisms

$$T_e \mathrm{Gr}_{\mathcal{G}} = \mathscr{L}ie(\mathrm{L}G)/\mathscr{L}ie(\mathrm{L}^+\mathcal{G}), \qquad T_e \mathrm{Gr}_{\mathcal{P}_\lambda} = \mathscr{L}ie(\mathrm{L}P_\lambda)/\mathscr{L}ie(\mathrm{L}^+\mathcal{P}_\lambda).$$

The first isomorphism implies that

$$(T_e \mathrm{Gr}_{\mathcal{G}})^+ = \big(\mathscr{L}ie(\mathrm{L}G)\big)^+/\big(\mathscr{L}ie(\mathrm{L}^+\mathcal{G})\big)^+$$

where $(\mathscr{L}ie(\mathrm{L}G))^+$ is the subspace of $\mathscr{L}ie(\mathrm{L}G)$ spanned by weight vectors of nonnegative weights, and similarly for $(\mathscr{L}ie(\mathrm{L}^+\mathcal{G}))^+$. Hence proving (3.5) amounts to proving an identification

$$\big(\mathscr{L}ie(\mathrm{L}G)\big)^+/\big(\mathscr{L}ie(\mathrm{L}^+\mathcal{G})\big)^+ = \mathscr{L}ie(\mathrm{L}P_\lambda)/\mathscr{L}ie(\mathrm{L}^+\mathcal{P}_\lambda). \tag{3.6}$$

By [BT84, §§3.8.1–3.8.2, §4.6.2], the product map $(u^-, t, u^+) \mapsto u^- \cdot t \cdot u^+$ gives an open immersion of O_F-schemes

$$\prod_{\alpha \in \Phi^{\mathrm{nd},-}} \mathcal{U}_\alpha \times \mathcal{T} \times \prod_{\alpha \in \Phi^{\mathrm{nd},+}} \mathcal{U}_\alpha \to \mathcal{G}, \tag{3.7}$$

where $\Phi^{\mathrm{nd}} \subset \Phi$ is the system of non-divisible, relative roots for (G, A), and $\Phi^{\mathrm{nd},+} = \Phi^+ \cap \Phi^{\mathrm{nd}}$, $\Phi^{\mathrm{nd},-} = -\Phi^{\mathrm{nd},+}$. (Here, for any α, $\mathcal{U}_\alpha$ is the "root group" associated with α.) After applying the loop functor L, resp. positive loop functor L^+, the product map is still formally étale (and therefore identifies the tangent spaces), so we obtain isomorphisms

$$\mathscr{L}ie(\mathrm{L}G) = \mathscr{L}ie(\mathrm{L}\mathcal{T}) \oplus \bigoplus_{\alpha \in \Phi^{\mathrm{nd}}} \mathscr{L}ie(\mathrm{L}\mathcal{U}_\alpha), \tag{3.8}$$

$$\mathscr{L}ie(\mathrm{L}^+\mathcal{G}) = \mathscr{L}ie(\mathrm{L}^+\mathcal{T}) \oplus \bigoplus_{\alpha \in \Phi^{\mathrm{nd}}} \mathscr{L}ie(\mathrm{L}^+\mathcal{U}_\alpha). \tag{3.9}$$

Since $\mathbb{G}_{\mathrm{m},\mathbb{F}}$ acts on $\mathscr{L}ie(\mathrm{L}\mathcal{U}_\alpha)$ with weights in $\{\langle \alpha, \lambda\rangle, 2\langle \alpha, \lambda\rangle\}$, passing to attractors yields

$$\mathscr{L}ie(\mathrm{L}G)^+ = \mathscr{L}ie(\mathrm{L}\mathcal{T}) \oplus \bigoplus_{\alpha \in \Phi^{\mathrm{nd}}_\lambda} \mathscr{L}ie(\mathrm{L}\mathcal{U}_\alpha),$$

$$\mathscr{L}ie(\mathrm{L}^+\mathcal{G})^+ = \mathscr{L}ie(\mathrm{L}^+\mathcal{T}) \oplus \bigoplus_{\alpha \in \Phi^{\mathrm{nd}}_\lambda} \mathscr{L}ie(\mathrm{L}^+\mathcal{U}_\alpha),$$

where $\Phi^{\mathrm{nd}}_\lambda = \{\alpha \in \Phi^{\mathrm{nd}} \mid \langle \lambda, \alpha\rangle \geq 0\}$.

On the other hand, passing to attractors in the open immersion (3.7), by [HR21, Corollary 2.3] we obtain an open immersion

$$\prod_{\alpha \in \Phi^{\mathrm{nd},-}_\lambda} \mathcal{U}_\alpha \times \mathcal{T} \times \prod_{\alpha \in \Phi^{\mathrm{nd},+}_\lambda} \mathcal{U}_\alpha \to \mathcal{P}_\lambda,$$

where $\Phi^{\mathrm{nd},\pm}_\lambda = \Phi^{\mathrm{nd},\pm} \cap \Phi^{\mathrm{nd}}_\lambda$. As above we deduce identifications

$$\mathscr{L}ie(\mathrm{L}P_\lambda) = \mathscr{L}ie(\mathrm{L}\mathcal{T}) \oplus \bigoplus_{\alpha \in \Phi^{\mathrm{nd}}_\lambda} \mathscr{L}ie(\mathrm{L}\mathcal{U}_\alpha), \tag{3.10}$$

$$\mathscr{L}ie(\mathrm{L}^+\mathcal{P}_\lambda) = \mathscr{L}ie(\mathrm{L}^+\mathcal{T}) \oplus \bigoplus_{\alpha \in \Phi^{\mathrm{nd}}_\lambda} \mathscr{L}ie(\mathrm{L}^+\mathcal{U}_\alpha). \tag{3.11}$$

Comparing (3.8)–(3.9) with (3.10)–(3.11) we deduce the identification (3.6), which finishes the proof. □

Proof of Proposition 3.2. In this proof we will use the following property of tangent spaces. Let X be an $\mathbb{F}$-scheme of finite type. For any $\mathbb{F}$-algebra R and any $y \in X(R) = (R \otimes_{\mathbb{F}} X)(R)$ we have the associated conormal sheaf ω_y, see [DG70, II, §4, 3.1], which is an R-module and has the property that if $J \subset R$ is an ideal such that $J^2 = 0$ there is a canonical identification between $\mathrm{Hom}_R(\omega_y, J)$ and the set of points $y' \in X(R)$ whose image in $X(R/J)$ is the image of y; see [DG70, II, §4, 3.2]. In particular, if $x \in X(\mathbb{F})$, ω_x identifies with the dual of the finite-dimensional $\mathbb{F}$-vector space T_xX, and if we denote by x_R the image of x in $X(R)$ we have $\omega_{x_R} = R \otimes_{\mathbb{F}} \omega_x$, see [DG70, I, §4, 1.4]. For any $\mathbb{F}$-algebra R and any ideal $J \subset R$ such that $J^2 = 0$, we deduce an identification between $J \otimes_{\mathbb{F}} T_xX$ and the set of points $y \in X(R)$ whose image in $X(R/J)$ is (the image of) x. This property of course extends to ind-schemes of ind-finite type over $\mathbb{F}$ by passing to the colimit of tangent spaces in any presentation.

Below we will consider various local artinian $\mathbb{F}$-algebras. If R is such an algebra, we will denote by $\mathrm{rad}(R)$ its Jacobson radical, i.e. its unique maximal ideal. We will also denote by $\mathrm{n}(R)$ the minimal positive integer n such that $\mathrm{rad}(R)^n = 0$. Note that if the field $R/\mathrm{rad}(R)$ is algebraically closed, then R is a strictly henselian local ring. Using Lemma 2.1 we deduce that in this case we have

$$\mathrm{Gr}_{\mathcal{P}_\lambda}(R) = \mathrm{L}P_\lambda(R)/\mathrm{L}^+\mathcal{P}_\lambda(R). \tag{3.12}$$

By [HR21, Proposition 4.7(i)], our maps are closed immersions. To finish the proof, in view of [HLR18, Lemma 8.6 and Remark 8.7] it therefore suffices to check that for any local artinian $\mathbb{F}$-algebra R such that $R/\mathrm{rad}(R)$ is algebraically closed, every R-valued point x_R of $\mathrm{Gr}^0_{\mathcal{G}}$, resp. $\mathrm{Gr}^+_{\mathcal{G}}$, already lies in $\mathrm{Gr}_{\mathcal{M}_\lambda}(R)$, resp. $\mathrm{Gr}_{\mathcal{P}_\lambda}(R)$. We shall treat the case of the attractor and leave the case of fixed points to the reader.

We proceed by induction on $\mathrm{n}(R)$. If $\mathrm{n}(R) = 1$, then R is an algebraically closed field. In this case, using base change we can assume that $R = \mathbb{F}$; this case is treated in [HR21, Proposition 4.7(iii)].

Now assume that $\mathrm{n}(R) > 1$, and let $J \subset R$ be an ideal such that $J^2 = 0$ and $\mathrm{n}(R/J) < \mathrm{n}(R)$. (For instance, one can take $J = \mathrm{rad}(R)^{\mathrm{n}(R)-1}$.) Consider the R/J-valued point $x_{R/J}$ of $\mathrm{Gr}^+_{\mathcal{G}}$ associated with x_R. By induction, the point $x_{R/J}$ lies in $\mathrm{Gr}_{\mathcal{P}_\lambda}(R/J)$. By (3.12), we can lift $x_{R/J}$ to an R/J-valued point $y_{R/J}$ of $\mathrm{L}P_\lambda$, and then further to an R-valued point y_R of $\mathrm{L}P_\lambda$ by formal smoothness. In other words, replacing x_R by $y_R^{-1}x_R$ we may assume that $x_{R/J}$ is the base point e of $\mathrm{Gr}^+_{\mathcal{G}}$. By the reminder at the beginning of the proof, x_R corresponds to an element in $T_e(\mathrm{Gr}^+_{\mathcal{G}}) \otimes_{\mathbb{F}} J$. By Lemma 3.4 and the same considerations, there exists an R-point of $\mathrm{Gr}_{\mathcal{P}_\lambda}$ whose image in $\mathrm{Gr}_{\mathcal{P}_\lambda}(R/J)$ is e and whose image under (3.4) is x_R, which finishes the proof. □

We conclude this subsection with a remark on the convolution schemes $\mathrm{Conv}_{\mathcal{G}}$, $\mathrm{Conv}_{\mathcal{M}_\lambda}$ and $\mathrm{Conv}_{\mathcal{P}_\lambda}$ (see §2.5). Since $\mathrm{L}^+\mathcal{G}$ acts on $\mathrm{Conv}_{\mathcal{G}}$ on the left, the homomorphism (3.2) yields a $\mathbb{G}_{\mathrm{m},\mathbb{F}}$-action on $\mathrm{Conv}_{\mathcal{G}}$, and one can consider the attractor and fixed-point ind-schemes for this action. The same reasoning that led to (3.3) and (3.4) lets us construct maps

$$\mathrm{Conv}_{\mathcal{M}_\lambda} \to \mathrm{Conv}^0_{\mathcal{G}}, \tag{3.13}$$

$$\mathrm{Conv}_{\mathcal{P}_\lambda} = (\mathrm{Conv}_{\mathcal{P}_\lambda})^+ \to \mathrm{Conv}^+_{\mathcal{G}}. \tag{3.14}$$

In view of the isomorphism (2.11) (which is $\mathrm{L}^+\mathcal{G}$-equivariant, and hence $\mathbb{G}_{\mathrm{m},\mathbb{F}}$-equivariant), we obtain the following immediate consequence of Proposition 3.2.

Proposition 3.5. *The morphisms* (3.13) *and* (3.14) *are isomorphisms.*

3.3. **Parabolic subgroups and cartesian diagrams.** Let us now consider two parabolic subgroups $P, P' \subset G$ containing A and such that $P \subset P'$. If we denote by M and M' their Levi factors containing A, then we have $M \subset M'$. Moreover, $P \cap M'$ is a parabolic subgroup of the reductive group M' containing A, and M is its Levi factor containing A. Let $\mathcal{P}$, $\mathcal{P}'$, $\mathcal{M}$, and $\mathcal{M}'$ be their scheme-theoretic closures inside $\mathcal{G}$ (cf. (3.1)).

Point (3) of the following lemma involves fiber products of ind-schemes. For this notion, see [Ric20, Lemma 1.10].

Lemma 3.6. (1) *The intersection $\mathcal{P} \cap \mathcal{M}'$ is the scheme-theoretic closure of the group scheme $P \cap M'$ inside $\mathcal{M}'$, and is smooth over O_F.*

(2) *The following commutative square of group schemes is cartesian:*

$$\begin{tikzcd} \mathcal{P} \arrow[r] \arrow[d] & \mathcal{P} \cap \mathcal{M}' \arrow[d] \\ \mathcal{P}' \arrow[r] & \mathcal{M}'. \end{tikzcd}$$

(3) *The following commutative squares of ind-schemes are cartesian:*

$$\begin{tikzcd} \mathrm{Gr}_{\mathcal{P}} \arrow[r] \arrow[d] & \mathrm{Gr}_{\mathcal{P}\cap\mathcal{M}'} \arrow[d] \\ \mathrm{Gr}_{\mathcal{P}'} \arrow[r] & \mathrm{Gr}_{\mathcal{M}'}, \end{tikzcd} \qquad \begin{tikzcd} \mathrm{Conv}_{\mathcal{P}} \arrow[r] \arrow[d] & \mathrm{Conv}_{\mathcal{P}\cap\mathcal{M}'} \arrow[d] \\ \mathrm{Conv}_{\mathcal{P}'} \arrow[r] & \mathrm{Conv}_{\mathcal{M}'}. \end{tikzcd}$$

Proof. (1) Let $\lambda \in \mathbb{X}_*(A)$ be a cocharacter such that $P = P_\lambda$ (see Remark 3.1). Then from the definitions one sees that $\mathcal{P} \cap \mathcal{M}'$ is the attractor associated with the $\mathbb{G}_{\mathrm{m},O_F}$-action on $\mathcal{M}'$ defined by λ, so our claim is a special case of (3.1).

(2) The morphism $\mathcal{P} \to \mathcal{P}'$ is a closed immersion, and hence so is the induced morphism $\mathcal{P} \to \mathcal{P}' \times_{\mathcal{M}'} (\mathcal{P} \cap \mathcal{M}')$; in other words, the associated morphism

$$\mathscr{O}(\mathcal{P}' \times_{\mathcal{M}'} (\mathcal{P} \cap \mathcal{M}')) \to \mathscr{O}(\mathcal{P})$$

is surjective. On the other hand this morphism becomes an isomorphism after tensor product with F, and the fiber product $\mathcal{P}' \times_{\mathcal{M}'} (\mathcal{P} \cap \mathcal{M}')$ is flat over $\mathrm{Spec}(O_F)$. (Indeed the projection $\mathcal{P}' \to \mathcal{M}'$ is smooth, so this fiber product is smooth over $\mathcal{P} \cap \mathcal{M}'$, which is itself smooth over $\mathrm{Spec}(O_F)$ by (1).) This morphism is therefore an isomorphism, which finishes the proof.

(3) Let us first consider the left-hand diagram. If R is an $\mathbb{F}$-algebra, $\mathrm{Gr}_{\mathcal{P}}(R)$ classifies pairs consisting of an $\mathcal{P}$-torsor on $\mathrm{Spec}(R[\![t]\!])$ and a trivialization on $\mathrm{Spec}(R(\!(t)\!))$. On the other hand, by (1),

$$(\mathrm{Gr}_{\mathcal{P}'} \times_{\mathrm{Gr}_{\mathcal{M}'}} \mathrm{Gr}_{\mathcal{P}\cap\mathcal{M}'})(R) = \mathrm{Gr}_{\mathcal{P}'}(R) \times_{\mathrm{Gr}_{\mathcal{M}'}(R)} \mathrm{Gr}_{\mathcal{P}\cap\mathcal{M}'}(R)$$

parametrizes an $\mathcal{P}'$-torsor on $\mathrm{Spec}(R[\![t]\!])$ with a trivialization on $\mathrm{Spec}(R(\!(t)\!))$, an $\mathcal{P} \cap \mathcal{M}'$-torsor on $\mathrm{Spec}(R[\![t]\!])$ with a trivialization on $\mathrm{Spec}(R(\!(t)\!))$, and an isomorphism between the induced $\mathcal{M}'$-bundles and their trivializations. Now from (2) we deduce that the datum of an $\mathcal{P}$-torsor is equivalent to that of an $\mathcal{P}'$-torsor, an $\mathcal{P} \cap \mathcal{M}'$-torsor, and an isomorphism between the induced $\mathcal{M}'$-bundles. The desired claim follows.

The fact that the right-hand diagram is cartesian is immediate from the similar property of the left-hand diagram and (2.11). □

3.4. **Semi-infinite orbits.** Let us now study further the case where the cocharacter in $\mathbb{X}_*(A)$ is such that the attractor and fixed point sets in G are given by

$$G^+ = B \qquad \text{and} \qquad G^0 = T.$$

(This is indeed possible thanks to Remark 3.1.) Let $\mathcal{B}$ be the scheme-theoretic closure of B in $\mathcal{G}$. Then, by (3.1), $\mathcal{B}$ is smooth and we have

$$\mathcal{G}^+ = \mathcal{B} \qquad \text{and} \qquad \mathcal{G}^0 = \mathcal{T}.$$

It is well known (see e.g. [PR08, §3.b]) that the underlying topological space of $\mathrm{Gr}_{\mathcal{T}}$ is discrete, with

$$|\mathrm{Gr}_{\mathcal{T}}| = \mathbb{X}_*(T)_I \tag{3.15}$$

(see (2.3)). Since the morphism $\mathrm{Gr}^+_{\mathcal{G}} \to \mathrm{Gr}^0_{\mathcal{G}}$ induces a bijection on the sets of connected components (see [Ric19, Proposition 1.17] and [HR21, Theorem 2.1]), we deduce a bijection between the set of connected components of $\mathrm{Gr}^+_{\mathcal{G}}$, i.e. of $\mathrm{Gr}_{\mathcal{B}}$, and $\mathbb{X}_*(T)_I$.

For $\lambda \in \mathbb{X}_*(T)_I$, we will denote by S_λ the associated connected component of $\mathrm{Gr}^+_{\mathcal{G}}$. Then the natural map $\mathrm{S}_\lambda \to \mathrm{Gr}_{\mathcal{G}}$ is representable by a locally closed immersion, and the natural map

$$\bigsqcup_{\lambda \in \mathbb{X}_*(T)_I} \mathrm{S}_\lambda \to \mathrm{Gr}_{\mathcal{G}}$$

is bijective. If we denote by $\mathrm{S}_{\leq\lambda}$ the ind-schematic closure of S_λ inside $\mathrm{Gr}_{\mathcal{G}}$, then we have

$$|\mathrm{S}_{\leq\lambda}| = \bigsqcup_{\mu\leq\lambda} |\mathrm{S}_\mu|. \tag{3.16}$$

(This property is proved in [AGLR22, Proposition 5.4] for Witt vector affine Grassmannians; the same proof goes through in our present setting.) Choosing a presentation $\mathrm{Gr}_{\mathcal{G}} = \mathrm{colim}_i X_i$ by $\mathrm{L}^+\mathcal{G}$-stable projective k-schemes (see §2.6), for any i we have

$$|\mathrm{Gr}^+_{\mathcal{G}} \times_{\mathrm{Gr}_{\mathcal{G}}} X_i| = \bigsqcup_{\lambda \in \mathbb{X}_*(T)^+_I} |\mathrm{S}_\lambda \times_{\mathrm{Gr}_{\mathcal{G}}} X_i|, \tag{3.17}$$

and $\mathrm{Gr}_{\mathcal{G}}^{+} \times_{\mathrm{Gr}_{\mathcal{G}}} X_i$ identifies canonically with the attractor for the induced $\mathbb{G}_{\mathrm{m},\mathbb{F}}$-action on X_i. The intersection $\mathrm{S}_\lambda \times_{\mathrm{Gr}_{\mathcal{G}}} X_i$ is a locally closed subscheme of X_i (in particular, a scheme of finite type by [GW20, Example 3.45]) and it is empty unless $t^\lambda \in X_i$, which happens only for a finite number of λ's.

If $\mu \in \mathbb{X}_*(T)_I$ and $\lambda \in \mathbb{X}_*(T)_I^+$ then we can consider the intersection

$$\mathrm{S}_\mu \cap \mathrm{Gr}_{\mathcal{G}}^{\leq \lambda},$$

a locally closed subscheme of the projective scheme $\mathrm{Gr}_{\mathcal{G}}^{\leq\lambda}$, which is in particular an $\mathbb{F}$-scheme of finite type. For the statement of the next lemma, recall that $\mathbb{X}_*(T)_I^+$ is a system of representatives for $\mathbb{X}_*(T)_I/W_0$, see Lemma 2.6(1).

Lemma 3.7. *If $\mu \in \mathbb{X}_*(T)_I$ and $\lambda \in \mathbb{X}_*(T)_I^+$, the scheme*

$$\mathrm{S}_\mu \cap \mathrm{Gr}_{\mathcal{G}}^{\leq \lambda}$$

is nonempty if and only if the only element $\mu' \in (W_0 \cdot \mu) \cap \mathbb{X}_*(T)_I^+$ satisfies $\mu' \leq \lambda$. In this case, this scheme is affine and equidimensional of dimension $\langle \mu + \lambda, \rho\rangle$.

Here we write $\langle \mu + \lambda, \rho\rangle$ for $\frac{1}{2}\langle \mu + \lambda, 2\rho\rangle$ (which is an integer in the case considered in the statement). Again, this result is proved in [AGLR22, Lemma 5.5] for ramified groups over p-adic fields, but the arguments also apply in our current setting.

Of course one can play the same game with the Borel subgroup B^- opposite to B (with respect to T), see Remark 3.1(2). The connected components of the associated affine Grassmannian will be denoted $(\mathrm{T}_\lambda : \lambda \in \mathbb{X}_*(T)_I)$.

3.5. **Semi-infinite orbits on convolution schemes.** We finish this section by explaining how to adapt the discussion of §3.4 to the setting of convolution schemes. In view of (2.11), the ind-scheme $\mathrm{Conv}_{\mathcal{T}}$ is discrete, and the connected components of both $\mathrm{Conv}_{\mathcal{T}}$ and $\mathrm{Conv}_{\mathcal{B}}$ are in bijection with $\mathbb{X}_*(T)_I \times \mathbb{X}_*(T)_I$. However, we will use a labeling of these components that is *not* compatible with (2.11): given $\lambda, \mu \in \mathbb{X}_*(T)_I$, we let

$$\mathrm{S}_\lambda \mathbin{\tilde{\times}} \mathrm{S}_\mu \subset \mathrm{Conv}_{\mathcal{B}}$$

be the connected component that identifies under (2.11) with $\mathrm{S}_\lambda \times \mathrm{S}_{\mu+\lambda} \subset \mathrm{Gr}_{\mathcal{B}} \times \mathrm{Gr}_{\mathcal{B}}$. This labeling has the advantage that it makes the statement of the following lemma (which follows from the same considerations as for Lemma 3.7) cleaner.

Lemma 3.8. *If $\mu, \mu' \in \mathbb{X}_*(T)_I$ and $\lambda, \lambda' \in \mathbb{X}_*(T)_I^+$, the scheme*

$$(\mathrm{S}_\mu \mathbin{\tilde{\times}} \mathrm{S}_{\mu'}) \cap \mathrm{Conv}_{\mathcal{G}}^{\leq(\lambda,\lambda')}$$

is nonempty if and only if the only elements $\nu \in (W_0 \cdot \mu) \cap \mathbb{X}_*(T)_I^+$ and $\nu' \in (W_0 \cdot \mu') \cap \mathbb{X}_*(T)_I^+$ satisfy $\nu \leq \lambda$ and $\nu' \leq \lambda'$. In this case, this scheme is affine and equidimensional of dimension $\langle \mu + \mu' + \lambda + \lambda', \rho\rangle$.

4. Sheaves on the Hecke stack

4.1. **Constructible sheaves on the affine Grassmannian and the Hecke stack.** From now on we fix a prime number ℓ invertible in $\mathbb{F}$, and a finite extension $\mathbb{K}$ of $\mathbb{Q}_\ell$. The ring of integers of $\mathbb{K}$ will be denoted $\mathbb{O}$, and the residue field of $\mathbb{O}$ will be denoted $\Bbbk$. We will use the notation Λ to denote one of the rings $\mathbb{K}$, $\mathbb{O}$, $\Bbbk$, when the choice does not matter. Below we will use the bounded constructible categories of étale Λ-sheaves on Artin stacks of finite type; see §B.1 for a brief review of the relevant ingredients, and for references on the construction.

Recall the notation of §2.6. For any $i \geq 0$ we have the algebraic stack of finite type $[\mathrm{L}^+_{n_i}\mathcal{G}\backslash X_i]_{\text{ét}}$ over $\mathbb{F}$, and we can consider the bounded constructible derived category

$$D^{\mathrm{b}}_{\mathrm{c}}([\mathrm{L}^+_{n_i}\mathcal{G}\backslash X_i]_{\text{ét}}, \Lambda).$$

For any $n \geq n_i$, as in [FS21, Proposition VI.1.10], the quotient of the surjective morphism $\mathrm{L}^+_n\mathcal{G} \to \mathrm{L}^+_{n_i}\mathcal{G}$ is an extension of copies of the additive group $\mathbb{G}_{\mathrm{a},\mathbb{F}}$, so that by standard arguments the pullback functor

$$D^{\mathrm{b}}_{\mathrm{c}}([\mathrm{L}^+_{n_i}\mathcal{G}\backslash X_i]_{\text{ét}}, \Lambda) \to D^{\mathrm{b}}_{\mathrm{c}}([\mathrm{L}^+_{n}\mathcal{G}\backslash X_i]_{\text{ét}}, \Lambda) \tag{4.1}$$

is an equivalence of categories. We can therefore set

$$D^{\mathrm{b}}_{\mathrm{c}}([\mathrm{L}^+\mathcal{G}\backslash X_i]_{\text{ét}}, \Lambda) := \lim_{n \geq n_i} D^{\mathrm{b}}_{\mathrm{c}}([\mathrm{L}^+_{n_i}\mathcal{G}\backslash X_i]_{\text{ét}}, \Lambda),$$

this limit being stationary. If $i \leq j$ we have an obvious pushforward functor $D^{\mathrm{b}}_{\mathrm{c}}([\mathrm{L}^+\mathcal{G}\backslash X_i]_{\text{ét}}, \Lambda) \to D^{\mathrm{b}}_{\mathrm{c}}([\mathrm{L}^+\mathcal{G}\backslash X_j]_{\text{ét}}, \Lambda)$, and we can therefore set

$$D^{\mathrm{b}}_{\mathrm{c}}(\mathrm{Hk}_{\mathcal{G}}, \Lambda) := \mathrm{colim}_i\, D^{\mathrm{b}}_{\mathrm{c}}([\mathrm{L}^+\mathcal{G}\backslash X_i]_{\text{ét}}, \Lambda).$$

Standard arguments show that this category does not depend (up to equivalence) on the choice of presentation $\mathrm{Gr}_{\mathcal{G}} = \mathrm{colim}_{i \geq 0} X_i$ as in §2.6.

Of course, in this construction one can forget the $\mathrm{L}^+\mathcal{G}$-action, and consider the category

$$D^{\mathrm{b}}_{\mathrm{c}}(\mathrm{Gr}_{\mathcal{G}}, \Lambda) := \mathrm{colim}_i\, D^{\mathrm{b}}_{\mathrm{c}}(X_i, \Lambda).$$

The quotient morphism $h \colon \mathrm{Gr}_{\mathcal{G}} \to \mathrm{Hk}_{\mathcal{G}}$ induces a pullback (or "forgetful") functor

$$h^* \colon D^{\mathrm{b}}_{\mathrm{c}}(\mathrm{Hk}_{\mathcal{G}}, \Lambda) \to D^{\mathrm{b}}_{\mathrm{c}}(\mathrm{Gr}_{\mathcal{G}}, \Lambda). \tag{4.2}$$

For any $i \geq 0$ one can consider the perverse t-structure on $D^{\mathrm{b}}_{\mathrm{c}}(X_i, \Lambda)$. These t-structures "glue" to define a (bounded) t-structure on $D^{\mathrm{b}}_{\mathrm{c}}(\mathrm{Gr}_{\mathcal{G}}, \Lambda)$, which once again does not depend on the choice of presentation $\mathrm{Gr}_{\mathcal{G}} = \mathrm{colim}_{i \geq 0} X_i$, and which will be called the perverse t-structure. Its heart will be denoted $\mathsf{Perv}(\mathrm{Gr}_{\mathcal{G}}, \Lambda)$.

If $i \geq 0$ and $n \geq n_i$, one can also consider the perverse t-structure on the category $D^{\mathrm{b}}_{\mathrm{c}}([\mathrm{L}^+_n\mathcal{G}\backslash X_i]_{\text{ét}}, \Lambda)$. This t-structure *does* depend on the choice of n; to remedy this we introduce a shift in this definition, so that the pullback functor

$$D^{\mathrm{b}}_{\mathrm{c}}([\mathrm{L}^+_n\mathcal{G}\backslash X_i]_{\text{ét}}, \Lambda) \to D^{\mathrm{b}}_{\mathrm{c}}(X_i, \Lambda)$$

becomes t-exact. With this normalization, for any $n \geq n_i$ the equivalence (4.1) is t-exact, so that we obtain an induced t-structure on $D^{\mathrm{b}}_{\mathrm{c}}([\mathrm{L}^+\mathcal{G}\backslash X_i]_{\text{ét}}, \Lambda)$. If $j \geq i$ the pushforward functor $D^{\mathrm{b}}_{\mathrm{c}}([\mathrm{L}^+\mathcal{G}\backslash X_i]_{\text{ét}}, \Lambda) \to D^{\mathrm{b}}_{\mathrm{c}}([\mathrm{L}^+\mathcal{G}\backslash X_j]_{\text{ét}}, \Lambda)$ is t-exact, so that once again these t-structures "glue" to define a (bounded) t-structure on $D^{\mathrm{b}}_{\mathrm{c}}(\mathrm{Hk}_{\mathcal{G}}, \Lambda)$, which is called the perverse t-structure, and whose heart will be denoted $\mathsf{Perv}(\mathrm{Hk}_{\mathcal{G}}, \Lambda)$. By construction of the perverse t-structure for stacks and our choice of normalization the functor (4.2) is t-exact; in fact it "detects perversity" in the sense that for $\mathscr{F} \in D^{\mathrm{b}}_{\mathrm{c}}(\mathrm{Hk}_{\mathcal{G}}, \Lambda)$ the complex $\mathscr{F}$ is concentrated in nonpositive perverse degrees, resp. concentrated in nonnegative perverse degrees, resp. perverse, if and only if so is $h^*(\mathscr{F})$.

The constructions in §B.1.4 provide canonical "extension of scalars" functors

$$\Bbbk \overset{L}{\otimes}_{\mathbb{O}} (-) \colon D^{\mathrm{b}}_{\mathrm{c}}(\mathrm{Hk}_{\mathcal{G}}, \mathbb{O}) \to D^{\mathrm{b}}_{\mathrm{c}}(\mathrm{Hk}_{\mathcal{G}}, \Bbbk),$$
$$\mathbb{K} \otimes_{\mathbb{O}} (-) \colon D^{\mathrm{b}}_{\mathrm{c}}(\mathrm{Hk}_{\mathcal{G}}, \mathbb{O}) \to D^{\mathrm{b}}_{\mathrm{c}}(\mathrm{Hk}_{\mathcal{G}}, \mathbb{K}), \tag{4.3}$$

and a canonical "restriction of scalars" functor

$$D^{\mathrm{b}}_{\mathrm{c}}(\mathrm{Hk}_{\mathcal{G}}, \Bbbk) \to D^{\mathrm{b}}_{\mathrm{c}}(\mathrm{Hk}_{\mathcal{G}}, \mathbb{O}) \tag{4.4}$$

which is right adjoint to $\Bbbk \otimes^L_{\mathbb{O}} (-)$. (This functor will usually be omitted from notation.) The right-hand functor in (4.3) and the functor in (4.4) are t-exact for the perverse t-structures, hence induce exact functors

$$\mathbb{K} \otimes_{\mathbb{O}} (-) \colon \mathsf{Perv}(\mathrm{Hk}_{\mathcal{G}}, \mathbb{O}) \to \mathsf{Perv}(\mathrm{Hk}_{\mathcal{G}}, \mathbb{K})$$

and

$$\mathsf{Perv}(\mathrm{Hk}_{\mathcal{G}}, \Bbbk) \to \mathsf{Perv}(\mathrm{Hk}_{\mathcal{G}}, \mathbb{O}).$$

On the other hand the left-hand functor in (4.3) is right t-exact; we therefore obtain a right exact functor

$${}^{\mathrm{p}}\mathscr{H}^0(\Bbbk \overset{L}{\otimes}_{\mathbb{O}} (-)) \colon \mathsf{Perv}(\mathrm{Hk}_{\mathcal{G}}, \mathbb{O}) \to \mathsf{Perv}(\mathrm{Hk}_{\mathcal{G}}, \Bbbk).$$

Similar comments apply for categories of sheaves on $\mathrm{Gr}_{\mathcal{G}}$, or for locally closed subschemes, or for the various stacks we have already considered, and we will use similar notation in these cases.

For any $\lambda \in \mathbb{X}_*(T)^+_I$, we can consider the perverse sheaves

$$\mathscr{J}_!(\lambda, \Lambda) := {}^{\mathrm{p}}\mathscr{H}^0(j^\lambda_! \underline{\Lambda}_{\mathrm{Gr}^\lambda_{\mathcal{G}}}[\langle \lambda, 2\rho \rangle]), \quad \mathscr{J}_*(\lambda, \Lambda) := {}^{\mathrm{p}}\mathscr{H}^0(j^\lambda_* \underline{\Lambda}_{\mathrm{Gr}^\lambda_{\mathcal{G}}}[\langle \lambda, 2\rho \rangle])$$

in $\mathsf{Perv}(\mathrm{Hk}_{\mathcal{G}}, \Lambda)$, where we use the notation of (2.8). By adjunction there exists a canonical morphism

$$\mathscr{J}_!(\lambda, \Lambda) \to \mathscr{J}_*(\lambda, \Lambda)$$

whose image is denoted $\mathscr{J}_{!*}(\lambda, \Lambda)$. In case $\Lambda \in \{\Bbbk, \mathbb{K}\}$ each $\mathscr{J}_{!*}(\lambda, \Lambda)$ is simple, and the assignment $\lambda \mapsto \mathscr{J}_{!*}(\lambda, \Lambda)$ induces a bijection between $\mathbb{X}_*(T)^+_I$ and the set of isomorphism classes of simple objects in $\mathsf{Perv}(\mathrm{Hk}_{\mathcal{G}}, \Lambda)$; see e.g. [LO09, §8].

4.2. **Convolution.** Recall now the ind-scheme $\mathrm{Conv}_{\mathcal{G}}$ defined in §2.5, and the stack $\mathrm{HkConv}_{\mathcal{G}}$ introduced in §2.6. Considerations similar to those of §4.1 allow to define the triangulated categories

$$D^{\mathrm{b}}_{\mathrm{c}}(\mathrm{Conv}_{\mathcal{G}}, \Lambda) \quad \text{and} \quad D^{\mathrm{b}}_{\mathrm{c}}(\mathrm{HkConv}_{\mathcal{G}}, \Lambda),$$

and the pullback functor

$$\tilde{h}^* \colon D^{\mathrm{b}}_{\mathrm{c}}(\mathrm{HkConv}_{\mathcal{G}}, \Lambda) \to D^{\mathrm{b}}_{\mathrm{c}}(\mathrm{Conv}_{\mathcal{G}}, \Lambda). \tag{4.5}$$

One can also endow these categories with perverse t-structures, in such a way that the functor (4.5) is t-exact.

Recall the maps

$$\mathrm{Hk}_{\mathcal{G}} \times \mathrm{Hk}_{\mathcal{G}} \xleftarrow{p} \mathrm{HkConv}_{\mathcal{G}} \xrightarrow{m} \mathrm{Hk}_{\mathcal{G}}$$

from §2.6. We can now define the convolution bifunctor

$$\star \colon D^{\mathrm{b}}_{\mathrm{c}}(\mathrm{Hk}_{\mathcal{G}}, \Lambda) \times D^{\mathrm{b}}_{\mathrm{c}}(\mathrm{Hk}_{\mathcal{G}}, \Lambda) \to D^{\mathrm{b}}_{\mathrm{c}}(\mathrm{Hk}_{\mathcal{G}}, \Lambda)$$

by setting

$$\mathscr{F} \star \mathscr{G} := m_* p^*(\mathscr{F} \overset{L}{\boxtimes}_{\Lambda} \mathscr{G}).$$

Our goal in this subsection is to show that this bifunctor is right t-exact (on both sides) for the perverse t-structure.

The morphism m fits into a cartesian square

$$\begin{tikzcd} \mathrm{Conv}_{\mathcal{G}} \arrow[r, "\tilde{h}"] \arrow[d, "m"'] & \mathrm{HkConv}_{\mathcal{G}} \arrow[d, "m"] \\ \mathrm{Gr}_{\mathcal{G}} \arrow[r, "h"] & \mathrm{Hk}_{\mathcal{G}}. \end{tikzcd}$$

For any $\lambda, \mu, \nu \in \mathbb{X}_*(T)^+_I$ such that $\lambda + \mu \leq \nu$, the left-hand morphism m restricts to a morphism

$$m^\nu_{\lambda,\mu} \colon \mathrm{Conv}^{\leq(\lambda,\mu)}_{\mathcal{G}} \to \mathrm{Gr}^{\leq\nu}_{\mathcal{G}}.$$

Recall also the notions of a stratified locally trivial morphism and a stratified semismall morphism from §B.2.

Proposition 4.1. *For any $\lambda, \mu, \nu \in \mathbb{X}_*(T)^+_I$ such that $\lambda+\mu \le \nu$, the morphism $m^\nu_{\lambda,\mu}$ is stratified locally trivial and stratified semismall with respect to the stratifications*

$$|\mathrm{Conv}_{\mathcal{G}}^{\le(\lambda,\mu)}| = \bigsqcup_{\substack{\lambda' \le \lambda \\ \mu' \le \mu}} |\mathrm{Conv}_{\mathcal{G}}^{(\lambda',\mu')}|, \qquad |\mathrm{Gr}_{\mathcal{G}}^{\le \nu}| = \bigsqcup_{\nu' \le \nu} |\mathrm{Gr}_{\mathcal{G}}^{\nu'}|,$$

see (2.9) *and* (2.13)*.*

Proof. Our morphism is proper, and stratified locally trivial by $\mathrm{L}^+\mathcal{G}$-equivariance. It remains to check the condition on dimensions of fibers, which we will obtain from Lemma B.1. In the present setting, the first assumption of that lemma follows from $\mathrm{L}^+\mathcal{G}$-equivariance. For the second one we observe that, by definition of convolution, for any $\lambda', \mu' \in \mathbb{X}_*(T)^+_I$ such that $\lambda' \le \lambda$ and $\mu' \le \mu$ we have

$$(m^\nu_{\lambda,\mu})_! \mathscr{IC}(\mathrm{Conv}_{\mathcal{G}}^{(\lambda',\mu')}, \underline{\mathbb{K}}) \cong \mathscr{J}_{!*}(\lambda', \mathbb{K}) \star \mathscr{J}_{!*}(\mu', \mathbb{K}).$$

Hence the fact that this complex is perverse follows from [Ric16, Theorem 5.11(i)] (see also [Zhu15, Corollary 2.8] in the tamely ramified case). □

Remark 4.2.
(1) The proofs of [Ric16, Theorem 5.11(i)] and [Zhu15, Corollary 2.8] are incorrect as written, because of the problem mentioned in Remark 2.7. However they can easily be fixed by adding a step of reduction to adjoint groups.
(2) The proof of Proposition 4.1 given here is suggested (with some precautions) in [Zhu15, Remark 2.9(i)] (see also Remark B.2 for comments on this method). A different proof of Proposition 4.1 can be obtained by copying the arguments of [MV07, Lemma 4.4] (see also [BR18, §1.6.3]) and using the results of Section 3.

We can finally reach the goal of this subsection.

Corollary 4.3. *The bifunctor $\star$ is right t-exact, in the sense that if $\mathscr{F}$ and $\mathscr{G}$ belong to ${}^{\mathrm{p}}D^{\le 0}(\mathrm{Hk}_{\mathcal{G}}, \Lambda)$, then so does $\mathscr{F} \star \mathscr{G}$.*

Proof. The bifunctor $(\mathscr{F}, \mathscr{G}) \mapsto \mathscr{F} \boxtimes^L_\Lambda \mathscr{G}$ is clearly right t-exact. On the other hand, by standard arguments involving t-exactness of shifted pullback along a smooth morphism the functor p^* is t-exact, and the functor m_* is t-exact by Proposition 4.1 (and the fact that h^* "detects perversity," see §4.1). The desired claim follows. □

It is a standard fact that the bifunctor $\star$ defines a monoidal structure on the category $D^{\mathrm{b}}_{\mathrm{c}}(\mathrm{Hk}_{\mathcal{G}}, \Lambda)$. As a consequence of Corollary 4.3, we obtain that the bifunctor

$$\star^0 \colon \mathsf{Perv}(\mathrm{Hk}_{\mathcal{G}}, \Lambda) \times \mathsf{Perv}(\mathrm{Hk}_{\mathcal{G}}, \Lambda) \to \mathsf{Perv}(\mathrm{Hk}_{\mathcal{G}}, \Lambda)$$

defined by

$$\mathscr{F} \star^0 \mathscr{G} = {}^{\mathrm{p}}\mathscr{H}^0(\mathscr{F} \star \mathscr{G})$$

defines a monoidal structure on $\mathsf{Perv}(\mathrm{Hk}_{\mathcal{G}}, \Lambda)$. It is clear that the "change of scalars" functors considered in §4.1 on the categories $D^{\mathrm{b}}_{\mathrm{c}}(\mathrm{Hk}_{\mathcal{G}}, \Lambda)$ commute with the bifunctor $\star$ in the obvious way; it follows that their counterparts on the categories $\mathsf{Perv}(\mathrm{Hk}_{\mathcal{G}}, \Lambda)$ admit canonical monoidal structures.

4.3. **Constant term functors.** Consider a parabolic subgroup $P \subset G$ containing A, and let $M \subset P$ be its Levi factor containing A. Let $\mathcal{P}$ and $\mathcal{M}$ be their respective scheme-theoretic closures in $\mathcal{G}$, see §3.1. In §3.2 we have considered a diagram

$$\mathrm{Gr}_{\mathcal{G}} \xleftarrow{i_{\mathcal{P}}} \mathrm{Gr}_{\mathcal{P}} \xrightarrow{q_{\mathcal{P}}} \mathrm{Gr}_{\mathcal{M}}. \tag{4.6}$$

These maps are equivariant for the action of $\mathrm{L}^+\mathcal{M}$ on the left, so we obtain an analogous diagram of quotient stacks. (The morphisms will be denoted by the same symbols.) Appending the obvious quotient map $[\mathrm{L}^+\mathcal{M}\backslash\mathrm{Gr}_{\mathcal{G}}]_{\text{ét}} \to [\mathrm{L}^+\mathcal{G}\backslash\mathrm{Gr}_{\mathcal{G}}]_{\text{ét}}$ on the left, we obtain the following diagram:

$$\mathrm{Hk}_{\mathcal{G}} \xleftarrow{h_{\mathcal{M},\mathcal{G}}} [\mathrm{L}^+\mathcal{M}\backslash\mathrm{Gr}_{\mathcal{G}}]_{\text{ét}} \xleftarrow{i_{\mathcal{P}}} [\mathrm{L}^+\mathcal{M}\backslash\mathrm{Gr}_{\mathcal{P}}]_{\text{ét}} \xrightarrow{q_{\mathcal{P}}} \mathrm{Hk}_{\mathcal{M}}. \tag{4.7}$$

We will use this diagram to define the *constant term functor*

$$\mathsf{CT}_{\mathcal{P},\mathcal{G}} \colon D^{\mathrm{b}}_{\mathrm{c}}(\mathrm{Hk}_{\mathcal{G}}, \Lambda) \to D^{\mathrm{b}}_{\mathrm{c}}(\mathrm{Hk}_{\mathcal{M}}, \Lambda)$$

as follows.

Recall from (2.5) that the set $\pi_0(\mathrm{Gr}_{\mathcal{M}})$ of connected components of $\mathrm{Gr}_{\mathcal{M}}$ is in canonical bijection with $(\mathbb{X}_*(T)/Q^\vee_M)_I$, where $Q^\vee_M \subset \mathbb{X}_*(T)$ is the coroot lattice of (M_{F^s}, T_{F^s}). Recall also that we denote by ρ the half-sum of the positive roots of (G_{F^s}, T_{F^s}) with respect to B_{F^s}. We will similarly denote by ρ_M the half-sum of the positive roots of (M_{F^s}, T_{F^s}) with respect to $(M \cap B)_{F^s}$; as for ρ, this element is fixed by the action of I. The $\mathbb{Z}$-linear map $\langle -, 2\rho - 2\rho_M\rangle \colon \mathbb{X}_*(T) \to \mathbb{Z}$ vanishes on $Q^\vee_M$ and is I-invariant, so it induces a $\mathbb{Z}$-linear map

$$\langle -, 2\rho - 2\rho_M\rangle \colon (\mathbb{X}_*(T)/Q^\vee_M)_I \to \mathbb{Z}.$$

If $X \subset \mathrm{Gr}_{\mathcal{M}}$ is the connected component corresponding to $\lambda \in (\mathbb{X}_*(T)/Q^\vee_M)_I$, we set $\mathrm{corr}_{M,G}(X) = \langle \lambda, 2\rho - 2\rho_M\rangle$. This defines a function

$$\mathrm{corr}_{M,G} \colon \pi_0(\mathrm{Gr}_{\mathcal{M}}) \to \mathbb{Z}.$$

For $\mathscr{F}$ in $D^{\mathrm{b}}_{\mathrm{c}}(\mathrm{Gr}_{\mathcal{M}}, \Lambda)$ or $D^{\mathrm{b}}_{\mathrm{c}}(\mathrm{Hk}_{\mathcal{M}}, \Lambda)$, one can make sense of the expression

$$\mathscr{F}[\mathrm{corr}_{M,G}]$$

as meaning $\mathscr{F}[\mathrm{corr}_{M,G}(X)]$ if $\mathscr{F}$ is supported on the connected component X, and then extending to arbitrary $\mathscr{F}$ by additivity.

We now define the constant term functor by the formula

$$\mathsf{CT}_{\mathcal{P},\mathcal{G}}(\mathscr{F}) = q_{\mathcal{P}!} i^*_{\mathcal{P}} h^*_{\mathcal{M},\mathcal{G}}(\mathscr{F})[\mathrm{corr}_{M,G}].$$

The rest of this section is devoted to the proof of some basic properties of these functors. All the proofs are similar to those of their classical counterparts for split groups, see [MV07, FS21, BR18]; still, we will generally give (sketches of) the arguments for completeness.

First of all, it is clear that the functor $\mathsf{CT}_{\mathcal{P},\mathcal{G}}$ commutes with the “change of scalars” functors considered in §4.1 (for derived categories) in the obvious sense.

Next, we explain an alternative formula for $\mathsf{CT}_{\mathcal{P},\mathcal{G}}$ involving the opposite parabolic P^- to P. Let $\mathcal{P}^-$ be the scheme theoretic closure of P^- in $\mathcal{G}$. Then we can consider the following counterpart of (4.7):

$$\mathrm{Hk}_{\mathcal{G}} \xleftarrow{h_{\mathcal{M},\mathcal{G}}} [\mathrm{L}^+\mathcal{M}\backslash\mathrm{Gr}_{\mathcal{G}}]_{\text{ét}} \xleftarrow{i_{\mathcal{P}^-}} [\mathrm{L}^+\mathcal{M}\backslash\mathrm{Gr}_{\mathcal{P}^-}]_{\text{ét}} \xrightarrow{q_{\mathcal{P}^-}} \mathrm{Hk}_{\mathcal{M}}. \tag{4.8}$$

Proposition 4.4. *For any $\mathscr{F}$ in $D^{\mathrm{b}}_{\mathrm{c}}(\mathrm{Hk}_{\mathcal{G}}, \Lambda)$ there is a natural isomorphism*

$$(q_{\mathcal{P}^-})_* i^!_{\mathcal{P}^-} h^*_{\mathcal{M},\mathcal{G}}(\mathscr{F})[\mathrm{corr}_{M,G}] \xrightarrow{\sim} \mathsf{CT}_{\mathcal{P},\mathcal{G}}(\mathscr{F}).$$

Proof. This is an application of Braden's theory of hyperbolic localization [Bra03], in the version explained in [Ric19]. Choose a cocharacter $\lambda \in \mathbb{X}_*(A)$ such that $P = P_\lambda$ (see Remark 3.1(1)), and then let $\mathbb{G}_{\mathrm{m},\mathbb{F}}$ act on $\mathrm{Gr}_{\mathcal{G}}$ via this cocharacter. There is a natural transformation $(q_{\mathcal{P}^-})_* \circ i^!_{\mathcal{P}^-} \to q_{\mathcal{P}!} \circ i^*_{\mathcal{P}}$ given by [Ric19, Construction 2.2]. Since the pullback functor $D^{\mathrm{b}}_{\mathrm{c}}(\mathrm{Hk}_{\mathcal{M}}, \Lambda) \to D^{\mathrm{b}}_{\mathrm{c}}(\mathrm{Gr}_{\mathcal{M}}, \Lambda)$ does not kill any nonzero object, one can check whether this morphism is an isomorphism on any given object after forgetting the $\mathrm{L}^+\mathcal{M}$-equivariance, i.e., working with the diagram (4.6) rather than (4.7). According to [Ric19, Theorem 2.6], our map is an isomorphism on any object in $D^{\mathrm{b}}_{\mathrm{c}}(\mathrm{Gr}_{\mathcal{G}}, \Lambda)$ that is $\mathbb{G}_{\mathrm{m}}$-monodromic in the sense of [Ric19, Definition 2.3]. But all objects in the image of $D^{\mathrm{b}}_{\mathrm{c}}(\mathrm{Hk}_{\mathcal{G}}, \Lambda) \to D^{\mathrm{b}}_{\mathrm{c}}(\mathrm{Gr}_{\mathcal{G}}, \Lambda)$ (or even of $D^{\mathrm{b}}_{\mathrm{c}}([\mathrm{L}^+\mathcal{M}\backslash \mathrm{Gr}_{\mathcal{G}}]_{\text{ét}}, \Lambda) \to D^{\mathrm{b}}_{\mathrm{c}}(\mathrm{Gr}_{\mathcal{G}}, \Lambda)$) are $\mathbb{G}_{\mathrm{m}}$-monodromic, so we are done. □

If $P \subset P'$ are parabolic subgroups of G containing A then as in §3.3 the intersection $P \cap M'$ is a parabolic subgroup of the reductive group M'. We can therefore consider the construction above for M', its special facet $\mathfrak{a}_{M'}$ (see §3.1), and the parabolic subgroup $P \cap M'$, and obtain the functor $\mathsf{CT}_{\mathcal{P}\cap\mathcal{M}',\mathcal{M}'}$.

Lemma 4.5. *Let $P \subset P'$ be parabolic subgroups of G containing A. There exists a canonical isomorphism of functors*

$$\mathsf{CT}_{\mathcal{P}\cap\mathcal{M}',\mathcal{M}'} \circ \mathsf{CT}_{\mathcal{P}',\mathcal{G}} \xrightarrow{\sim} \mathsf{CT}_{\mathcal{P},\mathcal{G}} \colon D^{\mathrm{b}}_{\mathrm{c}}(\mathrm{Hk}_{\mathcal{G}}, \Lambda) \to D^{\mathrm{b}}_{\mathrm{c}}(\mathrm{Hk}_{\mathcal{M}}, \Lambda).$$

Proof. This follows from the base change theorem, in view of Lemma 3.6(3). □

Proposition 4.6. *For any parabolic subgroup $P \subset G$ containing A, the functor*

$$\mathsf{CT}_{\mathcal{P},\mathcal{G}} \colon D^{\mathrm{b}}_{\mathrm{c}}(\mathrm{Hk}_{\mathcal{G}}, \Lambda) \to D^{\mathrm{b}}_{\mathrm{c}}(\mathrm{Hk}_{\mathcal{M}}, \Lambda)$$

is t-exact and conservative.

Proof. Lemma 4.5 reduces the proof of the proposition to the case where P is conjugate to B, which clearly reduces to the case $P = B$. Since $\mathsf{CT}_{\mathcal{B},\mathcal{G}}$ is a triangulated functor, to prove conservativity it suffices to prove that this functor does not kill any nonzero object. Let $\mathscr{F} \in D^{\mathrm{b}}_{\mathrm{c}}(\mathrm{Hk}_{\mathcal{G}}, \Lambda)$ be nonzero, and let $\lambda \in \mathbb{X}_*(T)^+_I$ be such that $\mathrm{Gr}^\lambda_{\mathcal{G}}$ is open in the support of $\mathscr{F}$. Then the restriction of $\mathscr{F}$ to $\mathrm{Gr}^\lambda_{\mathcal{G}}$ is nonzero, and if μ is the unique $W_{\mathfrak{a}}$-conjugate of λ which belongs to $-\mathbb{X}_*(T)^+_I$ we have $|\mathrm{S}_\mu \cap \mathrm{Gr}^\lambda_{\mathcal{G}}| = \{t^\mu\}$ by the proof of [AGLR22, Lemma 5.3]. It follows that $\mathsf{CT}_{\mathcal{B},\mathcal{G}}(\mathscr{F})$ is nonzero on the component of $\mathrm{Gr}_{\mathcal{T}}$ corresponding to μ, which finishes the proof of conservativity.

For t-exactness let us first consider the case $\Lambda \in \{\mathbb{k}, \mathbb{K}\}$. In this case, the subcategory ${}^{\mathrm{p}}D^{\leq 0}(\mathrm{Hk}_{\mathcal{G}}, \Lambda)$ is generated under extensions by the objects of the form $j^\lambda_! \underline{\Lambda}[\langle \lambda, 2\rho\rangle]$. Now, by the base change theorem, for any $\mu \in \mathbb{X}_*(T)^+_I$ we have

$$\big(\mathsf{CT}_{\mathcal{B},\mathcal{G}}(j^\lambda_! \underline{\Lambda}[\langle \lambda, 2\rho\rangle])\big)_{t^\mu} \cong R\Gamma_{\mathrm{c}}(\mathrm{Gr}^\lambda_{\mathcal{G}} \cap \mathrm{S}_\mu; \Lambda)[\langle \lambda + \mu, 2\rho\rangle].$$

By Lemma 3.7, $\mathrm{Gr}^\lambda_{\mathcal{G}} \cap \mathrm{S}_\mu$ has dimension $\langle \lambda + \mu, \rho\rangle$ when it is nonempty; hence $R\Gamma_{\mathrm{c}}(\mathrm{Gr}^\lambda_{\mathcal{G}} \cap \mathrm{S}_\mu; \Lambda)$ is concentrated in degrees at most $\langle \lambda + \mu, 2\rho\rangle$, which implies that $\mathsf{CT}_{\mathcal{B},\mathcal{G}}(j^\lambda_! \underline{\Lambda}[\langle \lambda, 2\rho\rangle])$ is indeed concentrated in nonpositive degrees. This proves that $\mathsf{CT}_{\mathcal{B},\mathcal{G}}$ is right t-exact. Left t-exactness follows using Verdier duality, Proposition 4.4 and Remark 3.1(2).

Finally we consider the case $\Lambda = \mathbb{O}$. Here also ${}^{\mathrm{p}}D^{\leq 0}(\mathrm{Hk}_{\mathcal{G}}, \mathbb{O})$ is generated under extensions by the objects of the form $j^\lambda_! \underline{\mathbb{O}}[\langle \lambda, 2\rho\rangle]$, so that the same proof as above shows that $\mathsf{CT}_{\mathcal{B},\mathcal{G}}$ is right t-exact. On the other hand, ${}^{\mathrm{p}}D^{\geq 0}(\mathrm{Hk}_{\mathcal{G}}, \mathbb{O})$ is generated under extensions by the objects of the form $j^\lambda_* \underline{\mathbb{O}}[\langle \lambda, 2\rho\rangle]$ or $j^\lambda_* \underline{\mathbb{k}}[\langle \lambda, 2\rho\rangle]$ for $\lambda \in \mathbb{X}_*(T)^+_I$. Now $\mathsf{CT}_{\mathcal{B},\mathcal{G}}(j^\lambda_* \underline{\mathbb{k}}[\langle \lambda, 2\rho\rangle])$ is concentrated in nonnegative degrees by the case of $\mathbb{k}$ treated above, and so is

$$\mathbb{k} \overset{L}{\otimes}_{\mathbb{O}} \mathsf{CT}_{\mathcal{B},\mathcal{G}}(j^\lambda_* \underline{\mathbb{O}}[\langle \lambda, 2\rho\rangle]) \cong \mathsf{CT}_{\mathcal{B},\mathcal{G}}(j^\lambda_* \underline{\mathbb{k}}[\langle \lambda, 2\rho\rangle]).$$

By standard arguments this implies that $\mathsf{CT}_{\mathcal{B},\mathcal{G}}(j_*^\lambda \underline{\mathbb{O}}[\langle\lambda, 2\rho\rangle])$ is concentrated in nonnegative degrees, and finishes the proof. □

Remark 4.7. Concretely, what t-exactness of the functor $\mathsf{CT}_{\mathcal{B},\mathcal{G}}$ means is that for $\mathscr{F} \in \mathsf{Perv}(\mathrm{Hk}_\mathcal{G}, \Lambda)$, $i \in \mathbb{Z}$ and $\mu \in \mathbb{X}_*(T)_I$ we have

$$\mathsf{H}^i_\mathrm{c}(\mathrm{S}_\mu, \mathscr{F}) = 0 \quad \text{unless } i = \langle\mu, 2\rho\rangle.$$

Similarly, by Proposition 4.4, we have

$$\mathsf{H}^i_{\mathrm{T}_\mu}(\mathrm{Gr}_\mathcal{G}, \mathscr{F}) = 0 \quad \text{unless } i = \langle\mu, 2\rho\rangle,$$

where $\mathsf{H}^\bullet_{\mathrm{T}_\mu}(\mathrm{Gr}_\mathcal{G}, -)$ denotes cohomology with support in T_μ.

Proposition 4.6 implies that $\mathsf{CT}_{\mathcal{P},\mathcal{G}}$ restricts to an exact functor

$$\mathsf{Perv}(\mathrm{Hk}_\mathcal{G}, \Lambda) \to \mathsf{Perv}(\mathrm{Hk}_\mathcal{M}, \Lambda),$$

which will be denoted similarly. These functors commute with the "change of scalars" functors for categories of perverse sheaves considered in §4.1.

4.4. **Compatibility with total cohomology.** Recall the pullback functor h^* from (4.2). We now define the functor

$$\mathsf{F}_\mathcal{G} \colon D^\mathrm{b}_\mathrm{c}(\mathrm{Hk}_\mathcal{G}, \Lambda) \to \mathsf{mod}_\Lambda$$

(where mod_Λ is the category of finitely generated Λ-modules) by

$$\mathsf{F}_\mathcal{G}(\mathscr{F}) := \bigoplus_{n\in\mathbb{Z}} \mathsf{H}^n(\mathrm{Gr}_\mathcal{G}, h^*\mathscr{F}).$$

(We ignore the $\mathbb{Z}$-grading on the right-hand side.) For any parabolic subgroup $P \subset G$ containing A, with Levi factor containing A denoted M, we can consider the analogous functor $\mathsf{F}_\mathcal{M} \colon D^\mathrm{b}_\mathrm{c}(\mathrm{Hk}_\mathcal{M}, \Lambda) \to \mathsf{mod}_\Lambda$.

Proposition 4.8. *For any P as above, there exists a canonical isomorphism*

$$\mathsf{F}_\mathcal{G} \cong \mathsf{F}_\mathcal{M} \circ \mathsf{CT}_{\mathcal{P},\mathcal{G}}$$

of functors from $\mathsf{Perv}(\mathrm{Hk}_\mathcal{G}, \Lambda)$ *to* mod_Λ.

Proof. Lemma 4.5 reduces the proof to the case $P = B$. In this case $\mathrm{Gr}_\mathcal{T}$ is discrete (see §3.4), and for $\mathscr{F}$ in $\mathsf{Perv}(\mathrm{Hk}_\mathcal{G}, \Lambda)$ we have

$$\mathsf{F}_\mathcal{T} \circ \mathsf{CT}_{\mathcal{B},\mathcal{G}}(\mathscr{F}) = \bigoplus_{\lambda\in\mathbb{X}_*(T)_I^+} \mathsf{H}^\bullet_\mathrm{c}(\mathrm{S}_\lambda, \mathscr{F}'_{|\mathrm{S}_\lambda})$$

where $\mathscr{F}' := h^*\mathscr{F}$. By Remark 4.7 we moreover have that

$$\mathsf{H}^n_\mathrm{c}(\mathrm{S}_\lambda, \mathscr{F}'_{|\mathrm{S}_\lambda}) = 0 \quad \text{unless } n = \langle\lambda, 2\rho\rangle. \tag{4.9}$$

What we have to construct is therefore a canonical identification

$$\mathsf{F}_\mathcal{G}(\mathscr{F}) = \bigoplus_{\lambda\in\mathbb{X}_*(T)_I^+} \mathsf{H}^{\langle\lambda,2\rho\rangle}_\mathrm{c}(\mathrm{S}_\lambda, \mathscr{F}'_{|\mathrm{S}_\lambda}). \tag{4.10}$$

Decomposing $\mathscr{F}$ as a direct sum of its restrictions to each connected component of $\mathrm{Gr}_\mathcal{G}$, we can assume that it is supported on one such component X. Choose a presentation $X = \mathrm{colim}_i X_i$ by $\mathrm{L}^+\mathcal{G}$-stable projective k-schemes, and some index i such that $\mathscr{F}$ is supported on X_i. Recall the decomposition

$$|X_i| = \bigsqcup_{\lambda\in Z} |\mathrm{S}_\lambda \times_{\mathrm{Gr}_\mathcal{G}} X_i|$$

where Z is the finite subset of $\mathbb{X}_*(T)^+_I$ consisting of the elements λ such that $\mathrm{S}_\lambda \times_{\mathrm{Gr}_\mathcal{G}} X_i \neq \varnothing$; see in particular (3.17). The parity of the integers $\langle \lambda, 2\rho \rangle$ is constant on this set since X_i is contained in a connected component of $\mathrm{Gr}_\mathcal{G}$; to fix notation we will assume that they are all even. For $n \in \mathbb{Z}$ we then set

$$X_i^n = \bigcup_{\substack{\lambda \in Z \\ \langle \lambda, 2\rho \rangle \leq 2n}} \overline{\mathrm{S}_\lambda \times_{\mathrm{Gr}_\mathcal{G}} X_i}$$

where $\overline{\mathrm{S}_\lambda \times_{\mathrm{Gr}_\mathcal{G}} X_i}$ is the scheme-theoretic closure of $\mathrm{S}_\lambda \times_{\mathrm{Gr}_\mathcal{G}} X_i$. Then for some integers $n_1 < n_2$ we have a finite filtration

$$\varnothing = X_i^{n_1} \subset X_i^{n_1+1} \subset \cdots \subset X_i^{n_2-1} \subset X_i^{n_2} = X_i$$

by closed subschemes, and by (3.16) we have

$$X_i^n \smallsetminus X_i^{n-1} = \bigsqcup_{\substack{\lambda \in Z \\ \langle \lambda, 2\rho \rangle = 2n}} \mathrm{S}_\lambda \times_{\mathrm{Gr}_\mathcal{G}} X_i$$

for any $n \in \{n_1+1, \ldots, n_2\}$. For $n \in \{n_1, \ldots, n_2\}$, resp. $n \in \{n_1+1, \ldots, n_2\}$, we will denote by

$$a_{\leq n} \colon X_i^n \to X_i, \quad \text{resp.} \quad a_n \colon X_i^n \smallsetminus X_i^{n-1} \to X_i$$

the immersions. Then for any $n \in \{n_1+1, \ldots, n_2\}$ we have a canonical distinguished triangle

$$(a_n)_! a_n^* \mathscr{F}' \to (a_{\leq n})_! a_{\leq n}^* \mathscr{F}' \to (a_{\leq n-1})_! a_{\leq n-1}^* \mathscr{F}' \xrightarrow{[1]} .$$

Using (4.9) and parity arguments one proves by induction on n that we have canonical isomorphisms

$$\mathsf{H}^q(X_i^n, \mathscr{F}'_{|X_i^n}) = \begin{cases} \bigoplus_{\substack{\lambda \in Z \\ \langle \lambda, 2\rho \rangle = q}} \mathsf{H}^q_\mathrm{c}(\mathrm{S}_\lambda, \mathscr{F}'_{|\mathrm{S}_\lambda}) & \text{if } q \text{ is even and } q \leq 2n; \\ 0 & \text{otherwise.} \end{cases}$$

Taking $n = n_2$ we deduce (4.10), which finishes the proof. □

Remark 4.9. (1) As in the usual geometric Satake context (see [BR18, Theorem 1.10.4]), Proposition 4.8 (in case $P = B$) implies that the functor $\mathsf{F}_\mathcal{G} \colon \mathsf{Perv}(\mathrm{Hk}_\mathcal{G}, \Lambda) \to \mathsf{mod}_\Lambda$ is exact.

(2) Recall the "change of scalars" functors from §4.1. It is clear that for $\mathscr{F}$ in $\mathsf{Perv}(\mathrm{Hk}_\mathcal{G}, \mathbb{O})$ we have a canonical isomorphism

$$\mathbb{K} \otimes_\mathbb{O} \mathsf{F}_\mathcal{G}(\mathscr{F}) \cong \mathsf{F}_\mathcal{G}(\mathbb{K} \otimes_\mathbb{O} \mathscr{F}).$$

On the other hand, there exists a canonical morphism

$$\Bbbk \otimes_\mathbb{O} \mathsf{F}_\mathcal{G}(\mathscr{F}) \to \mathsf{F}_\mathcal{G}({}^\mathrm{p}\mathscr{H}^0(\Bbbk \otimes_\mathbb{O} \mathscr{F})). \tag{4.11}$$

It is clear that the analogous morphism for the category $\mathsf{Perv}(\mathrm{Hk}_\mathcal{T}, \mathbb{O})$ is an isomorphism. Using Proposition 4.8 we deduce that (4.11) is an isomorphism.

The proof of Proposition 4.8 depends crucially on the fact that we are working with perverse sheaves; one cannot expect it to hold for general non-perverse objects in $D^\mathrm{b}_\mathrm{c}(\mathrm{Hk}_\mathcal{G}, \Lambda)$. Nevertheless, the following statement shows that Proposition 4.8 does generalize to direct sums of shifts of perverse sheaves.

Corollary 4.10. *Let $M \in D^\mathrm{b}_\mathrm{c}(\mathrm{pt}, \Lambda)$, and assume that $\mathsf{H}^i(M) = 0$ for i odd. For $\mathscr{F} \in \mathsf{Perv}(\mathrm{Hk}_\mathcal{G}, \Lambda)$, there is a natural isomorphism*

$$\mathsf{F}_\mathcal{G}(\mathscr{F} \overset{L}{\otimes}_\Lambda M) \cong \mathsf{F}_\mathcal{M}(\mathsf{CT}_{\mathcal{P},\mathcal{G}}(\mathscr{F} \overset{L}{\otimes}_\Lambda M))$$

Proof. For brevity, let $\mathscr{F}' = h^*\mathscr{F}$. It is enough to prove the claim in the case where $\mathscr{F}$ is supported on a single connected component of $\mathrm{Gr}_{\mathcal{G}}$. In this case, by (the proof of) Proposition 4.8, $\mathsf{H}^i(\mathrm{Gr}_{\mathcal{G}}, \mathscr{F}')$ can be nonzero only for i of a single parity. To fix notation, we assume that $\mathsf{H}^i(\mathrm{Gr}_{\mathcal{G}}, \mathscr{F}') \neq 0$ only for i even.

By an appropriate version of the universal coefficient theorem, for any $n \in \mathbb{Z}$ there is a natural short exact sequence

$$0 \to \bigoplus_{i+j=n} \mathsf{H}^i(\mathrm{Gr}_{\mathcal{G}}, \mathscr{F}') \otimes_\Lambda \mathsf{H}^j(M) \to \mathsf{H}^n(\mathrm{Gr}_{\mathcal{G}}, \mathscr{F}' \overset{L}{\otimes}_\Lambda M) \\ \to \bigoplus_{i+j=n+1} \mathrm{Tor}_1^\Lambda(\mathsf{H}^i(\mathrm{Gr}_{\mathcal{G}}, \mathscr{F}'), \mathsf{H}^j(M)) \to 0.$$

Our assumptions imply that the first term vanishes when n is odd, and the last term vanishes when n is even. We deduce that

$$\mathsf{H}^n(\mathrm{Gr}_{\mathcal{G}}, \mathscr{F}' \overset{L}{\otimes}_\Lambda M) \cong \begin{cases} \bigoplus_{i+j=n} \mathsf{H}^i(\mathrm{Gr}_{\mathcal{G}}, \mathscr{F}') \otimes_\Lambda \mathsf{H}^j(M) & \text{if } n \text{ is even,} \\ \bigoplus_{i+j=n+1} \mathrm{Tor}_1^\Lambda(\mathsf{H}^i(\mathrm{Gr}_{\mathcal{G}}, \mathscr{F}'), \mathsf{H}^j(M)) & \text{if } n \text{ is odd,} \end{cases}$$

and hence that there is a natural isomorphism

$$\mathsf{F}_{\mathcal{G}}(\mathscr{F} \overset{L}{\otimes}_\Lambda M) \cong (\mathsf{F}_{\mathcal{G}}(\mathscr{F}) \otimes_\Lambda \mathsf{H}^\bullet(M)) \oplus \mathrm{Tor}_1^\Lambda(\mathsf{F}_{\mathcal{G}}(\mathscr{F}), \mathsf{H}^\bullet(M)). \tag{4.12}$$

Applying this isomorphism with the group $\mathcal{G}$ replaced by $\mathcal{M}$ and with $\mathscr{F}$ replaced by $\mathsf{CT}_{\mathcal{P},\mathcal{G}}(\mathscr{F})$, we obtain an isomorphism

$$\mathsf{F}_{\mathcal{M}}(\mathsf{CT}_{\mathcal{P},\mathcal{G}}(\mathscr{F} \overset{L}{\otimes}_\Lambda M)) \cong \mathsf{F}_{\mathcal{M}}(\mathsf{CT}_{\mathcal{P},\mathcal{G}}(\mathscr{F}) \overset{L}{\otimes}_\Lambda M) \\ \cong (\mathsf{F}_{\mathcal{M}}(\mathsf{CT}_{\mathcal{P},\mathcal{G}}(\mathscr{F})) \otimes_\Lambda \mathsf{H}^\bullet(M)) \oplus \mathrm{Tor}_1^\Lambda(\mathsf{F}_{\mathcal{M}}(\mathsf{CT}_{\mathcal{P},\mathcal{G}}(\mathscr{F})), \mathsf{H}^\bullet(M)). \tag{4.13}$$

By Proposition 4.8, the right-hand sides of (4.12) and (4.13) are naturally isomorphic, so the left-hand sides are as well. ☐

4.5. **Constant term functors for convolution schemes.** Let $P \subset G$ be a parabolic subgroup containing A, and let $M \subset P$ be its Levi factor containing A. Let $\mathcal{P}$ and $\mathcal{M}$ be the scheme-theoretic closures of P and M, respectively, in $\mathcal{G}$. Then we have a diagram

$$\mathrm{Conv}_{\mathcal{G}} \xleftarrow{\tilde{\imath}_{\mathcal{P}}} \mathrm{Conv}_{\mathcal{P}} \xrightarrow{\tilde{q}_{\mathcal{P}}} \mathrm{Conv}_{\mathcal{M}}$$

similar to (4.6), along with the companion diagram

$$\mathrm{HkConv}_{\mathcal{G}} \xleftarrow{\tilde{h}_{\mathcal{M},\mathcal{G}}} [\mathrm{L}^+\mathcal{M} \backslash \mathrm{Conv}_{\mathcal{G}}]_{\text{ét}} \xleftarrow{\tilde{\imath}_{\mathcal{P}}} [\mathrm{L}^+\mathcal{M} \backslash \mathrm{Conv}_{\mathcal{P}}]_{\text{ét}} \xrightarrow{\tilde{q}_{\mathcal{P}}} \mathrm{HkConv}_{\mathcal{M}}.$$

As in §3.5 (where we considered the case $P = B$) the connected components of $\mathrm{Conv}_{\mathcal{M}}$ (and hence also those of $\mathrm{Conv}_{\mathcal{P}}$) are in bijection with $(\mathbb{X}_*(T)/Q_M^\vee)_I \times (\mathbb{X}_*(T)/Q_M^\vee)_I$. (This parametrization differs from that obtained via the identification with $\mathrm{Gr}_{\mathcal{M}} \times \mathrm{Gr}_{\mathcal{M}}$ from (2.11) by the map $(\lambda, \mu) \mapsto (\lambda, \lambda + \mu)$.) Define a function

$$\widetilde{\mathrm{corr}}_{M,G} \colon \pi_0(\mathrm{Conv}_{\mathcal{M}}) \to \mathbb{Z}$$

as follows: if $X \subset \mathrm{Conv}_{\mathcal{M}}$ is the connected component corresponding to $(\lambda, \mu) \in (\mathbb{X}_*(T)/Q_M^\vee)_I \times (\mathbb{X}_*(T)/Q_M^\vee)_I$, then

$$\widetilde{\mathrm{corr}}_{M,G}(X) = \langle \lambda + \mu, 2\rho - 2\rho_M \rangle.$$

Define the *convolution constant term functor*

$$\widetilde{\mathsf{CT}}_{\mathcal{P},\mathcal{G}} \colon D^{\mathrm{b}}_{\mathrm{c}}(\mathrm{HkConv}_{\mathcal{G}}, \Lambda) \to D^{\mathrm{b}}_{\mathrm{c}}(\mathrm{HkConv}_{\mathcal{M}}, \Lambda)$$

by a recipe similar to that used for $\mathsf{CT}_{\mathcal{P},\mathcal{G}}$: specifically, we set

$$\widetilde{\mathsf{CT}}_{\mathcal{P},\mathcal{G}} = \tilde{q}_{\mathcal{P}!}\tilde{i}^*_{\mathcal{P}}\tilde{h}^*_{\mathcal{M},\mathcal{G}}(\mathscr{F})[\widetilde{\mathrm{corr}}_{M,G}].$$

Using the opposite parabolic P^- and its scheme-theoretic closure $\mathcal{P}^-$, we have the following counterpart of (4.8):

$$\mathrm{HkConv}_{\mathcal{G}} \xleftarrow{\tilde{h}_{\mathcal{M},\mathcal{G}}} [\mathrm{L}^+\mathcal{M}\backslash\mathrm{Conv}_{\mathcal{G}}]_{\text{ét}} \xleftarrow{\tilde{i}_{\mathcal{P}^-}} [\mathrm{L}^+\mathcal{M}\backslash\mathrm{Conv}_{\mathcal{P}^-}]_{\text{ét}} \xrightarrow{\tilde{q}_{\mathcal{P}^-}} \mathrm{HkConv}_{\mathcal{M}}.$$

The proofs of the next two statements are essentially identical to those of Proposition 4.4 and Lemma 4.5, and we omit them.

Proposition 4.11. *For any $\mathscr{F}$ in $D^{\mathrm{b}}_{\mathrm{c}}(\mathrm{HkConv}_{\mathcal{G}},\Lambda)$ there is a natural isomorphism*

$$(\tilde{q}_{\mathcal{P}^-})_*\tilde{i}^!_{\mathcal{P}^-}\tilde{h}^*_{\mathcal{M},\mathcal{G}}(\mathscr{F})[\widetilde{\mathrm{corr}}_{M,G}] \xrightarrow{\sim} \widetilde{\mathsf{CT}}_{\mathcal{P},\mathcal{G}}(\mathscr{F}).$$

Lemma 4.12. *Let $P \subset P'$ be parabolic subgroups of G containing A, and let $M \subset M'$ be their respective Levi factors containing A. There exists a canonical isomorphism of functors*

$$\widetilde{\mathsf{CT}}_{\mathcal{P}\cap\mathcal{M}',\mathcal{M}'} \circ \widetilde{\mathsf{CT}}_{\mathcal{P}',\mathcal{G}} \xrightarrow{\sim} \widetilde{\mathsf{CT}}_{\mathcal{P},\mathcal{G}} \colon D^{\mathrm{b}}_{\mathrm{c}}(\mathrm{HkConv}_{\mathcal{G}},\Lambda) \to D^{\mathrm{b}}_{\mathrm{c}}(\mathrm{HkConv}_{\mathcal{M}},\Lambda).$$

For the next statement we need to restrict the domain of $\widetilde{\mathsf{CT}}_{\mathcal{P},\mathcal{G}}$ as follows: define

$$D^{\mathrm{b}}_{\mathrm{c}}(\mathrm{Conv}_{\mathcal{G}},\Lambda)_{\mathrm{sph}} \subset D^{\mathrm{b}}_{\mathrm{c}}(\mathrm{Conv}_{\mathcal{G}},\Lambda)$$

to be the full subcategory consisting of objects that are constructible with respect to the stratification

$$|\mathrm{Conv}_{\mathcal{G}}| = \bigsqcup_{\lambda,\mu\in\mathbb{X}_*(T)^+_I} |\mathrm{Conv}^{(\lambda,\mu)}_{\mathcal{G}}|. \tag{4.14}$$

Similarly, define

$$D^{\mathrm{b}}_{\mathrm{c}}(\mathrm{HkConv}_{\mathcal{G}},\Lambda)_{\mathrm{sph}} \subset D^{\mathrm{b}}_{\mathrm{c}}(\mathrm{HkConv}_{\mathcal{G}},\Lambda)$$

to be the full subcategory consisting of objects whose image in $D^{\mathrm{b}}_{\mathrm{c}}(\mathrm{Conv}_{\mathcal{G}},\Lambda)$ lies in the subcategory $D^{\mathrm{b}}_{\mathrm{c}}(\mathrm{Conv}_{\mathcal{G}},\Lambda)_{\mathrm{sph}}$. The triangulated categories $D^{\mathrm{b}}_{\mathrm{c}}(\mathrm{Conv}_{\mathcal{G}},\Lambda)_{\mathrm{sph}}$ and $D^{\mathrm{b}}_{\mathrm{c}}(\mathrm{HkConv}_{\mathcal{G}},\Lambda)_{\mathrm{sph}}$ both inherit a perverse t-structure; their hearts are denoted by

$$\mathsf{Perv}(\mathrm{Conv}_{\mathcal{G}},\Lambda)_{\mathrm{sph}} \qquad \text{and} \qquad \mathsf{Perv}(\mathrm{HkConv}_{\mathcal{G}},\Lambda)_{\mathrm{sph}},$$

respectively.

Proposition 4.13. *The functor*

$$\widetilde{\mathsf{CT}}_{\mathcal{B},\mathcal{G}} \colon D^{\mathrm{b}}_{\mathrm{c}}(\mathrm{HkConv}_{\mathcal{G}},\Lambda)_{\mathrm{sph}} \to D^{\mathrm{b}}_{\mathrm{c}}(\mathrm{HkConv}_{\mathcal{T}},\Lambda)$$

is t-exact and conservative.

Proof. The proof is essentially identical to that of Proposition 4.6, replacing the reference to Lemma 3.7 by a reference to Lemma 3.8. □

In concrete terms, Proposition 4.13 says that for $\mathscr{F} \in \mathsf{Perv}(\mathrm{HkConv}_{\mathcal{G}},\Lambda)_{\mathrm{sph}}$, $i \in \mathbb{Z}$ and $\lambda,\mu \in \mathbb{X}_*(T)_I$ we have

$$\mathsf{H}^i_{\mathrm{c}}(\mathrm{S}_\lambda \mathbin{\tilde{\times}} \mathrm{S}_\mu, \tilde{h}^*\mathscr{F}) = 0 \qquad \text{unless } i = \langle \lambda+\mu, 2\rho\rangle, \tag{4.15}$$

where $\tilde{h}\colon \mathrm{Conv}_{\mathcal{G}} \to \mathrm{HkConv}_{\mathcal{G}}$ is the quotient map, see §2.6.

Remark 4.14. It is likely that Proposition 4.13 holds for any $\widetilde{\mathsf{CT}}_{\mathcal{P},\mathcal{G}}$, not just the special case $P = B$. In fact, the general case would follow from Lemma 4.12 if we also had the following claim:

For any $\mathscr{F} \in D^{\mathrm{b}}_{\mathrm{c}}(\mathrm{HkConv}_{\mathcal{G}},\Lambda)_{\mathrm{sph}}$, the object $\widetilde{\mathsf{CT}}_{\mathcal{P},\mathcal{G}}(\mathscr{F})$ lies in $D^{\mathrm{b}}_{\mathrm{c}}(\mathrm{HkConv}_{\mathcal{M}},\Lambda)_{\mathrm{sph}}$.

This claim is probably true, and not difficult to prove, but as we will not need it in the sequel we do not pursue it here.

In the next statement, we denote by $\mathrm{pr}_{1,\mathcal{G}}$ and $\mathrm{pr}_{1,\mathcal{T}}$ the two versions of the "first projection" map

$$\mathrm{HkConv}_{\mathcal{G}} \to \mathrm{Hk}_{\mathcal{G}} \qquad \text{and} \qquad \mathrm{HkConv}_{\mathcal{T}} \to \mathrm{Hk}_{\mathcal{T}},$$

see §2.5–2.6.

Proposition 4.15. *For $\mathscr{F} \in \mathsf{Perv}(\mathrm{HkConv}_{\mathcal{G}}, \Lambda)_{\mathrm{sph}}$, there is a natural isomorphism*

$$\mathsf{F}_{\mathcal{T}}(\mathsf{CT}_{\mathcal{B},\mathcal{G}}(\mathrm{pr}_{1,\mathcal{G}!}\mathscr{F})) \cong \mathsf{F}_{\mathcal{T}}(\mathrm{pr}_{1,\mathcal{T}!}\widetilde{\mathsf{CT}}_{\mathcal{B},\mathcal{G}}(\mathscr{F})).$$

Proof. The proof is similar in spirit to that of Proposition 4.8. We may assume without loss of generality that $\mathscr{F}$ is supported on a single connected component X of $\mathrm{Conv}_{\mathcal{G}}$. As in that proof, we have

$$\begin{aligned}\mathsf{F}_{\mathcal{T}}(\mathsf{CT}_{\mathcal{B},\mathcal{G}}(\mathrm{pr}_{1,\mathcal{G}!}\mathscr{F})) &= \bigoplus_{\lambda\in\mathbb{X}_*(T)_I^+} \mathsf{H}^\bullet_{\mathrm{c}}(\mathrm{S}_\lambda, (h^*\,\mathrm{pr}_{1,\mathcal{G}!}\mathscr{F})_{|\mathrm{S}_\lambda}) \\ &= \bigoplus_{\lambda\in\mathbb{X}_*(T)_I^+} \mathsf{H}^\bullet_{\mathrm{c}}(\mathrm{pr}_{1,\mathcal{G}}^{-1}(\mathrm{S}_\lambda), \mathscr{F}'_{|\,\mathrm{pr}_{1,\mathcal{G}}^{-1}(\mathrm{S}_\lambda)}) = \bigoplus_{\lambda\in\mathbb{X}_*(T)_I^+} \mathsf{H}^\bullet_{\mathrm{c}}(\mathrm{S}_\lambda \,\widetilde{\times}\, \mathrm{Gr}_{\mathcal{G}}, \mathscr{F}'_{|\mathrm{S}_\lambda \widetilde{\times} \mathrm{Gr}_{\mathcal{G}}})\end{aligned}$$

where $\mathscr{F}' := \tilde{h}^*\mathscr{F}$.

We now fix $\lambda \in \mathbb{X}_*(T)_I^+$. The subset $\mathrm{S}_\lambda \,\widetilde{\times}\, \mathrm{Gr}_{\mathcal{G}} \subset \mathrm{Conv}_{\mathcal{G}}$ is a union of subsets of the form $\mathrm{S}_\lambda \,\widetilde{\times}\, \mathrm{S}_\mu$. Moreover, the parity of $\langle \lambda+\mu, 2\rho\rangle$ is constant among the subsets of this form that are contained in X; to fix notation we assume that these numbers are even.

Consider the restriction $\mathscr{F}'_{|\mathrm{S}_\lambda \widetilde{\times} \mathrm{Gr}_{\mathcal{G}}}$ of $\mathscr{F}'$ to $\mathrm{S}_\lambda \,\widetilde{\times}\, \mathrm{Gr}_{\mathcal{G}}$. Because $\mathscr{F}'$ lies in $\mathsf{Perv}(\mathrm{Conv}_{\mathcal{G}}, \Lambda)_{\mathrm{sph}}$, this object is supported on a subscheme of the form $\mathrm{S}_\lambda \,\widetilde{\times}\, X'$, where $X' \subset \mathrm{Gr}_{\mathcal{G}}$ is some $\mathrm{L}^+\mathcal{G}$-stable projective $\mathbb{F}$-scheme contained in a single component of $\mathrm{Gr}_{\mathcal{G}}$. We set $Y = \mathrm{S}_\lambda \,\widetilde{\times}\, X'$. We also let Z be the finite subset of $\mathbb{X}_*(T)_I^+$ consisting of the elements $\mu \in \mathbb{X}_*(T)_I$ such that that $\mathrm{S}_\mu \times_{\mathrm{Gr}_{\mathcal{G}}} X' \neq \varnothing$.

For $n \in \mathbb{Z}$ we set

$$Y^n = \bigcup_{\substack{\mu\in Z \\ \langle\lambda+\mu,2\rho\rangle\leq 2n}} \mathrm{S}_\lambda \,\widetilde{\times}\, (\overline{\mathrm{S}_\mu \times_{\mathrm{Gr}_{\mathcal{G}}} X'}).$$

Then for some integers $n_1 < n_2$ we have a finite filtration

$$\varnothing = Y^{n_1} \subset Y^{n_2} \subset \cdots \subset Y^{n_2-1} \subset Y^{n_2} = Y$$

by closed subschemes, as well as a decomposition

$$Y^n \smallsetminus Y^{n-1} = \bigsqcup_{\substack{\mu\in Z \\ \langle\lambda+\mu,2\rho\rangle = 2n}} \mathrm{S}_\lambda \,\widetilde{\times}\, (\mathrm{S}_\mu \times_{\mathrm{Gr}_{\mathcal{G}}} X').$$

Let $a_{\leq n}\colon Y^n \to Y$ and $a_n\colon Y^n \smallsetminus Y^{n-1} \to Y$ be the immersions, so that for $n \in \{n_1+1, \ldots, n_2\}$ we have a distinguished triangle

$$(a_n)_! a_n^* \mathscr{F}' \to (a_{\leq n})_! a_{\leq n}^* \mathscr{F}' \to (a_{\leq n-1})_! a_{\leq n-1}^* \mathscr{F}' \xrightarrow{[1]}.$$

Using (4.15) and parity arguments one proves by induction on n that we have canonical isomorphisms

$$\mathsf{H}^q(Y^n, \mathscr{F}'_{|Y^n}) = \begin{cases} \bigoplus_{\substack{\mu\in Z_\lambda \\ \langle\lambda+\mu,2\rho\rangle=q}} \mathsf{H}^q(\mathrm{S}_\lambda \,\widetilde{\times}\, \mathrm{S}_\mu, \mathscr{F}'_{|\mathrm{S}_\lambda \widetilde{\times} \mathrm{S}_\mu}) & \text{if } q \text{ is even and } q \leq 2n; \\ 0 & \text{otherwise.}\end{cases}$$

Taking $n = n_2$ we deduce an isomorphism

$$\mathsf{H}^\bullet_c(\mathrm{S}_\lambda \mathbin{\widetilde{\times}} \mathrm{Gr}_{\mathcal{G}}, \mathscr{F}'_{|\mathrm{S}_\lambda \widetilde{\times} \mathrm{Gr}_{\mathcal{G}}}) = \bigoplus_{\mu \in \mathbb{X}_*(T)^+_I} \mathsf{H}^{\langle \lambda+\mu, 2\rho \rangle}_c(\mathrm{S}_\lambda \mathbin{\widetilde{\times}} \mathrm{S}_\mu, \mathscr{F}'_{|\mathrm{S}_\lambda \widetilde{\times} \mathrm{S}_\mu}).$$

Summing the previous isomorphisms over λ we deduce a natural isomorphism

$$\mathsf{F}_{\mathcal{T}}(\mathsf{CT}_{\mathcal{B},\mathcal{G}}(\mathrm{pr}_{1,\mathcal{G}!} \mathscr{F})) = \bigoplus_{\lambda, \mu \in \mathbb{X}_*(T)^+_I} \mathsf{H}^{\langle \lambda+\mu, 2\rho \rangle}_c(\mathrm{S}_\lambda \mathbin{\widetilde{\times}} \mathrm{S}_\mu, \mathscr{F}'_{|\mathrm{S}_\lambda \widetilde{\times} \mathrm{S}_\mu}).$$

The right-hand side identifies with $\mathsf{F}_{\mathcal{T}}(\mathrm{pr}_{1,\mathcal{T}!} \widetilde{\mathsf{CT}}_{\mathcal{B},\mathcal{G}}(\mathscr{F}))$, so we are done. □

Corollary 4.16. *Let $\mathscr{F}, \mathscr{G} \in \mathsf{Perv}(\mathrm{Hk}_{\mathcal{G}}, \Lambda)$, and let $Y \subset \mathrm{Conv}_{\mathcal{G}}$ be a locally closed sub-ind-scheme that is a union of subsets of the form $\mathrm{S}_\lambda \mathbin{\widetilde{\times}} \mathrm{S}_\mu$. Then there is a canonical isomorphism*

$$\mathsf{H}^\bullet_c\big(Y, (\tilde{h}^* p^*(\mathscr{F} \overset{L}{\boxtimes}_\Lambda \mathscr{G}))_{|Y}\big) \cong \bigoplus_{\substack{\lambda,\mu \in \mathbb{X}_*(T)^+_I \\ \mathrm{S}_\lambda \widetilde{\times} \mathrm{S}_\mu \subset Y}} \mathsf{H}^{\langle \lambda+\mu, 2\rho \rangle}_c\big(\mathrm{S}_\lambda \mathbin{\widetilde{\times}} \mathrm{S}_\mu, (\tilde{h}^* p^*(\mathscr{F} \overset{L}{\boxtimes}_\Lambda \mathscr{G}))_{|\mathrm{S}_\lambda \widetilde{\times} \mathrm{S}_\mu}\big).$$

Proof. For brevity, set $\mathscr{F}' = h^* \mathscr{F}$ and $\mathscr{G}' = h^* \mathscr{G}$. It is sufficient to treat the case where $\mathscr{G}'$ is supported on a single connected component of $\mathrm{Gr}_{\mathcal{G}}$, and we assume this as well. Let us first consider the special case where $Y = \mathrm{Conv}_{\mathcal{G}}$. By proper base change using the cartesian square (2.17), we have

$$\mathrm{pr}_{1,\mathcal{G}!}\, p^*(\mathscr{F} \overset{L}{\boxtimes}_\Lambda \mathscr{G}) \cong \mathscr{F} \overset{L}{\otimes}_\Lambda R\Gamma(\mathrm{Gr}_{\mathcal{G}}, \mathscr{G}').$$

Since $\mathscr{G}'$ is supported on a single connected component, as in the proof of Corollary 4.10, the nonzero cohomology groups $\mathsf{H}^i(\mathrm{Gr}_{\mathcal{G}}, \mathscr{G}')$ all have i of the same parity. Then, by Corollary 4.10, we obtain that

$$\mathsf{F}_{\mathcal{G}}\big(\mathrm{pr}_{1,\mathcal{G}!}\, p^*(\mathscr{F} \overset{L}{\boxtimes}_\Lambda \mathscr{G})\big) \cong \mathsf{F}_{\mathcal{T}}\big(\mathsf{CT}_{\mathcal{B},\mathcal{G}}(\mathrm{pr}_{1,\mathcal{G}!}\, p^*(\mathscr{F} \overset{L}{\boxtimes}_\Lambda \mathscr{G}))\big). \tag{4.16}$$

Combining this with Proposition 4.15, we obtain

$$\begin{aligned} \mathsf{H}^\bullet_c(\mathrm{Conv}_{\mathcal{G}}, \tilde{h}^* p^*(\mathscr{F} \overset{L}{\boxtimes}_\Lambda \mathscr{G})) &= \mathsf{H}^\bullet_c(\mathrm{Gr}_{\mathcal{G}}, h^* \mathrm{pr}_{1,\mathcal{G}!}\, p^*(\mathscr{F} \overset{L}{\boxtimes}_\Lambda \mathscr{G})) \\ &\cong \mathsf{F}_{\mathcal{T}}(\mathsf{CT}_{\mathcal{B},\mathcal{G}}(\mathrm{pr}_{1,\mathcal{G}!}\, p^*(\mathscr{F} \overset{L}{\boxtimes}_\Lambda \mathscr{G}))) \\ &\cong \bigoplus_{\lambda,\mu \in \mathbb{X}_*(T)^+_I} \mathsf{H}^{\langle \lambda+\mu, 2\rho \rangle}_c(\mathrm{S}_\lambda \mathbin{\widetilde{\times}} \mathrm{S}_\mu, \tilde{h}^* p^*(\mathscr{F} \overset{L}{\boxtimes}_\Lambda \mathscr{G})_{|\mathrm{S}_\lambda \widetilde{\times} \mathrm{S}_\mu}), \end{aligned} \tag{4.17}$$

which establishes the desired identification in this case.

Here is another interpretation of this equality. For $\lambda, \mu \in \mathbb{X}_*(T)^+_I$, let $\mathrm{S}_{\not\geq(\lambda,\mu)} \subset \mathrm{Conv}_{\mathcal{G}}$ be the closed subset defined by

$$\mathrm{S}_{\not\geq(\lambda,\mu)} = \bigcup_{\substack{\lambda',\mu' \in \mathbb{X}_*(T)^+_I \\ \lambda' \not\geq \lambda \text{ or } \lambda'+\mu' \not\geq \lambda+\mu}} \mathrm{S}_{\lambda'} \mathbin{\widetilde{\times}} \mathrm{S}_{\mu'}.$$

This is a closed subset of $\mathrm{Conv}_{\mathcal{G}}$. There is a filtration $F_{\bullet,\bullet}$ of the Λ-module

$$\mathsf{H}^\bullet_c(\mathrm{Conv}_{\mathcal{G}}, \tilde{h}^* p^*(\mathscr{F} \overset{L}{\boxtimes}_\Lambda \mathscr{G}))$$

indexed by $\mathbb{X}_*(T)^+_I \times \mathbb{X}_*(T)^+_I$, given by

$$F_{\lambda,\mu} := \ker(\mathsf{H}^\bullet_c(\mathrm{Conv}_{\mathcal{G}}, \tilde{h}^* p^*(\mathscr{F} \overset{L}{\boxtimes}_\Lambda \mathscr{G})) \to \mathsf{H}^\bullet_c(\mathrm{S}_{\not\geq(\lambda,\mu)}, \tilde{h}^* p^*(\mathscr{F} \overset{L}{\boxtimes}_\Lambda \mathscr{G})_{|\mathrm{S}_{\not\geq(\lambda,\mu)}})).$$

Then (4.17) says that this filtration admits a canonical splitting.

We now return to the setting of a general Y as in the statement of the corollary. Let Z be the set of pairs $(\lambda,\mu) \in \mathbb{X}_*(T)_I^+ \times \mathbb{X}_*(T)_I^+$ such that $\mathrm{S}_\lambda \mathbin{\widetilde{\times}} \mathrm{S}_\mu \subset Y$, and let

$$Z^- = \left\{ (\lambda,\mu) \in \mathbb{X}_*(T)_I^+ \times \mathbb{X}_*(T)_I^+ \;\middle|\; \begin{array}{c} \text{there exists } (\lambda',\mu') \in Z \text{ with} \\ \lambda \geq \lambda' \text{ and } \lambda+\mu \geq \lambda'+\mu' \end{array} \right\}.$$

The fact that Y is locally closed implies that $Z^- \smallsetminus Z$ is an upper closed set with respect to the partial order on $\mathbb{X}_*(T)_I^+ \times \mathbb{X}_*(T)_I^+$. As a consequence, the module $\mathsf{H}^\bullet_{\mathrm{c}}(Y, \tilde{h}^* p^*(\mathscr{F} \boxtimes^L_\Lambda \mathscr{G})_{|Y})$ can be identified with a subquotient of the filtration defined above: namely,

$$\mathsf{H}^\bullet_{\mathrm{c}}(Y, \tilde{h}^* p^*(\mathscr{F} \overset{L}{\boxtimes}_\Lambda \mathscr{G})_{|Y}) = \sum_{(\lambda,\mu)\in Z^-} F_{\lambda,\mu} \Big/ \sum_{(\lambda,\mu)\in Z^-\smallsetminus Z} F_{\lambda,\mu}.$$

Since the filtration is canonically split, the result follows. □

Proposition 4.17. *For $\mathscr{F},\mathscr{G} \in \mathsf{Perv}(\mathrm{Hk}_\mathcal{G},\Lambda)$, there is a natural isomorphism*

$$\mathsf{CT}_{\mathcal{B},\mathcal{G}}(\mathscr{F} \star^0 \mathscr{G}) \cong m_{\mathcal{T}!}\widetilde{\mathsf{CT}}_{\mathcal{B},\mathcal{G}}(p^*({}^{\mathrm{p}}\mathscr{H}^0(\mathscr{F} \overset{L}{\boxtimes}_\Lambda \mathscr{G}))).$$

Proof. The statement is a natural isomorphism in $\mathsf{Perv}(\mathrm{Hk}_\mathcal{T},\Lambda)$. Below, we will prove a closely related statement: we will show that in $\mathsf{Perv}(\mathrm{Gr}_\mathcal{T},\Lambda)$, there is a natural isomorphism

$$h^*\mathsf{CT}_{\mathcal{B},\mathcal{G}}(\mathscr{F} \star \mathscr{G}) \cong h^* m_{\mathcal{T}!}\widetilde{\mathsf{CT}}_{\mathcal{B},\mathcal{G}}(p^*(\mathscr{F} \overset{L}{\boxtimes}_\Lambda \mathscr{G})). \tag{4.18}$$

Let us explain how to deduce the proposition from (4.18). By t-exactness of the various functors above (see Propositions 4.1, 4.6, and 4.13, and also the discussion in §B.2), (4.18) implies that

$$h^*\mathsf{CT}_{\mathcal{B},\mathcal{G}}(\mathscr{F} \star^0 \mathscr{G}) \cong h^* m_{\mathcal{T}!}\widetilde{\mathsf{CT}}_{\mathcal{B},\mathcal{G}}(p^*({}^{\mathrm{p}}\mathscr{H}^0(\mathscr{F} \overset{L}{\boxtimes}_\Lambda \mathscr{G}))),$$

and this in turn implies the proposition because $h^* \colon \mathsf{Perv}(\mathrm{Hk}_\mathcal{T},\Lambda) \to \mathsf{Perv}(\mathrm{Gr}_\mathcal{T},\Lambda)$ is fully faithful.

Let us prove (4.18). Since $\mathrm{Gr}_\mathcal{T}$ is discrete, we may further reduce the problem to that of comparing stalks of these complexes at each point of $\mathrm{Gr}_\mathcal{T}$. Let us fix an element $\nu \in \mathbb{X}_*(T)_I$, and focus on the stalks $(h^*\mathsf{CT}_{\mathcal{B},\mathcal{G}}(\mathscr{F} \star \mathscr{G}))_{t^\nu}$ and $(h^* m_{\mathcal{T}!}\widetilde{\mathsf{CT}}_{\mathcal{B},\mathcal{G}}(p^*(\mathscr{F} \boxtimes^L_\Lambda \mathscr{G})))_{t^\nu}$.

Note that $m_\mathcal{G}^{-1}(\mathrm{S}_\nu)$ is the union of all $\mathrm{S}_\lambda \mathbin{\widetilde{\times}} \mathrm{S}_\mu$ where $\lambda+\mu=\nu$. Therefore, using Corollary 4.16, we have

$$\begin{aligned} (h^*\mathsf{CT}_{\mathcal{B},\mathcal{G}}(\mathscr{F} \star \mathscr{G}))_{t^\nu} &\cong \mathsf{H}^{\langle \nu,2\rho\rangle}_{\mathrm{c}}(\mathrm{S}_\nu, (m_{\mathcal{G}!} p^*(\mathscr{F} \overset{L}{\boxtimes}_\Lambda \mathscr{G}))_{|\mathrm{S}_\nu}) \\ &\cong \mathsf{H}^{\langle \nu,2\rho\rangle}_{\mathrm{c}}(m_\mathcal{G}^{-1}(\mathrm{S}_\nu), p^*(\mathscr{F} \overset{L}{\boxtimes}_\Lambda \mathscr{G})_{|m_\mathcal{G}^{-1}(\mathrm{S}_\nu)}) \\ &\cong \bigoplus_{\substack{\lambda,\mu \in \mathbb{X}_*(T)_I^+ \\ \lambda+\mu=\nu}} \mathsf{H}^{\langle \nu,2\rho\rangle}_{\mathrm{c}}(\mathrm{S}_\lambda \mathbin{\widetilde{\times}} \mathrm{S}_\mu, p^*(\mathscr{F} \overset{L}{\boxtimes}_\Lambda \mathscr{G})_{|\mathrm{S}_\lambda \widetilde{\times} \mathrm{S}_\mu}). \end{aligned}$$

On the other hand, the stalk $(h^* m_{\mathcal{T}!}\widetilde{\mathsf{CT}}_{\mathcal{B},\mathcal{G}}(p^*(\mathscr{F} \boxtimes^L_\Lambda \mathscr{G})))_{t^\nu}$ is given by

$$\begin{aligned} (h^* m_{\mathcal{T}!}\widetilde{\mathsf{CT}}_{\mathcal{B},\mathcal{G}}(p^*(\mathscr{F} \overset{L}{\boxtimes}_\Lambda \mathscr{G})))_{t^\nu} &\cong \bigoplus_{\lambda+\mu=\nu} (h^*\widetilde{\mathsf{CT}}_{\mathcal{B},\mathcal{G}}(p^*(\mathscr{F} \overset{L}{\boxtimes}_\Lambda \mathscr{G})))_{[\lambda,\mu]} \\ &\cong \bigoplus_{\lambda+\mu=\nu} \mathsf{H}^{\langle \lambda+\mu,2\rho\rangle}_{\mathrm{c}}(\mathrm{S}_\lambda \mathbin{\widetilde{\times}} \mathrm{S}_\mu, p^*(\mathscr{F} \overset{L}{\boxtimes}_\Lambda \mathscr{G})_{|\mathrm{S}_\lambda \widetilde{\times} \mathrm{S}_\mu}) \end{aligned}$$

where $[\lambda,\mu] \in \mathrm{Conv}_\mathcal{T}$ is the point corresponding to $(\lambda,\lambda+\mu)$ under the identifications (2.11) (for $\mathcal{T}$) and (3.15). The result follows. □

Combining Proposition 4.17 with Proposition 4.8, we obtain the following immediate consequence.

Proposition 4.18. *For $\mathscr{F}, \mathscr{G} \in \mathsf{Perv}(\mathrm{Hk}_{\mathcal{G}}, \Lambda)$, there is a natural isomorphism*

$$\mathsf{F}_{\mathcal{G}}(\mathscr{F} \star^0 \mathscr{G}) \cong \mathsf{F}_{\mathcal{T}}(m_{\mathcal{T}!}\widetilde{\mathsf{CT}}_{\mathcal{B},\mathcal{G}}(p^*({}^{\mathrm{p}}\mathscr{H}^0(\mathscr{F} \overset{L}{\boxtimes}_{\Lambda} \mathscr{G})))).$$

Remark 4.19. The preceding proposition depends on Corollary 4.10, whose proof makes crucial use of the fact that Λ has global dimension ≤ 1. In the analytic setting mentioned in Remark 1.3, one may wish to consider coefficient rings of global dimension > 1 (but still finite). To handle this situation, one can modify the argument as follows:

- First, prove Corollary 4.10 under the additional assumption that the cohomology modules $\mathsf{H}^i(M)$ are flat over Λ.
- Then, prove Corollary 4.16, Proposition 4.17, and Proposition 4.18 under the additional assumption that $\mathsf{F}_{\mathcal{G}}(\mathscr{G})$ is flat over Λ.
- The results in Section 6 imply that every perverse sheaf admits a presentation $\mathscr{P}_1 \to \mathscr{P}_2 \twoheadrightarrow \mathscr{G}$ where $\mathsf{F}_{\mathcal{G}}(\mathscr{P}_1)$ and $\mathsf{F}_{\mathcal{G}}(\mathscr{P}_2)$ are flat over Λ. Because the functors in the statement of Proposition 4.17 are right exact, one can uniquely fill in the dotted arrow in the diagram below:

$$\begin{tikzcd}
\mathsf{CT}_{\mathcal{B},\mathcal{G}}(\mathscr{F} \star^0 \mathscr{P}_1) \arrow[r] \arrow[d, "\substack{\text{flat case of}\\ \text{Prop. 4.17}}"'] & \mathsf{CT}_{\mathcal{B},\mathcal{G}}(\mathscr{F} \star^0 \mathscr{P}_2) \arrow[r, two heads] \arrow[d, "\substack{\text{flat case of}\\ \text{Prop. 4.17}}"'] & \mathsf{CT}_{\mathcal{B},\mathcal{G}}(\mathscr{F} \star^0 \mathscr{G}) \arrow[r] \arrow[d, dashed] & 0 \\
m_{\mathcal{T}!}\widetilde{\mathsf{CT}}_{\mathcal{B},\mathcal{G}}(p^{*\,\mathrm{p}}\mathscr{H}^0(\mathscr{F} \boxtimes^L_\Lambda \mathscr{P}_1)) \arrow[r] & m_{\mathcal{T}!}\widetilde{\mathsf{CT}}_{\mathcal{B},\mathcal{G}}(p^{*\,\mathrm{p}}\mathscr{H}^0(\mathscr{F} \boxtimes^L_\Lambda \mathscr{P}_2)) \arrow[r, two heads] & m_{\mathcal{T}!}\widetilde{\mathsf{CT}}_{\mathcal{B},\mathcal{G}}(p^{*\,\mathrm{p}}\mathscr{H}^0(\mathscr{F} \boxtimes^L_\Lambda \mathscr{G})) \arrow[r] & 0
\end{tikzcd}$$

  In this way, one can deduce Propositions 4.17 and 4.18 in general.

5. Monoidality

5.1. **Statement and strategy.** The goal of this section is to equip the total cohomology functor

$$\mathsf{F}_{\mathcal{G}} \colon \mathsf{Perv}(\mathrm{Hk}_{\mathcal{G}}, \Lambda) \to \mathsf{mod}_{\Lambda}$$

(see §4.4) with a monoidal structure, i.e., with a natural isomorphism

$$\phi \colon \mathsf{F}_{\mathcal{G}}(\mathscr{F}) \otimes_{\Lambda} \mathsf{F}_{\mathcal{G}}(\mathscr{G}) \xrightarrow{\sim} \mathsf{F}_{\mathcal{G}}(\mathscr{F} \star^0 \mathscr{G}) \tag{5.1}$$

satisfying appropriate associativity and identity equations. In the "classical" geometric Satake context (see [MV07, BR18]) the monoidal structure on the fiber functor is constructed using the Beĭlinson–Drinfeld Grassmannian and the interpretation of convolution as fusion, see [MV07, §6] or [BR18, §1.8] for details. In our present setting we have no analogue of the Beĭlinson–Drinfeld Grassmannian; we therefore have to argue differently.

As a warm-up, we treat an easy special case: that in which $\mathcal{G}$ is replaced by $\mathcal{T}$.

Lemma 5.1. *For $\mathscr{F}, \mathscr{G} \in \mathsf{Perv}(\mathrm{Hk}_{\mathcal{T}}, \Lambda)$ there is a natural isomorphism*

$$\mathsf{F}_{\mathcal{T}}(\mathscr{F} \star^0 \mathscr{G}) \cong \mathsf{F}_{\mathcal{T}}(\mathscr{F}) \otimes_{\Lambda} \mathsf{F}_{\mathcal{T}}(\mathscr{G})$$

making $\mathsf{F}_{\mathcal{T}} \colon \mathsf{Perv}(\mathrm{Hk}_{\mathcal{T}}, \Lambda) \to \mathsf{mod}_{\Lambda}$ into a monoidal functor.

Proof. Recall that the underlying topological spaces of $\mathrm{Gr}_{\mathcal{T}}$ and $\mathrm{Conv}_{\mathcal{T}}$ are discrete, associated with the sets $\mathbb{X}_*(T)_I$ and $\mathbb{X}_*(T)_I \times \mathbb{X}_*(T)_I$ respectively; see in particular (3.15). Using the identifications (2.14) and (2.18), along with the commutative diagram (2.19), we see that

$$\begin{aligned}\mathsf{F}_{\mathcal{T}}(\mathscr{F} \star^0 \mathscr{G}) &\cong \mathsf{H}^0(\mathrm{Gr}_{\mathcal{T}}, {}^{\mathrm{p}}\mathscr{H}^0(h^* m_! p^*(\mathscr{F} \overset{L}{\boxtimes}_{\Lambda} \mathscr{G}))) \\ &\cong \mathsf{H}^0(\mathrm{Gr}_{\mathcal{T}} \times \mathrm{Gr}_{\mathcal{T}}, {}^{\mathrm{p}}\mathscr{H}^0((h^*\mathscr{F}) \overset{L}{\boxtimes}_{\Lambda} (h^*\mathscr{G}))).\end{aligned}$$

Using again that $\mathrm{Gr}_{\mathcal{T}}$ is discrete we see that the last expression is isomorphic to

$$\mathsf{H}^0(\mathrm{Gr}_{\mathcal{T}}, h^*\mathscr{F}) \otimes_\Lambda \mathsf{H}^0(\mathrm{Gr}_{\mathcal{T}}, h^*\mathscr{G}) = \mathsf{F}_{\mathcal{T}}(\mathscr{F}) \otimes_\Lambda \mathsf{F}_{\mathcal{T}}(\mathscr{G}),$$

as desired. □

In view of Lemma 5.1 and Proposition 4.8 (for $P = B$), to construct a monoidal structure on $\mathsf{F}_{\mathcal{G}}$ it suffices to construct a monoidal structure on the functor

$$\mathsf{CT}_{\mathcal{B},\mathcal{G}} \colon \mathsf{Perv}(\mathrm{Hk}_{\mathcal{G}}, \Lambda) \to \mathsf{Perv}(\mathrm{Hk}_{\mathcal{T}}, \Lambda)$$

with respect to the convolution product $\star^0$. This is exactly what we do in the rest of this section.

5.2. **A Künneth lemma.**

Lemma 5.2. *Consider the diagram*

$$\begin{tikzcd}
 & {[\mathrm{L}^+\mathcal{T}\backslash\mathrm{Gr}_{\mathcal{B}}]_{\text{ét}} \times \mathrm{Hk}_{\mathcal{B}}} & \\
{[\mathrm{L}^+\mathcal{T}\backslash\mathrm{Conv}_{\mathcal{B}}]_{\text{ét}}} \arrow[ur, "p_{\mathcal{B}}"] \arrow[dr, "\tilde{q}_{\mathcal{B}}"'] & & {[\mathrm{L}^+\mathcal{T}\backslash\mathrm{Gr}_{\mathcal{B}}]_{\text{ét}} \times \mathrm{Gr}_{\mathcal{B}}} \arrow[ul, "\mathrm{id}\times h_{\mathcal{B}}"'] \arrow[dl, "q_{\mathcal{B}}\times q_{\mathcal{B}}"] \\
 & {\mathrm{HkConv}_{\mathcal{T}} \overset{(2.18)}{=} \mathrm{Hk}_{\mathcal{T}} \times \mathrm{Gr}_{\mathcal{T}}.} &
\end{tikzcd}$$

For $\mathscr{F} \in D^{\mathrm{b}}_{\mathrm{c}}([\mathrm{L}^+\mathcal{T}\backslash\mathrm{Gr}_{\mathcal{B}}]_{\text{ét}}, \Lambda)$ and $\mathscr{G} \in D^{\mathrm{b}}_{\mathrm{c}}(\mathrm{Hk}_{\mathcal{B}}, \Lambda)$, there is a natural isomorphism

$$\tilde{q}_{\mathcal{B}!} p^*_{\mathcal{B}}(\mathscr{F} \overset{L}{\boxtimes}_\Lambda \mathscr{G}) \cong (q_{\mathcal{B}!}\mathscr{F}) \overset{L}{\boxtimes}_\Lambda (q_{\mathcal{B}!} h^*_{\mathcal{B}}\mathscr{G}).$$

Proof. It is enough to prove this when $\mathscr{F}$ and $\mathscr{G}$ are each supported on a single connected component of $\mathrm{Gr}_{\mathcal{B}}$, say S_λ and S_μ, respectively for some $\lambda, \mu \in \mathbb{X}_*(T)_I$. Note that all maps in the diagram induce bijections on the sets of connected components. Taking appropriate connected components, our diagram restricts to

$$\begin{tikzcd}
 & {[\mathrm{L}^+\mathcal{T}\backslash\mathrm{S}_\lambda]_{\text{ét}} \times [\mathrm{L}^+\mathcal{B}\backslash\mathrm{S}_\mu]_{\text{ét}}} & \\
{[\mathrm{L}^+\mathcal{T}\backslash\mathrm{S}_\lambda \mathbin{\widetilde{\times}} \mathrm{S}_\mu]_{\text{ét}}} \arrow[ur, "p_{\mathcal{B}}"] \arrow[dr, "\tilde{q}_{\mathcal{B}}"'] & & {[\mathrm{L}^+\mathcal{T}\backslash\mathrm{S}_\lambda]_{\text{ét}} \times \mathrm{S}_\mu} \arrow[ul, "\mathrm{id}\times h_{\mathcal{B}}"'] \arrow[dl, "q_{\mathcal{B}}\times q_{\mathcal{B}}"] \\
 & {[\mathrm{L}^+\mathcal{T}\backslash\mathrm{pt}]_{\text{ét}}} &
\end{tikzcd}$$

where pt is the point given by (t^λ, t^μ). By proper base change using the cartesian square (2.17), we have $\mathrm{pr}_{1,\mathcal{B}!}(p^*_{\mathcal{B}}(\mathscr{F} \boxtimes^L_\Lambda \mathscr{G})) \cong \mathscr{F} \otimes^L_\Lambda R\Gamma_{\mathrm{c}}(\mathrm{Gr}_{\mathcal{B}}, h^*_{\mathcal{B}}\mathscr{G})$, and hence a natural isomorphism

$$\tilde{q}_{\mathcal{B}!} p^*_{\mathcal{B}}(\mathscr{F} \overset{L}{\boxtimes}_\Lambda \mathscr{G}) \cong (q_{\mathcal{B}!}\mathscr{F}) \overset{L}{\boxtimes}_\Lambda (q_{\mathcal{B}!} h^*_{\mathcal{B}}\mathscr{G}),$$

as desired. □

5.3. **Constant term functors and external product.** From Lemma 5.2 we will deduce the following compatibility statement between the constant term functors and the external product.

Corollary 5.3. *For $\mathscr{F}, \mathscr{G} \in D^{\mathrm{b}}_{\mathrm{c}}(\mathrm{Hk}_{\mathcal{G}}, \Lambda)$ there is a natural isomorphism*

$$\widetilde{\mathsf{CT}}_{\mathcal{B},\mathcal{G}}\big(p^*_{\mathcal{G}}(\mathscr{F} \overset{L}{\boxtimes}_\Lambda \mathscr{G})\big) \cong p^*_{\mathcal{T}}\big(\mathsf{CT}_{\mathcal{B},\mathcal{G}}(\mathscr{F}) \overset{L}{\boxtimes}_\Lambda \mathsf{CT}_{\mathcal{B},\mathcal{G}}(\mathscr{G})\big). \tag{5.2}$$

For $\mathscr{F}, \mathscr{G} \in \mathsf{Perv}(\mathrm{Hk}_{\mathcal{G}}, \Lambda)$, this induces a natural isomorphism

$$\widetilde{\mathsf{CT}}_{\mathcal{B},\mathcal{G}}\big(p^*_{\mathcal{G}}\,{}^{\mathrm{p}}\mathscr{H}^0(\mathscr{F} \overset{L}{\boxtimes}_\Lambda \mathscr{G})\big) \cong p^*_{\mathcal{T}}\,{}^{\mathrm{p}}\mathscr{H}^0\big(\mathsf{CT}_{\mathcal{B},\mathcal{G}}(\mathscr{F}) \overset{L}{\boxtimes}_\Lambda \mathsf{CT}_{\mathcal{B},\mathcal{G}}(\mathscr{G})\big). \tag{5.3}$$

Proof. Let $w\colon \mathrm{Hk}_{\mathcal{B}} \to \mathrm{Hk}_{\mathcal{G}}$ be the map induced by the inclusion $\mathcal{B} \subset \mathcal{G}$. Our arguments will exploit the following diagram, in which the left and middle squares commute:

$$\begin{tikzcd}
\mathrm{Hk}_{\mathcal{G}} \times \mathrm{Hk}_{\mathcal{G}} & {[\mathrm{L}^+\mathcal{T}\backslash \mathrm{Gr}_{\mathcal{G}}]_{\text{ét}} \times \mathrm{Hk}_{\mathcal{G}}} \arrow[l, "h_{\mathcal{T},\mathcal{G}}\times \mathrm{id}"'] & {[\mathrm{L}^+\mathcal{T}\backslash \mathrm{Gr}_{\mathcal{B}}]_{\text{ét}} \times \mathrm{Hk}_{\mathcal{B}}} \arrow[l, "i_{\mathcal{B}}\times w"'] & {[\mathrm{L}^+\mathcal{T}\backslash \mathrm{Gr}_{\mathcal{B}}]_{\text{ét}} \times \mathrm{Gr}_{\mathcal{B}}} \arrow[l, "\mathrm{id}\times h_{\mathcal{B}}"'] \arrow[d, "q_{\mathcal{B}}\times q_{\mathcal{B}}"] \\
\mathrm{HkConv}_{\mathcal{G}} \arrow[u, "p_{\mathcal{G}}"] & {[\mathrm{L}^+\mathcal{T}\backslash \mathrm{Conv}_{\mathcal{G}}]_{\text{ét}}} \arrow[l, "\tilde{h}_{\mathcal{T},\mathcal{G}}"'] \arrow[u, "p_{\mathcal{B}}"] & {[\mathrm{L}^+\mathcal{T}\backslash \mathrm{Conv}_{\mathcal{B}}]_{\text{ét}}} \arrow[l, "\tilde{i}_{\mathcal{B}}"'] \arrow[u, "p_{\mathcal{B}}"] \arrow[r, "\tilde{q}_{\mathcal{B}}"] & \mathrm{HkConv}_{\mathcal{T}} \overset{(2.18)}{=} \mathrm{Hk}_{\mathcal{T}} \times \mathrm{Gr}_{\mathcal{T}}.
\end{tikzcd}$$

Using the commutativity and Lemma 5.2 we obtain isomorphisms

$$\begin{aligned}
\widetilde{\mathsf{CT}}_{\mathcal{B},\mathcal{G}}(p_{\mathcal{G}}^*(\mathscr{F} \overset{L}{\boxtimes}_{\Lambda} \mathscr{G})) &:= \tilde{q}_{\mathcal{B}!}\tilde{i}_{\mathcal{B}}^*\tilde{h}_{\mathcal{T},\mathcal{G}}^* p_{\mathcal{G}}^*(\mathscr{F} \overset{L}{\boxtimes}_{\Lambda} \mathscr{G}) \\
&\cong \tilde{q}_{\mathcal{B}!}p_{\mathcal{B}}^*((i_{\mathcal{B}}^* h_{\mathcal{T},\mathcal{G}}^*\mathscr{F}) \overset{L}{\boxtimes}_{\Lambda} w^*\mathscr{G}) \cong (q_{\mathcal{B}!}i_{\mathcal{B}}^* h_{\mathcal{T},\mathcal{G}}^*\mathscr{F}) \overset{L}{\boxtimes}_{\Lambda} (q_{\mathcal{B}!}h_{\mathcal{B}}^* w^*\mathscr{G}).
\end{aligned}$$

Next we observe that

$$q_{\mathcal{B}!}i_{\mathcal{B}}^* h_{\mathcal{T},\mathcal{G}}^*\mathscr{F} =: \mathsf{CT}_{\mathcal{B},\mathcal{G}}(\mathscr{F}) \quad \text{and} \quad q_{\mathcal{B}!}h_{\mathcal{B}}^* w^*\mathscr{G} \cong h_{\mathcal{T}}^*\mathsf{CT}_{\mathcal{B},\mathcal{G}}(\mathscr{G}).$$

Finally, under the isomorphism $\mathrm{HkConv}_{\mathcal{T}} \overset{(2.18)}{=} \mathrm{Hk}_{\mathcal{T}} \times \mathrm{Gr}_{\mathcal{T}}$ the map $p_{\mathcal{T}}$ corresponds to $\mathrm{id} \times h_{\mathcal{T}}$, which allows us to convert the isomorphisms above into (5.2) (and thereby to finish the proof).

In view of Proposition 4.6, we obtain (5.3) by applying ${}^{\mathrm{p}}\mathscr{H}^0$ to (5.2). □

5.4. **End of the construction.** We are now ready to exhibit a monoidal structure on $\mathsf{CT}_{\mathcal{B},\mathcal{G}}$. As explained in §5.1, from there one obtains a monoidal structure (5.1) on the functor $\mathsf{F}_{\mathcal{G}}$.

Proposition 5.4. *For $\mathscr{F}, \mathscr{G} \in \mathsf{Perv}(\mathrm{Hk}_{\mathcal{G}}, \Lambda)$, there is a natural isomorphism*

$$\mathsf{CT}_{\mathcal{B},\mathcal{G}}(\mathscr{F} \star^0 \mathscr{G}) \cong \mathsf{CT}_{\mathcal{B},\mathcal{G}}(\mathscr{F}) \star^0 \mathsf{CT}_{\mathcal{B},\mathcal{G}}(\mathscr{G}). \tag{5.4}$$

that makes $\mathsf{CT}_{\mathcal{B},\mathcal{G}}$ into a monoidal functor.

Proof. By Proposition 4.17 and Corollary 5.3, the following diagram commutes up to natural isomorphism:

$$\begin{tikzcd}
\mathsf{Perv}(\mathrm{Hk}_{\mathcal{G}}, \Lambda) \times \mathsf{Perv}(\mathrm{Hk}_{\mathcal{G}}, \Lambda) \arrow[r, "\mathsf{CT}_{\mathcal{B},\mathcal{G}} \times \mathsf{CT}_{\mathcal{B},\mathcal{G}}"] \arrow[d, "p_{\mathcal{G}}^* {}^{\mathrm{p}}\mathscr{H}^0(-\boxtimes^L_{\Lambda}-)"] \arrow[dd, bend right=60, "(-)\star^0(-)"'] \arrow[dr, phantom, "\text{Cor. 5.3}"] & \mathsf{Perv}(\mathrm{Hk}_{\mathcal{T}}, \Lambda) \times \mathsf{Perv}(\mathrm{Hk}_{\mathcal{T}}, \Lambda) \arrow[d, "p_{\mathcal{T}}^* {}^{\mathrm{p}}\mathscr{H}^0(-\boxtimes^L_{\Lambda}-)"'] \arrow[dd, bend left=60, "(-)\star^0(-)"] \\
\mathsf{Perv}(\mathrm{HkConv}_{\mathcal{G}}, \Lambda)_{\mathrm{sph}} \arrow[r, "\widetilde{\mathsf{CT}}_{\mathcal{B},\mathcal{G}}"] \arrow[dr, phantom, "\text{Prop. 4.17}"] & \mathsf{Perv}(\mathrm{HkConv}_{\mathcal{T}}, \Lambda)_{\mathrm{sph}} \arrow[d, "m_{\mathcal{T}!}"] \\
\mathsf{Perv}(\mathrm{Hk}_{\mathcal{G}}, \Lambda) \arrow[r, "\mathsf{CT}_{\mathcal{B},\mathcal{G}}"'] & \mathsf{Perv}(\mathrm{Hk}_{\mathcal{T}}, \Lambda)
\end{tikzcd}$$

(Note that Proposition 4.17 asserts the commutativity of a pentagon in this diagram, and *not* of the square that would be obtained by including the arrow $m_{\mathcal{G}!}\colon \mathsf{Perv}(\mathrm{HkConv}_{\mathcal{G}}, \Lambda)_{\mathrm{sph}} \to \mathsf{Perv}(\mathrm{Hk}_{\mathcal{G}}, \Lambda)$. For this reason, $m_{\mathcal{G}!}$ is omitted from the picture.) Considering the outer square of this diagram, we obtain the isomorphism (5.4). We leave it to the reader to check that (5.4) is compatible with the associativity and identity constraints. □

Remark 5.5. By construction, the monoidal structure on $\mathsf{F}_{\mathcal{G}}$ is characterized by the fact that it makes the following diagram commute:

$$\begin{tikzcd}
\mathsf{F}_{\mathcal{T}}(m_{\mathcal{T}!}{}^{\mathrm{p}}\mathscr{H}^0(\widetilde{\mathsf{CT}}_{\mathcal{B},\mathcal{G}}(p_{\mathcal{G}}^*(\mathscr{F} \boxtimes^L_{\Lambda} \mathscr{G})))) \arrow[r, "\text{Cor. 5.3}", "\sim"'] \arrow[dd, "\text{Prop. 4.18}"', "\wr"] & \mathsf{F}_{\mathcal{T}}(\mathsf{CT}_{\mathcal{B},\mathcal{G}}(\mathscr{F}) \star^0 \mathsf{CT}_{\mathcal{B},\mathcal{G}}(\mathscr{G})) \arrow[d, "\text{Lem. 5.1}", "\wr"'] \\
 & \mathsf{F}_{\mathcal{T}}(\mathsf{CT}_{\mathcal{B},\mathcal{G}}(\mathscr{F})) \otimes_{\Lambda} \mathsf{F}_{\mathcal{T}}(\mathsf{CT}_{\mathcal{B},\mathcal{G}}(\mathscr{G})) \arrow[d, "\text{Prop. 4.8}", "\wr"'] \\
\mathsf{F}_{\mathcal{G}}(\mathscr{F} \star^0 \mathscr{G}) \arrow[r, "(5.1)", "\sim"'] & \mathsf{F}_{\mathcal{G}}(\mathscr{F}) \otimes_{\Lambda} \mathsf{F}_{\mathcal{G}}(\mathscr{G}).
\end{tikzcd}$$

Remark 5.6. There is another construction of a monoidal structure on the constant term functor which does not use the Beĭlinson–Drinfeld Grassmannian in [Yu22, Proof of Proposition 4.4]. (The geometric setting considered in that reference is different from ours, but shares similar formal properties.) Unfortunately, this proof is based on the false claim that the filtrations on the total cohomology arising from the semi-infinite orbits and their opposites are complementary to each other.[4] This seems to be a common misconception in the literature and appears in several places including [MV07, Proof of Theorem 3.6], [Zhu17, Proof of Theorem 5.3.9(3)] and [HR21, Proof of Theorem 3.16]. The proof of Proposition 5.4 replaces the false argument with the direct computation in Corollary 4.16.

6. A bialgebra governing perverse sheaves

6.1. **Statement.** The main result of this section is the following Theorem 6.1, whose proof will be finished in §6.7. Given a Λ-coalgebra C, we denote by comod_C its category of right comodules which are finitely generated over Λ. If C is a bialgebra, then the tensor product $\otimes_{\Lambda}$ equips this category with a monoidal structure.

Theorem 6.1. *For $\Lambda \in \{\mathbb{K}, \mathbb{O}, \Bbbk\}$ there exists a canonical Λ-bialgebra $\mathrm{B}_{\mathcal{G}}(\Lambda)$ and an equivalence of monoidal categories*

$$\mathsf{S}_{\mathcal{G}} \colon \big(\mathsf{Perv}(\mathrm{Hk}_{\mathcal{G}}, \Lambda), \star^0\big) \xrightarrow{\sim} \big(\mathsf{comod}_{\mathrm{B}_{\mathcal{G}}(\Lambda)}, \otimes_{\Lambda}\big).$$

Moreover, $\mathrm{B}_{\mathcal{G}}(\mathbb{O})$ is flat over $\mathbb{O}$ and there exist canonical isomorphisms of $\Bbbk$- and $\mathbb{K}$-bialgebras

$$\Bbbk \otimes_{\mathbb{O}} \mathrm{B}_{\mathcal{G}}(\mathbb{O}) \xrightarrow{\sim} \mathrm{B}_{\mathcal{G}}(\Bbbk), \quad \mathbb{K} \otimes_{\mathbb{O}} \mathrm{B}_{\mathcal{G}}(\mathbb{O}) \xrightarrow{\sim} \mathrm{B}_{\mathcal{G}}(\mathbb{K}) \tag{6.1}$$

respectively, compatible with the change-of-scalars functors

$$\begin{gathered}
{}^{\mathrm{p}}\mathscr{H}^0(\Bbbk \overset{L}{\otimes}_{\mathbb{O}} (-)) \colon \mathsf{Perv}(\mathrm{Hk}_{\mathcal{G}}, \mathbb{O}) \to \mathsf{Perv}(\mathrm{Hk}_{\mathcal{G}}, \Bbbk), \\
\mathbb{K} \otimes_{\mathbb{O}} (-) \colon \mathsf{Perv}(\mathrm{Hk}_{\mathcal{G}}, \mathbb{O}) \to \mathsf{Perv}(\mathrm{Hk}_{\mathcal{G}}, \mathbb{K}), \\
\mathsf{Perv}(\mathrm{Hk}_{\mathcal{G}}, \Bbbk) \to \mathsf{Perv}(\mathrm{Hk}_{\mathcal{G}}, \mathbb{O})
\end{gathered}$$

in the natural way.

The proof of Theorem 6.1 is very similar to that of the corresponding claim in the context of the “ordinary” geometric Satake equivalence; see [MV07, Section 11] and [BR18, §1.13.1]. We will not repeat the proofs that can be copied from these references.

[4] This problem was pointed out to the fourth named author by S. Lysenko several years ago. It was also discussed during Scholze’s geometrization lectures in the winter term 2020/21 where the third named author was present.

6.2. **Weight functors.** Recall the functor

$$\mathsf{F}_{\mathcal{G}} \colon \mathsf{Perv}(\mathrm{Hk}_{\mathcal{G}}, \Lambda) \to \mathsf{mod}_\Lambda$$

considered in §4.4. We also have a similar functor

$$\mathsf{F}_{\mathcal{T}} \colon \mathsf{Perv}(\mathrm{Hk}_{\mathcal{T}}, \Lambda) \to \mathsf{mod}_\Lambda$$

for the group $\mathcal{T}$. Since $\mathrm{Gr}_{\mathcal{T}}$ is discrete with underlying set $\mathbb{X}_*(T)_I$ (see §3.4), we have a canonical identification

$$\mathsf{Perv}(\mathrm{Hk}_{\mathcal{T}}, \Lambda) = \mathsf{mod}_\Lambda^{\mathbb{X}_*(T)_I} \tag{6.2}$$

where the right-hand side denotes the category of finitely generated $\mathbb{X}_*(T)_I$-graded Λ-modules. Via this identification, the functor $\mathsf{F}_{\mathcal{T}}$ sends an $\mathbb{X}_*(T)_I$-graded Λ-modules to the underlying Λ-module.

Using (6.2), the functor $\mathsf{CT}_{\mathcal{B},\mathcal{G}}$ (see §4.3) can be seen as a functor

$$\mathsf{Perv}(\mathrm{Hk}_{\mathcal{G}}, \Lambda) \to \mathsf{mod}_\Lambda^{\mathbb{X}_*(T)_I}.$$

If we denote, for any $\lambda \in \mathbb{X}_*(T)_I$, by $\mathsf{F}_{\mathcal{G},\lambda}$ the composition of this functor with the functor $\mathsf{mod}_\Lambda^{\mathbb{X}_*(T)_I} \to \mathsf{mod}_\Lambda$ sending an $\mathbb{X}_*(T)_I$-graded Λ-module to its λ-component, then by Proposition 4.8 (for $\mathcal{P} = \mathcal{B}$) we have a canonical isomorphism of functors

$$\mathsf{F}_{\mathcal{G}} \cong \bigoplus_{\lambda \in \mathbb{X}_*(T)_I} \mathsf{F}_{\mathcal{G},\lambda}. \tag{6.3}$$

The functor $\mathsf{F}_{\mathcal{G},\lambda}$ is called the *weight functor* associated with λ. Explicitly, for any $\lambda \in \mathbb{X}_*(T)_I$ and $\mathscr{F} \in \mathsf{Perv}(\mathrm{Hk}_{\mathcal{G}}, \Lambda)$ we have

$$\mathsf{F}_{\mathcal{G},\lambda}(\mathscr{F}) = \mathsf{H}_c^{\langle \lambda, 2\rho \rangle}(\mathrm{S}_\lambda, (h^*\mathscr{F})_{|\mathrm{S}_\lambda}) \cong \mathsf{H}_{\mathrm{T}_\lambda}^{\langle \lambda, 2\rho \rangle}(h^*\mathscr{F}),$$

where the isomorphism is provided by Proposition 4.4 (again, for $\mathcal{P} = \mathcal{B}$).

6.3. **Preliminaries on standard and costandard perverse sheaves.** Recall, for $\mu \in \mathbb{X}_*(T)_I^+$, the objects $\mathscr{J}_!(\mu, \Lambda)$ and $\mathscr{J}_*(\mu, \Lambda)$ considered in §4.1.

Lemma 6.2. *In case $\Lambda = \mathbb{K}$, the category $\mathsf{Perv}(\mathrm{Hk}_{\mathcal{G}}, \Lambda)$ is semisimple. In particular, the natural morphism $\mathscr{J}_!(\mu, \Lambda) \to \mathscr{J}_*(\mu, \Lambda)$ is an isomorphism.*

Proof. Like in the setting of the ordinary geometric Satake equivalence (see [BR18, §1.4]), the claim follows from the fact that the parity of the dimension of Schubert varieties is constant on each connected component of $\mathrm{Gr}_{\mathcal{G}}$, see (2.5) and (2.10), and that the simple objects in $\mathsf{Perv}(\mathrm{Hk}_{\mathcal{G}}, \Lambda)$ are parity complexes in the sense of [JMW14]. □

Lemma 6.3. (1) *For $\mu \in \mathbb{X}_*(T)_I^+$ and $\nu \in \mathbb{X}_*(T)_I$, the Λ-module*

$$\mathsf{F}_{\mathcal{G},\nu}(\mathscr{J}_!(\mu, \Lambda)), \quad \textit{resp.} \quad \mathsf{F}_{\mathcal{G},\nu}(\mathscr{J}_*(\mu, \Lambda)),$$

is free with a canonical basis parametrized by the irreducible components of $\mathrm{Gr}_{\mathcal{G}}^\mu \cap \mathrm{S}_\nu$, resp. $\mathrm{Gr}_{\mathcal{G}}^\mu \cap \mathrm{T}_\nu$.

(2) *For any $\mu \in \mathbb{X}_*(T)_I^+$ there exist canonical isomorphisms*

$$\mathscr{J}_!(\mu, \mathbb{K}) \cong \mathbb{K} \overset{L}{\otimes}_{\mathbb{O}} \mathscr{J}_!(\mu, \mathbb{O}), \qquad \mathscr{J}_*(\mu, \mathbb{K}) \cong \mathbb{K} \overset{L}{\otimes}_{\mathbb{O}} \mathscr{J}_*(\mu, \mathbb{O}),$$

$$\mathscr{J}_!(\mu, \mathbb{k}) \cong \mathbb{k} \overset{L}{\otimes}_{\mathbb{O}} \mathscr{J}_!(\mu, \mathbb{O}), \qquad \mathscr{J}_*(\mu, \mathbb{k}) \cong \mathbb{k} \overset{L}{\otimes}_{\mathbb{O}} \mathscr{J}_*(\mu, \mathbb{O}).$$

(3) *In case $\Lambda = \mathbb{O}$, for any $\mu \in \mathbb{X}_*(T)_I^+$ the canonical morphism*

$$\mathscr{J}_!(\mu, \Lambda) \to \mathscr{J}_*(\mu, \Lambda)$$

is injective.

Proof. (1) The proof is the same as for [BR18, Proposition 1.11.1].

(2) The proof is the same as for [BR18, Proposition 1.11.3].

(3) By the same considerations as in [BR18, Lemma 1.11.5], the claim follows from Lemma 6.2. □

6.4. **Representability.** Consider a closed subscheme $Z \subset \mathrm{Gr}_{\mathcal{G}}$ whose underlying topological subspace is a union of finitely many $\mathrm{L}^+\mathcal{G}$-orbits. We can then consider the quotient stack $[\mathrm{L}^+\mathcal{G}\backslash Z]_{\text{ét}}$, and the corresponding full subcategory

$$\mathsf{Perv}([\mathrm{L}^+\mathcal{G}\backslash Z]_{\text{ét}}, \Lambda) \subset \mathsf{Perv}(\mathrm{Hk}_{\mathcal{G}}, \Lambda).$$

In fact, the action of $\mathrm{L}^+\mathcal{G}$ on Z factors through an action of $\mathrm{L}^+_n\mathcal{G}$ for some $n \geq 0$, and we then have

$$\mathsf{Perv}([\mathrm{L}^+\mathcal{G}\backslash Z]_{\text{ét}}, \Lambda) = \mathsf{Perv}([\mathrm{L}^+_n\mathcal{G}\backslash Z]_{\text{ét}}, \Lambda).$$

Fix some $\nu \in \mathbb{X}_*(T)_I$ such that $\mathrm{T}_\nu \cap Z \neq \varnothing$, and consider the immersion

$$i_{Z,\nu} \colon \mathrm{T}_\nu \cap Z \to Z$$

and the action and projection morphisms

$$a_{Z,\nu}, p_{Z,\nu} \colon \mathrm{L}^+_n\mathcal{G} \times Z \to Z.$$

One checks as in [BR18, Proposition 1.12.1] that the complex

$$(a_{Z,\nu})_!(p_{Z,\nu})^!(i_{Z,\nu})_!\underline{\Lambda}_{\mathrm{T}_\nu \cap Z}[-\langle \nu, 2\rho \rangle] \tag{6.4}$$

is concentrated in nonpositive perverse degrees, and that if we set

$$\mathscr{P}_Z(\nu, \Lambda) := {}^{\mathrm{p}}\mathscr{H}^0((a_{Z,\nu})_!(p_{Z,\nu})^!(i_{Z,\nu})_!\underline{\Lambda}_{\mathrm{T}_\nu \cap Z}[-\langle \nu, 2\rho \rangle]),$$

the perverse sheaf $\mathscr{P}_Z(\nu, \Lambda)$ is a projective object in $\mathsf{Perv}([\mathrm{L}^+\mathcal{G}\backslash Z]_{\text{ét}}, \Lambda)$ which represents the restriction of $\mathsf{F}_{\mathcal{G},\nu}$ to this subcategory. In particular, this object does not depend on the choice of n.

Set

$$\mathbb{X}_Z = \{\nu \in \mathbb{X}_*(T)_I \mid Z \cap \mathrm{T}_\nu \neq \varnothing\} = \bigcup_{\substack{\lambda \in \mathbb{X}_*(T)^+_I \\ |\mathrm{Gr}^\lambda_{\mathcal{G}}| \subset |Z|}} W_0 \cdot \lambda,$$

where the equality follows from Lemma 3.7. (This set is clearly finite.) In view of (6.3), setting

$$\mathscr{P}_Z(\Lambda) := \bigoplus_{\nu \in \mathbb{X}_Z} \mathscr{P}_Z(\nu, \Lambda)$$

one obtains a projective object in $\mathsf{Perv}([\mathrm{L}^+\mathcal{G}\backslash Z]_{\text{ét}}, \Lambda)$ which represents the restriction of $\mathsf{F}_{\mathcal{G}}$ to this subcategory. In fact, as in [BR18, §1.12.1], this object is a projective generator of the category $\mathsf{Perv}([\mathrm{L}^+\mathcal{G}\backslash Z]_{\text{ét}}, \Lambda)$.

We finish this subsection with the discussion of a property which will be used in Appendix A. Consider a *locally closed* subscheme $X \subset \mathrm{Gr}_{\mathcal{G}}$ whose underlying topological subspace is a union of finitely many $\mathrm{L}^+\mathcal{G}$-orbits. Choose closed subschemes $Y \subset Z \subset \mathrm{Gr}_{\mathcal{G}}$ whose underlying topological subspace is a union of finitely many $\mathrm{L}^+\mathcal{G}$-orbits and such that $X = Z \smallsetminus Y$, and denote by $j \colon X \to Z$ the open embedding. If $\nu \in \mathbb{X}_Z \smallsetminus \mathbb{X}_Y$ (or, in other words, if $t^\nu \in |X|$), the same considerations as in [BR18, Remark 1.5.8(2)] show that for any $\mathscr{G} \in \mathsf{Perv}([\mathrm{L}^+\mathcal{G}\backslash X]_{\text{ét}}, \Lambda)$ we have

$$\mathsf{H}^k_{X \cap \mathrm{T}_\nu}(X, \mathscr{G}) = 0 \quad \text{unless } k = \langle \nu, 2\rho \rangle, \tag{6.5}$$

so that the functor

$$\mathsf{F}^X_{\mathcal{G},\nu} := \mathsf{H}^{\langle \nu, 2\rho \rangle}_{X \cap \mathrm{T}_\nu}(X, -) \colon \mathsf{Perv}([\mathrm{L}^+\mathcal{G}\backslash X]_{\text{ét}}, \Lambda) \to \mathsf{mod}_\Lambda$$

is exact (by consideration of an appropriate long exact sequence).

Lemma 6.4. *The functor $\mathsf{F}^X_{\mathcal{G},\nu}$ is represented by the perverse sheaf $j^*\mathscr{P}_Z(\nu,\Lambda)$.*

Proof. For $\mathscr{F} \in \mathsf{Perv}([\mathrm{L}^+\mathcal{G}\backslash X]_{\text{ét}},\Lambda)$, by adjunction we have

$$\mathrm{Hom}(j^*\mathscr{P}_Z(\nu,\Lambda),\mathscr{F}) \cong \mathrm{Hom}(\mathscr{P}_Z(\nu,\Lambda),{}^{\mathrm{p}}\mathscr{H}^0(j_*\mathscr{F})).$$

Since $\mathscr{P}_Z(\nu,\Lambda)$ represents $\mathsf{F}_{\mathcal{G},\nu}$ on $\mathsf{Perv}([\mathrm{L}^+\mathcal{G}\backslash Z]_{\text{ét}},\Lambda)$, the right-hand side identifies with

$$\mathsf{F}_{\mathcal{G},\nu}({}^{\mathrm{p}}\mathscr{H}^0(j_*\mathscr{F})) = \mathsf{H}^{\langle\nu,2\rho\rangle}_{\mathrm{T}_\nu}(\mathrm{Gr}_{\mathcal{G}},{}^{\mathrm{p}}\mathscr{H}^0(j_*\mathscr{F})).$$

Now, by Remark 4.7 and standard considerations involving perverse truncation triangles (as in [BR18, Lemma 1.10.7]), one sees that

$$\mathsf{H}^{\langle\nu,2\rho\rangle}_{\mathrm{T}_\nu}(\mathrm{Gr}_{\mathcal{G}},{}^{\mathrm{p}}\mathscr{H}^0(j_*\mathscr{F})) = \mathsf{H}^{\langle\nu,2\rho\rangle}_{\mathrm{T}_\nu}(\mathrm{Gr}_{\mathcal{G}},j_*\mathscr{F}).$$

By base change the right-hand side identifies with $\mathsf{H}^{\langle\nu,2\rho\rangle}_{X\cap\mathrm{T}_\nu}(X,\mathscr{F})$, which finishes the proof. □

It follows from Lemma 6.4 and the comments preceding it that the perverse sheaf

$$\mathscr{P}_X(\nu,\Lambda) = j^*\mathscr{P}_Z(\nu,\Lambda)$$

does not depend on the choice of Y and Z, and is projective.

Consider now the projective object

$$\mathscr{P}_X(\Lambda) = \bigoplus_{\substack{\nu\in\mathbb{X}\\ t^\nu\in|X|}} \mathscr{P}_X(\nu,\Lambda) \quad \in \mathsf{Perv}([\mathrm{L}^+\mathcal{G}\backslash X]_{\text{ét}},\Lambda).$$

The same arguments as in the proof of Proposition 4.6 show that the functor $\mathrm{Hom}(\mathscr{P}_X(\Lambda),-)$ does not kill any nonzero object; it follows that $\mathscr{P}$ is a projective generator of the category $\mathsf{Perv}([\mathrm{L}^+\mathcal{G}\backslash X]_{\text{ét}},\Lambda)$.

6.5. **Structure of projective objects.** Let us now consider two $\mathrm{L}^+\mathcal{G}$-stable closed subschemes $Y,Z \subset \mathrm{Gr}_{\mathcal{G}}$ whose underlying topological subspace is a union of finitely many $\mathrm{L}^+\mathcal{G}$-orbits and such that $Y \subset Z$. If we denote by $i\colon Y \to Z$ the closed immersion, then as in [BR18, Proposition 1.12.2] one checks that there exists a canonical isomorphism

$$\mathscr{P}_Y(\Lambda) \cong {}^{\mathrm{p}}\mathscr{H}^0(i^*\mathscr{P}_Z(\Lambda)),$$

and that the morphism

$$\mathscr{P}_Z(\Lambda) \to {}^{\mathrm{p}}\mathscr{H}^0(i_*i^*\mathscr{P}_Z(\Lambda)) = i_*\mathscr{P}_Y(\Lambda)$$

induced by adjunction is surjective.

Next, using Lemma 6.3 and the same arguments as for [BR18, Proposition 1.12.3], one checks that, for any closed subscheme $Z \subset \mathrm{Gr}_{\mathcal{G}}$ whose underlying topological subspace is a union of finitely many $\mathrm{L}^+\mathcal{G}$-orbits:

(1) the object $\mathscr{P}_Z(\Lambda)$ admits a finite filtration with associated graded
$$\bigoplus_{\substack{\mu\in\mathbb{X}_*(T)^+_I\\ |\mathrm{Gr}^\mu_{\mathcal{G}}|\subset|Z|}} \mathsf{F}_{\mathcal{G}}(\mathscr{J}_*(\mu,\Lambda)) \otimes_\Lambda \mathscr{J}_!(\mu,\Lambda);$$

(2) there exist canonical isomorphisms
$$\mathscr{P}_Z(\mathbb{K}) \cong \mathbb{K} \otimes_{\mathbb{O}} \mathscr{P}_Z(\mathbb{O}), \quad \mathscr{P}_Z(\Bbbk) \cong \Bbbk \overset{L}{\otimes}_{\mathbb{O}} \mathscr{P}_Z(\mathbb{O});$$

(3) the $\mathbb{O}$-module $\mathsf{F}_{\mathcal{G}}(\mathscr{P}_Z(\mathbb{O}))$ is free of finite rank, and there exist canonical isomorphisms
$$\mathsf{F}_{\mathcal{G}}(\mathscr{P}_Z(\mathbb{K})) \cong \mathbb{K} \otimes_{\mathbb{O}} \mathsf{F}_{\mathcal{G}}(\mathscr{P}_Z(\mathbb{O})), \quad \mathsf{F}_{\mathcal{G}}(\mathscr{P}_Z(\Bbbk)) \cong \Bbbk \otimes_{\mathbb{O}} \mathsf{F}_{\mathcal{G}}(\mathscr{P}_Z(\mathbb{O})).$$

Let us note for later use the following consequence of (2).

Lemma 6.5. *For any $\mu \in \mathbb{X}_*(T)^+_I$ such that $|\mathrm{Gr}^\mu_{\mathcal{G}}| \subset |Z|$ there exist canonical isomorphisms*

$$\mathscr{P}_Z(\mu, \mathbb{K}) \cong \mathbb{K} \otimes_{\mathbb{O}} \mathscr{P}_Z(\mu, \mathbb{O}), \quad \mathscr{P}_Z(\mu, \Bbbk) \cong \Bbbk \overset{L}{\otimes}_{\mathbb{O}} \mathscr{P}_Z(\mu, \mathbb{O}).$$

Proof. The first isomorphism is clear from the construction of the objects $\mathscr{P}_Z(\mu, \mathbb{K})$ and $\mathscr{P}_Z(\mu, \mathbb{O})$ and t-exactness of the functor $\mathbb{K} \otimes_{\mathbb{O}} (-)$. For the second one, property (2) above implies that $\Bbbk \otimes^L_{\mathbb{O}} \mathscr{P}_Z(\mu, \mathbb{O})$ is a perverse sheaf. On the other hand, since the complex (6.4) is concentrated in nonpositive perverse degrees, and since its version over $\Bbbk$ is obtained from the version over $\mathbb{O}$ by application of the functor $\Bbbk \otimes^L_{\mathbb{O}} (-)$, we have $\mathscr{P}_Z(\mu, \Bbbk) \cong {}^{\mathrm{p}}\mathscr{H}^0(\Bbbk \otimes^L_{\mathbb{O}} \mathscr{P}_Z(\mu, \mathbb{O}))$. The desired claim follows. □

Remark 6.6. In case Λ is a field (i.e. if $\Lambda = \mathbb{K}$ or $\Lambda = \Bbbk$), as in [BR18, Proposition 1.12.4] one can check that for any $\mathrm{L}^+\mathcal{G}$-stable closed subscheme $Z \subset \mathrm{Gr}_{\mathcal{G}}$ whose underlying topological subspace is a union of finitely many $\mathrm{L}^+\mathcal{G}$-orbits the category $\mathsf{Perv}([\mathrm{L}^+\mathcal{G}\backslash Z]_{\text{ét}}, \Lambda)$ is a highest weight category with weight poset

$$\{\lambda \in \mathbb{X}_*(T)^+_I \mid |\mathrm{Gr}^\lambda_{\mathcal{G}}| \subset |Z|\}$$

and standard, resp. costandard, objects the objects $\mathscr{J}_!(\lambda, \Lambda)$, resp. $\mathscr{J}_*(\lambda, \Lambda)$. (For generalities on highest weight categories, see [Ric, §7].)

More generally, for any $\mathrm{L}^+\mathcal{G}$-stable *locally closed* subscheme $X \subset \mathrm{Gr}_{\mathcal{G}}$ whose underlying topological subspace is a union of finitely many $\mathrm{L}^+\mathcal{G}$-orbits the category $\mathsf{Perv}([\mathrm{L}^+\mathcal{G}\backslash X]_{\text{ét}}, \Lambda)$ is a highest weight category with weight poset

$$\{\lambda \in \mathbb{X}_*(T)^+_I \mid |\mathrm{Gr}^\lambda_{\mathcal{G}}| \subset |X|\}$$

and standard, resp. costandard, objects the objects $\mathscr{J}_!(\lambda, \Lambda)_{|X}$, resp. $\mathscr{J}_*(\lambda, \Lambda)_{|X}$. In fact, writing $X = Z \smallsetminus Y$ with $Y \subset Z \subset \mathrm{Gr}_{\mathcal{G}}$ as above, $\mathsf{Perv}([\mathrm{L}^+\mathcal{G}\backslash X]_{\text{ét}}, \Lambda)$ is the Serre quotient of the category $\mathsf{Perv}([\mathrm{L}^+\mathcal{G}\backslash Z]_{\text{ét}}, \Lambda)$ by the Serre subcategory $\mathsf{Perv}([\mathrm{L}^+\mathcal{G}\backslash Y]_{\text{ét}}, \Lambda)$, so that the claim follows from [Ric, Lemma 7.8].

6.6. **Construction of $\mathrm{B}_{\mathcal{G}}(\Lambda)$.** If Z is as in §6.4, we set

$$\mathrm{A}_Z(\Lambda) := \mathrm{End}_{\mathsf{Perv}([\mathrm{L}^+\mathcal{G}\backslash Z]_{\text{ét}}, \Lambda)}(\mathscr{P}_Z(\Lambda))^{\mathrm{op}}.$$

Then, since $\mathscr{P}_Z(\Lambda)$ represents the restriction of $\mathsf{F}_{\mathcal{G}}$, we have

$$\mathrm{A}_Z(\Lambda) \cong \mathsf{F}_{\mathcal{G}}(\mathscr{P}_Z(\Lambda))$$

as Λ-modules, hence $\mathrm{A}_Z(\Lambda)$ is free of finite rank over Λ by (3) in §6.5. Since $\mathscr{P}_Z(\Lambda)$ is a projective generator of $\mathsf{Perv}([\mathrm{L}^+\mathcal{G}\backslash Z]_{\text{ét}}, \Lambda)$, by a variant of the Gabriel–Popescu theorem (see e.g. [BR18, Proposition 1.13.1]) one sees that there exists an equivalence of abelian categories

$$\mathsf{S}'_Z \colon \mathsf{Perv}([\mathrm{L}^+\mathcal{G}\backslash Z]_{\text{ét}}, \Lambda) \xrightarrow{\sim} \mathsf{mod}_{\mathrm{A}_Z(\Lambda)}$$

(where the right-hand side is the category of finitely generated $\mathrm{A}_Z(\Lambda)$-modules) whose composition with the forgetful functor $\mathsf{mod}_{\mathrm{A}_Z(\Lambda)} \to \mathsf{mod}_\Lambda$ is the restriction of $\mathsf{F}_{\mathcal{G}}$. If we set

$$\mathrm{B}_Z(\Lambda) := \mathrm{Hom}_\Lambda(\mathrm{A}_Z(\Lambda), \Lambda),$$

we therefore obtain a Λ-coalgebra such that there exists a canonical equivalence of abelian categories

$$\mathsf{S}_Z \colon \mathsf{Perv}([\mathrm{L}^+\mathcal{G}\backslash Z]_{\text{ét}}, \Lambda) \xrightarrow{\sim} \mathsf{comod}_{\mathrm{B}_Z(\Lambda)}$$

whose composition with the forgetful functor $\mathsf{comod}_{\mathrm{B}_Z(\Lambda)} \to \mathsf{mod}_\Lambda$ is the restriction of $\mathsf{F}_{\mathcal{G}}$.

In the setting of §6.5, if $Y \subset Z$ the functor ${}^{\mathrm{p}}\mathscr{H}^0 \circ i^*$ induces a morphism $f'_{Y,Z} \colon \mathrm{A}_Z(\Lambda) \to \mathrm{A}_Y(\Lambda)$ such that the diagram

$$\begin{tikzcd} \mathsf{Perv}([\mathrm{L}^+\mathcal{G}\backslash Y]_{\text{ét}}, \Lambda) \arrow[r, "\mathsf{S}_Y"] \arrow[d, "i_*"'] & \mathsf{mod}_{\mathrm{A}_Y(\Lambda)} \arrow[d] \\ \mathsf{Perv}([\mathrm{L}^+\mathcal{G}\backslash Z]_{\text{ét}}, \Lambda) \arrow[r, "\mathsf{S}_Z"] & \mathsf{mod}_{\mathrm{A}_Z(\Lambda)} \end{tikzcd}$$

commutes, where the right-hand vertical arrow is the restriction-of-scalars functor associated with $f'_{Y,Z}$. Passing to duals we deduce a morphism of coalgebras $f_{Y,Z} \colon \mathrm{B}_Y(\Lambda) \to \mathrm{B}_Z(\Lambda)$ such that the diagram

$$\begin{tikzcd} \mathsf{Perv}([\mathrm{L}^+\mathcal{G}\backslash Y]_{\text{ét}}, \Lambda) \arrow[r, "\mathsf{S}_Y"] \arrow[d, "i_*"'] & \mathsf{comod}_{\mathrm{B}_Y(\Lambda)} \arrow[d] \\ \mathsf{Perv}([\mathrm{L}^+\mathcal{G}\backslash Z]_{\text{ét}}, \Lambda) \arrow[r, "\mathsf{S}_Z"] & \mathsf{comod}_{\mathrm{B}_Z(\Lambda)} \end{tikzcd}$$

commutes, where the right-hand vertical arrow is the functor induced by $f_{Y,Z}$.

Finally we set

$$\mathrm{B}_{\mathcal{G}}(\Lambda) = \mathrm{colim}_Z \, \mathrm{B}_Z(\Lambda)$$

where Z runs over the closed subschemes of $\mathrm{Gr}_{\mathcal{G}}$ whose underlying topological subspace is a finite union of $\mathrm{L}^+\mathcal{G}$-orbits. Then $\mathrm{B}_{\mathcal{G}}(\Lambda)$ is a Λ-coalgebra, and since

$$\mathsf{Perv}(\mathrm{Hk}_{\mathcal{G}}, \Lambda) = \mathrm{colim}_Z \, \mathsf{Perv}([\mathrm{L}^+\mathcal{G}\backslash Z]_{\text{ét}}, \Lambda)$$

we obtain an equivalence of categories

$$\mathsf{S}_{\mathcal{G}} \colon \mathsf{Perv}(\mathrm{Hk}_{\mathcal{G}}, \Lambda) \xrightarrow{\sim} \mathsf{comod}_{\mathrm{B}_{\mathcal{G}}(\Lambda)}$$

whose composition with the forgetful functor $\mathsf{comod}_{\mathrm{B}_{\mathcal{G}}(\Lambda)} \to \mathsf{mod}_\Lambda$ is $\mathsf{F}_{\mathcal{G}}$. Note that in case $\Lambda = \mathbb{O}$, $\mathrm{B}_{\mathcal{G}}(\mathbb{O})$ is flat over $\mathbb{O}$, as a colimit of flat $\mathbb{O}$-modules.

Property (3) in §6.5 ensures that there exist canonical isomorphisms

$$\mathbb{K} \otimes_{\mathbb{O}} \mathrm{B}_{\mathcal{G}}(\mathbb{O}) \xrightarrow{\sim} \mathrm{B}_{\mathcal{G}}(\mathbb{K}), \quad \Bbbk \otimes_{\mathbb{O}} \mathrm{B}_{\mathcal{G}}(\mathbb{O}) \xrightarrow{\sim} \mathrm{B}_{\mathcal{G}}(\Bbbk)$$

such that the obvious diagrams involving the change-of-scalars functors commute.

6.7. **Algebra structure.** To conclude the proof of Theorem 6.1 we need to endow $\mathrm{B}_{\mathcal{G}}(\Lambda)$ with an algebra structure such that $\mathsf{S}_{\mathcal{G}}$ is monoidal. Fix three closed subschemes $X, Y, Z \subset \mathrm{Gr}_{\mathcal{G}}$ whose underlying topological subspaces are unions of finitely many $\mathrm{L}^+\mathcal{G}$-orbits, and such that the restriction of the morphism m from §2.5 to $X \mathbin{\widetilde{\times}} Y$ factors through a morphism $X \mathbin{\widetilde{\times}} Y \to Z$. The tensor product of identity morphisms provides a canonical element in

$$\begin{aligned} A_X(\Lambda) \otimes_\Lambda A_Y(\Lambda) &\cong \mathsf{F}_{\mathcal{G}}(\mathscr{P}_X(\Lambda)) \otimes_\Lambda \mathsf{F}_{\mathcal{G}}(\mathscr{P}_Y(\Lambda)) \\ &\cong \mathsf{F}_{\mathcal{G}}(\mathscr{P}_X(\Lambda) \star^0 \mathscr{P}_Y(\Lambda)) \cong \mathrm{Hom}_{\mathsf{Perv}(\mathrm{Hk}_{\mathcal{G}},\Lambda)}(\mathscr{P}_Z(\Lambda), \mathscr{P}_X(\Lambda) \star^0 \mathscr{P}_Y(\Lambda)) \end{aligned}$$

(where the second isomorphism is provided by the monoidal structure on $\mathsf{F}_{\mathcal{G}}$ constructed in Section 5). Applying the functor $\mathsf{F}_{\mathcal{G}}$ we deduce a morphism $A_Z(\Lambda) \to A_X(\Lambda) \otimes_\Lambda A_Y(\Lambda)$ and then, dualizing, a morphism

$$\mathrm{B}_X(\Lambda) \otimes_\Lambda \mathrm{B}_Y(\Lambda) \to \mathrm{B}_Z(\Lambda).$$

Passing to colimits we finally deduce a morphism

$$\mathrm{B}_{\mathcal{G}}(\Lambda) \otimes_\Lambda \mathrm{B}_{\mathcal{G}}(\Lambda) \to \mathrm{B}_{\mathcal{G}}(\Lambda).$$

It is not difficult to check that this map defines an associative multiplication morphism, with unit element the image of the unit element in $\mathrm{B}_{\mathrm{Gr}^0_{\mathcal{G}}}(\Lambda) = \Lambda$. Moreover, combined with the comultiplication considered above, this multiplication morphism endows $\mathrm{B}_{\mathcal{G}}(\Lambda)$ with a bialgebra structure, such that the equivalence of categories $\mathsf{S}_{\mathcal{G}}$ admits a canonical monoidal structure.

This finishes the proof of Theorem 6.1.

6.8. **Morphisms related to constant term functors.** We finish this section with the construction of morphisms of coalgebras induced by the constant term functors.

Let us consider a parabolic subgroup $P \subset G$ containing A, and its Levi factor M containing A. We can then consider the Λ-coalgebra $\mathrm{B}_{\mathcal{G}}(\Lambda)$ associated with $\mathcal{G}$, in other words with G and its special facet $\mathfrak{a}$, but also the Λ-coalgebra $\mathrm{B}_{\mathcal{M}}(\Lambda)$ associated with $\mathcal{M}$, i.e. with the reductive group M and its special facet $\mathfrak{a}_M$ (see §3.1). Here, $\mathcal{M}$ is the scheme-theoretic closure of M in $\mathcal{G}$, see (3.1).

Proposition 6.7. *For any parabolic subgroup P containing A, there exists a canonical morphism of coalgebras*

$$\mathrm{res}_{\mathcal{P},\mathcal{G}} \colon \mathrm{B}_{\mathcal{G}}(\Lambda) \to \mathrm{B}_{\mathcal{M}}(\Lambda)$$

such that the diagram

$$\begin{tikzcd} \mathsf{Perv}(\mathrm{Hk}_{\mathcal{G}}, \Lambda) \arrow[r, "\mathsf{S}_{\mathcal{G}}"] \arrow[d, "\mathsf{CT}_{\mathcal{P},\mathcal{G}}"'] & \mathsf{comod}_{\mathrm{B}_{\mathcal{G}}(\Lambda)} \arrow[d] \\ \mathsf{Perv}(\mathrm{Hk}_{\mathcal{M}}, \Lambda) \arrow[r, "\mathsf{S}_{\mathcal{M}}"] & \mathsf{comod}_{\mathrm{B}_{\mathcal{M}}(\Lambda)} \end{tikzcd}$$

commutes, where the right vertical arrow is the functor induced by $\mathrm{res}_{\mathcal{P},\mathcal{G}}$. *Moreover these morphisms are compatible with change of scalars in the obvious way, and with parabolic restriction in the sense that given parabolic subgroups $P \subset P' \subset G$ containing A, with Levi factors $M \subset M'$ containing A, we have*

$$\mathrm{res}_{\mathcal{P}\cap\mathcal{M}',\mathcal{M}'} \circ \mathrm{res}_{\mathcal{P}',\mathcal{G}} = \mathrm{res}_{\mathcal{P},\mathcal{G}}, \tag{6.6}$$

where $\mathcal{M}'$ is the scheme-theoretic closure of M' in $\mathcal{G}$.

Proof. The existence of $\mathrm{res}_{\mathcal{P},\mathcal{G}}$ follows from Proposition 4.8, using the standard fact that any exact functor between categories of comodules (finitely generated over Λ) over some Λ-coalgebras compatible with the forgetful functor to mod_{Λ} is induced by a morphism of coalgebras; see e.g. [BR18, Proposition 1.2.6(2)] (where the assumption that the coalgebras are defined over a field is not necessary). The compatibility with change of scalars is obvious. The equality (6.6) follows from Lemma 4.5. □

Remark 6.8. We emphasize that at this stage $\mathrm{res}_{\mathcal{P},\mathcal{G}}$ is only a morphism of *coalgebras*. We will later prove that it is also compatible with products, hence a morphism of bialgebras (see Proposition 9.2), but this fact is not clear for now.

7. The absolute case

7.1. **Absolute variants.** Recall from §2.1 that this paper treats two parallel geometric settings: the "ramified" case (including "unramified groups") involving the group $\mathcal{G}$ (where geometric objects live over $\mathbb{F}$), and the "absolute" case involving the group $F^s[\![z]\!] \otimes_F G$ (where geometric objects live over the field F^s). So far in this paper, we have considered only the ramified case. In fact, up to replacing F^s by its algebraic closure,[5] the absolute case is a special case of the ramified one.

In this section, we consider the absolute setting. We point out that all the constructions and results of Sections 2–6 are applicable in the absolute case. We adapt notation from those sections

[5]Note that, if F^{alg} is an algebraic closure of F^s the extension $F^s \to F^{\mathrm{alg}}$ is purely inseparable, hence the morphism $\mathrm{Spec}(F^{\mathrm{alg}}) \to \mathrm{Spec}(F^s)$ is a universal homeomorphism, see [Sta22, Tag 0BR5]; it therefore induces equivalences between the appropriate categories of sheaves.

by replacing the subscript "$\mathcal{G}$" by "G" throughout. For instance, we write Gr_G instead of $\mathrm{Gr}_{\mathcal{G}}$, F_G instead of $F_{\mathcal{G}}$, L^+G instead of $\mathrm{L}^+\mathcal{G}$, and so on. (To be completely formal, these objects should be be denoted $\mathrm{Gr}_{F^s[\![z]\!]\otimes_F G}$, $\mathsf{F}_{F^s[\![z]\!]\otimes_F G}$, $\mathrm{L}^+(F^s[\![z]\!]\otimes_F G)$, etc.)

The absolute counterparts of most statements from Sections 2–6 have appeared in the literature on the "usual" geometric Satake equivalence, in particular in [MV07, BR18]. However, there is one significant point of departure: the monoidal structure on F_G from Section 5 is constructed very differently from the one described in [MV07], and it is a priori not clear whether they coincide.

The main result of this section (Theorem 7.6) asserts that when $\Lambda = \mathbb{K}$, they do coincide. The most important consequence is that in this special case, the monoidal structure from Section 5 admits a *commutativity constraint*. We will see later in the paper (Corollary 8.9 and Proposition 9.1) how to transfer this commutativity to the ramified case and to general Λ.

7.2. **The fusion monoidal structure via equivariant cohomology.** Let us denote by

$$\phi_{\mathrm{fus}}\colon \mathsf{F}_G(\mathscr{F})\otimes_\Lambda \mathsf{F}_G(\mathscr{G}) \xrightarrow{\sim} \mathsf{F}_G(\mathscr{F}\star^0\mathscr{G}) \tag{7.1}$$

the natural isomorphism constructed in [MV07, Proposition 6.4] or [BR18, Proposition 1.10.11]. The subscript "fus" refers to the "fusion product," which is used in the construction of this isomorphism. We will not review the details of the construction in general.

However, in the special case where $\Lambda = \mathbb{K}$, ϕ_{fus} has an alternative description in terms of equivariant cohomology, as explained in [AR, §3.3.4]. In this subsection, we recall this alternative description.[6]

For the remainder of this subsection, we assume that $\Lambda = \mathbb{K}$. Let $R_{G,\mathbb{K}}$ be the L^+G-equivariant cohomology of a point, i.e., the graded $\mathbb{K}$-algebra given by

$$R_{G,\mathbb{K}} := \bigoplus_{n\in\mathbb{Z}} \mathsf{H}^n_{\mathrm{L}^+G}(\mathrm{Spec}(F^s),\mathbb{K}) = \bigoplus_{n\in\mathbb{Z}} \mathrm{Hom}_{D^{\mathrm{b}}_{\mathrm{c}}([\mathrm{Spec}(F^s)/\mathrm{L}^+G]_{\mathrm{\acute{e}t}},\mathbb{K})}(\underline{\mathbb{K}},\underline{\mathbb{K}}[n]).$$

The ring $\mathbb{K}$, considered as a graded object concentrated in degree 0, has a natural structure of graded $R_{G,\mathbb{K}}$-module.

More generally, for any Levi subgroup $M \subset G$, we can define the ring $R_{M,\mathbb{K}}$ in the same way, and there is an obvious map of equivariant cohomology rings

$$R_{G,\mathbb{K}} \to R_{M,\mathbb{K}}. \tag{7.2}$$

It is well known that in the case of a torus, we have that $R_{T,\mathbb{K}}$ is the graded symmetric algebra on

$$\mathsf{H}^2_{\mathrm{L}^+T}(\mathrm{Spec}(F^s),\mathbb{K}) \cong \mathbb{K}\otimes_{\mathbb{Z}} \mathbb{X}^*(T).$$

The (absolute) Weyl group W_{abs} of (G_{F^s}, T_{F^s}) acts on this ring, and the map (7.2) (in the special case $M = T$) induces an isomorphism

$$R_{G,\mathbb{K}} \xrightarrow{\sim} R_{T,\mathbb{K}}^{W_{\mathrm{abs}}}.$$

For $\mathscr{F}\in D^{\mathrm{b}}_{\mathrm{c}}(\mathrm{Hk}_G,\mathbb{K})$ and $n\in\mathbb{Z}$ we set

$$\mathsf{H}^n_{\mathrm{L}^+G}(\mathrm{Gr}_G,\mathscr{F}) := \mathrm{Hom}_{D^{\mathrm{b}}_{\mathrm{c}}(\mathrm{Hk}_G,\mathbb{K})}(\underline{\mathbb{K}}_X,\mathscr{F}[n]),$$

where $\underline{\mathbb{K}}_X$ is (the object whose image in $D^{\mathrm{b}}_{\mathrm{c}}(\mathrm{Gr}_G,\mathbb{K})$ is) the constant sheaf on some L^+G-stable closed subscheme X that contains the support of $\mathscr{F}$. As above, the same definition applies when G is replaced by a Levi subgroup M, e.g. by T.

---

[6] In fact, the equivariant cohomology description is available for any Λ in which the torsion primes of G are invertible. As the case $\Lambda = \mathbb{K}$ is sufficient for our proof of Theorem 7.6, we will not consider more general rings here.

Let $\mathsf{gMod}_{R_{G,\mathbb{K}}}$ be the category of graded $R_{G,\mathbb{K}}$-modules. Define the functor

$$\mathsf{F}_G^{\mathrm{eq}} \colon D_{\mathrm{c}}^{\mathrm{b}}(\mathrm{Hk}_G, \mathbb{K}) \to \mathsf{gMod}_{R_{G,\mathbb{K}}}$$

by

$$\mathsf{F}_G^{\mathrm{eq}}(\mathscr{F}) := \bigoplus_{n \in \mathbb{Z}} \mathsf{H}^n_{\mathrm{L}^+G}(\mathrm{Gr}_G, \mathscr{F}).$$

Let $M \subset G$ be a Levi subgroup, and recall the natural morphism

$$h_{M,G} \colon [\mathrm{L}^+M \backslash \mathrm{Gr}_G]_{\text{ét}} \to \mathrm{Hk}_G.$$

Then we can also consider L^+M-equivariant cohomology of complexes on Hk_G: for $\mathscr{F}$ in $D_{\mathrm{c}}^{\mathrm{b}}(\mathrm{Hk}_G, \mathbb{K})$ we set

$$\mathsf{F}_G^{\mathrm{eq},M}(\mathscr{F}) := \bigoplus_{n \in \mathbb{Z}} \mathsf{H}^n_{\mathrm{L}^+M}(\mathrm{Gr}_G, h_{M,G}^* \mathscr{F}).$$

The next statements are well known; see e.g. [Zhu17, §5.2] (see also [AR, §3.3.4] for analogues in an "analytic" setting).

Lemma 7.1. *For any $\mathscr{F} \in \mathsf{Perv}(\mathrm{Hk}_G, \mathbb{K})$, $\mathsf{F}_G^{\mathrm{eq}}(\mathscr{F})$ is a finitely generated projective $R_{G,\mathbb{K}}$-module. Moreover, there is a natural isomorphism*

$$\mathsf{F}_G(\mathscr{F}) \cong \mathbb{K} \otimes_{R_{G,\mathbb{K}}} \mathsf{F}_G^{\mathrm{eq}}(\mathscr{F}) \tag{7.3}$$

and, for any Levi subgroup $M \subset G$, a natural isomorphism

$$R_{M,\mathbb{K}} \otimes_{R_{G,\mathbb{K}}} \mathsf{F}_G^{\mathrm{eq}}(\mathscr{F}) \cong \mathsf{F}_G^{\mathrm{eq},M}(h_{M,G}^* \mathscr{F}). \tag{7.4}$$

Let $\mathsf{gproj}_{R_{G,\mathbb{K}}}$ be the category of finitely generated projective $R_{G,\mathbb{K}}$-modules. In view of Lemma 7.1, one can regard $\mathsf{F}_G^{\mathrm{eq}}$ as a functor

$$\mathsf{F}_G^{\mathrm{eq}} \colon \mathsf{Perv}(\mathrm{Hk}_G, \mathbb{K}) \to \mathsf{gproj}_{R_{G,\mathbb{K}}}.$$

Proposition 7.2. *For $\mathscr{F}, \mathscr{G} \in \mathsf{Perv}(\mathrm{Hk}_G, \mathbb{K})$, there is a natural isomorphism*

$$\mathsf{F}_G^{\mathrm{eq}}(\mathscr{F} \star \mathscr{G}) \cong \mathsf{F}_G^{\mathrm{eq}}(\mathscr{F}) \otimes_{R_{G,\mathbb{K}}} \mathsf{F}_G^{\mathrm{eq}}(\mathscr{G}),$$

making $\mathsf{F}_G^{\mathrm{eq}} \colon D_{\mathrm{c}}^{\mathrm{b}}(\mathrm{Hk}_G, \mathbb{K}) \to \mathsf{gproj}_{R_{G,\mathbb{K}}}$ into a monoidal functor.

The preceding two results give rise to a monoidal structure on

$$\mathsf{F}_G \colon (\mathsf{Perv}(\mathrm{Hk}_G, \mathbb{K}), \star) \to (\mathsf{mod}_{\mathbb{K}}, \otimes_{\mathbb{K}})$$

as follows: Lemma 7.1 expresses F_G as the composition of the functor

$$\mathbb{K} \otimes_{R_{G,\mathbb{K}}} (-) \colon \mathsf{gproj}_{R_{G,\mathbb{K}}} \to \mathsf{mod}_{\mathbb{K}}$$

with $\mathsf{F}_G^{\mathrm{eq}} \colon \mathsf{Perv}(\mathrm{Hk}_G, \mathbb{K}) \to \mathsf{gproj}_{R_{G,\mathbb{K}}}$. The former is obviously monoidal, and the latter is monoidal by Proposition 7.2, so F_G is monoidal as well.

7.3. **Comparison of the monoidal structures.** In this subsection we compare the monoidal structures on F_G constructed in (5.1) and (7.1).

Proposition 7.3. *For $\mathscr{F} \in \mathsf{Perv}(\mathrm{Hk}_G, \mathbb{K})$, there is a natural isomorphism*

$$R_{T,\mathbb{K}} \otimes_{R_{G,\mathbb{K}}} \mathsf{F}_G^{\mathrm{eq}}(\mathscr{F}) \cong \mathsf{F}_T^{\mathrm{eq}}(\mathsf{CT}_{B,G}(\mathscr{F})). \tag{7.5}$$

Moreover, this isomorphism has the property that the following diagram commutes, where the upper vertical arrows are induced by the augmentation map $R_{T,\mathbb{K}} \to \mathbb{K}$:

$$\begin{tikzcd}
R_{T,\mathbb{K}} \otimes_{R_{G,\mathbb{K}}} \mathsf{F}_G^{\mathrm{eq}}(\mathscr{F}) \arrow[r, "(7.5)", "\sim"'] \arrow[d] & \mathsf{F}_T^{\mathrm{eq}}(\mathsf{CT}_{B,G}(\mathscr{F})) \arrow[d] \\
\mathbb{K} \otimes_{R_{G,\mathbb{K}}} \mathsf{F}_G^{\mathrm{eq}}(\mathscr{F}) \arrow[d, "(7.3)"', "\wr"] & \mathbb{K} \otimes_{R_{T,\mathbb{K}}} \mathsf{F}_T^{\mathrm{eq}}(\mathsf{CT}_{B,G}(\mathscr{F})) \arrow[d, "\wr"', "(7.3)"] \\
\mathsf{F}_G(\mathscr{F}) \arrow[r, "\text{Prop. 4.8}", "\sim"'] & \mathsf{F}_T(\mathsf{CT}_{B,G}(\mathscr{F})).
\end{tikzcd}$$

Proof. Recall the functor $\mathsf{F}_G^{\mathrm{eq},T}$ considered above. In view of (7.4), to prove the proposition it suffices to establish a version of the commutative diagram above in which the upper left-hand corner is replaced by $\mathsf{F}_G^{\mathrm{eq},T}(\mathscr{F})$. The desired isomorphism

$$\mathsf{F}_G^{\mathrm{eq},T}(\mathscr{F}) \cong \mathsf{F}_T^{\mathrm{eq}}(\mathsf{CT}_{B,G}(\mathscr{F}))$$

can be obtained by copying the proof of Proposition 4.8 and replacing all occurrences of ordinary cohomology by L^+T-equivariant cohomology. (See [YZ11, Lemma 2.2] for similar considerations.) The commutativity of the diagram is then immediate from the construction. □

Proposition 7.4. *For $\mathscr{F}, \mathscr{G} \in \mathsf{Perv}(\mathrm{Hk}_G, \mathbb{K})$, there is a natural isomorphism*

$$R_{T,\mathbb{K}} \otimes_{R_{G,\mathbb{K}}} \mathsf{F}_G^{\mathrm{eq}}(\mathscr{F} \star \mathscr{G}) \cong \mathsf{F}_T^{\mathrm{eq}}(m_{T!}\widetilde{\mathsf{CT}}_{B,G}(p^*(\mathscr{F} \overset{L}{\boxtimes}_{\mathbb{K}} \mathscr{G}))). \tag{7.6}$$

Moreover, this isomorphism has the property that the following diagram commutes:

$$\begin{tikzcd}
R_{T,\mathbb{K}} \otimes_{R_{G,\mathbb{K}}} \mathsf{F}_G^{\mathrm{eq}}(\mathscr{F} \star \mathscr{G}) \arrow[r, "(7.6)", "\sim"'] \arrow[d] & \mathsf{F}_T^{\mathrm{eq}}(m_{T!}\widetilde{\mathsf{CT}}_{B,G}(p^*(\mathscr{F} \boxtimes^L_{\mathbb{K}} \mathscr{G}))) \arrow[d] \\
\mathbb{K} \otimes_{R_{G,\mathbb{K}}} \mathsf{F}_G^{\mathrm{eq}}(\mathscr{F} \star \mathscr{G}) \arrow[d, "(7.3)"', "\wr"] & \mathbb{K} \otimes_{R_{T,\mathbb{K}}} \mathsf{F}_T^{\mathrm{eq}}(m_{T!}\widetilde{\mathsf{CT}}_{B,G}(p^*(\mathscr{F} \boxtimes^L_{\mathbb{K}} \mathscr{G}))) \arrow[d, "\wr"', "(7.3)"] \\
\mathsf{F}_G(\mathscr{F} \star \mathscr{G}) \arrow[r, "\text{Prop. 4.18}", "\sim"'] & \mathsf{F}_T(m_{T!}\widetilde{\mathsf{CT}}_{B,G}(p^*(\mathscr{F} \boxtimes^L_{\mathbb{K}} \mathscr{G}))).
\end{tikzcd}$$

Proof. This is very similar to the proof of Proposition 7.3, replacing the details from the proof of Proposition 4.8 by those from Proposition 4.18. □

The following lemma is clear from constructions.

Lemma 7.5. *For $\mathscr{F}, \mathscr{G} \in \mathsf{Perv}(\mathrm{Hk}_G, \mathbb{K})$, the following diagram commutes:*

$$\begin{tikzcd}
\mathsf{F}_T^{\mathrm{eq}}(m_{T!}\widetilde{\mathsf{CT}}_{B,G}(p_G^*(\mathscr{F} \boxtimes^L_{\mathbb{K}} \mathscr{G}))) \arrow[r, "\text{Cor. 5.3}", "\sim"'] \arrow[dd, "\text{Prop. 7.4}"', "\wr"] & \mathsf{F}_T^{\mathrm{eq}}(\mathsf{CT}_{B,G}(\mathscr{F}) \star^0 \mathsf{CT}_{B,G}(\mathscr{G})) \arrow[d, "\wr"', "\text{Prop. 7.2}"] \\
& \mathsf{F}_T^{\mathrm{eq}}(\mathsf{CT}_{B,G}(\mathscr{F})) \otimes_{R_{T,\mathbb{K}}} \mathsf{F}_T^{\mathrm{eq}}(\mathsf{CT}_{B,G}(\mathscr{G})) \arrow[d, "\wr"', "\text{Prop. 7.3}"] \\
R_{T,\mathbb{K}} \otimes_{R_{G,\mathbb{K}}} \mathsf{F}_G^{\mathrm{eq}}(\mathscr{F} \star \mathscr{G}) \arrow[r, "\text{Prop. 7.2}", "\sim"'] & R_{T,\mathbb{K}} \otimes_{R_{G,\mathbb{K}}} \mathsf{F}_G^{\mathrm{eq}}(\mathscr{F}) \otimes_{R_{G,\mathbb{K}}} \mathsf{F}_G^{\mathrm{eq}}(\mathscr{G}).
\end{tikzcd}$$

For the next statement, we come back to the setting of a general ring of coefficients Λ as in §4.1.

Theorem 7.6. *The monoidal structures on* $\mathsf{F}_G \colon \mathsf{Perv}(\mathrm{Hk}_G, \Lambda) \to \mathsf{mod}_\Lambda$ *considered in* (5.1) *and* (7.1) *agree.*

Proof. We must show that for $\mathscr{F}, \mathscr{G} \in \mathsf{Perv}(\mathrm{Hk}_G, \Lambda)$, the two isomorphisms

$$\phi, \phi_{\mathrm{fus}} \colon \mathsf{F}_G(\mathscr{F}) \otimes_\Lambda \mathsf{F}_G(\mathscr{G}) \xrightarrow{\sim} \mathsf{F}_G(\mathscr{F} \star^0 \mathscr{G})$$

are equal. Before proving this in general, we consider a number of special cases.

Case 1. $\Lambda = \mathbb{K}$ and $G = T$. In this case, the monoidal structure in (7.1) is provided by Lemma 5.1, and the agreement with the structure provided by Proposition 7.2 is clear.

Case 2. $\Lambda = \mathbb{K}$, and G is arbitrary. Apply the functor $\mathbb{K} \otimes_{R_{T,\mathbb{K}}} (-)$ to the commutative diagram in Lemma 7.5. Using Lemma 7.1, the commutative diagrams in Propositions 7.3 and 7.4, and Case 1 above, the resulting diagram can be written as

$$\begin{tikzcd}
\mathsf{F}_T(m_{T!}\widetilde{\mathsf{CT}}_{B,G}(p_G^*(\mathscr{F} \boxtimes^L_{\mathbb{K}} \mathscr{G}))) \arrow[r, "\text{Cor. 5.3}", "\sim"'] \arrow[dd, "\text{Prop. 4.18}"', "\wr"] & \mathsf{F}_T(\mathsf{CT}_{B,G}(\mathscr{F}) \star^0 \mathsf{CT}_{B,G}(\mathscr{G})) \arrow[d, "\text{Lem. 5.1}", "\wr"'] \\
& \mathsf{F}_T(\mathsf{CT}_{B,G}(\mathscr{F})) \otimes_{\mathbb{K}} \mathsf{F}_T(\mathsf{CT}_{B,G}(\mathscr{G})) \arrow[d, "\text{Prop. 4.8}", "\wr"'] \\
\mathsf{F}_G(\mathscr{F} \star \mathscr{G}) \arrow[r, "(7.1)", "\sim"'] & \mathsf{F}_G(\mathscr{F}) \otimes_{\mathbb{K}} \mathsf{F}_G(\mathscr{G}).
\end{tikzcd}$$

The result follows by comparison with the commutative diagram in Remark 5.5.

Case 3. $\Lambda = \mathbb{O}$, and $\mathsf{F}_G(\mathscr{F})$ and $\mathsf{F}_G(\mathscr{G})$ are free over $\mathbb{O}$. This case follows from Case 2 by compatibility of our constructions with appropriate change-of-scalars functors.

Case 4. $\Lambda = \mathbb{O}$, and $\mathscr{F}$ and $\mathscr{G}$ are both direct sums of objects of the form $\mathscr{P}_Z(\mathbb{O})$. By Property (3) in §6.5, this is a special case of Case 3.

Case 5. $\Lambda = \Bbbk$, and $\mathscr{F}$ and $\mathscr{G}$ are both direct sums of objects of the form $\mathscr{P}_Z(\Bbbk)$. This case follows from Case 4 by Property (2) in §6.5 and compatibility with change of scalars.

Case 6. $\Lambda = \mathbb{O}$ or $\Bbbk$, and $\mathscr{F}$ and $\mathscr{G}$ are arbitrary. For $\Lambda = \mathbb{O}$ or $\Bbbk$, any object in $\mathsf{Perv}(\mathrm{Hk}_G, \Lambda)$ is a quotient of a direct sum of objects of the form $\mathscr{P}_Z(\Lambda)$, so the result in this case follows from Cases 4 and 5. □

7.4. **"Absolute" geometric Satake equivalence.** As mentioned already in §7.1 the structures we have considered on the category $D^{\mathrm{b}}_{\mathrm{c}}(\mathrm{Hk}_{\mathcal{G}}, \Lambda)$ above have well known (and older) counterparts for the group $F^s[\![z]\!] \otimes_{F^s} G_{F^s}$. In particular we have the stack Hk_G over $\mathrm{Spec}(F^s)$, the convolution product $\star$ on the category $D^{\mathrm{b}}_{\mathrm{c}}(\mathrm{Hk}_G, \Lambda)$ defining a monoidal structure, the induced monoidal structure on the subcategory $\mathsf{Perv}(\mathrm{Hk}_G, \Lambda)$ of perverse sheaves obtained by setting

$$\mathscr{F} \star^0 \mathscr{G} := {}^{\mathrm{p}}\mathscr{H}^0(\mathscr{F} \star \mathscr{G}),$$

and the total cohomology functor F_G. Given a parabolic subgroup $Q \subset G_{F^s}$ containing the maximal torus T_{F^s}, with Levi factor containing T_{F^s} denoted L, we also have the constant term functor

$$\mathsf{CT}_{Q,G} \colon D^{\mathrm{b}}_{\mathrm{c}}(\mathrm{Hk}_G, \Lambda) \to D^{\mathrm{b}}_{\mathrm{c}}(\mathrm{Hk}_L, \Lambda)$$

which is t-exact and admits a canonical monoidal structure.

On the other hand, denote by $G^\vee_{\mathbb{Z}}$ the unique pinned reductive group scheme over $\mathbb{Z}$ whose root datum is the dual of the root datum of G_{F^s}. Then, we set

$$G^\vee_\Lambda := \Lambda \otimes_{\mathbb{Z}} G^\vee_{\mathbb{Z}}.$$

We will denote by $\mathsf{Rep}(G^\vee_\Lambda)$ the category of (algebraic) representations of this group scheme on finitely generated Λ-modules.

Theorem 7.7. *Fix a compatible system of ℓ^n-th roots of unity in $\mathbb{F}$ for all $n \geq 1$. Then, there exists a canonical equivalence of monoidal categories*

$$\big(\mathsf{Perv}(\mathrm{Hk}_G, \Lambda), \star^0\big) \cong \big(\mathsf{Rep}(G^\vee_\Lambda), \otimes_\Lambda\big)$$

under which the functor F_G corresponds to the obvious forgetful functor $\mathsf{Rep}(G^\vee_\Lambda) \to \mathsf{mod}_\Lambda$. Moreover, under these equivalences the "change of scalars" functors analogous so those considered in §4.1 correspond to the obvious functors on categories of representations.

Proof sketch. This statement is essentially the usual version of the geometric Satake equivalence from [MV07]. Let us indicate the main steps of the proof.

First, for any connected reductive group H over F^s with Borel subgroup $B_H \subset H$ and maximal torus $T_H \subset B_H$, the absolute version of Theorem 6.1 gives us a bialgebra $\mathrm{B}_H(\Lambda)$ and an equivalence of monoidal categories

$$\big(\mathsf{Perv}(\mathrm{Hk}_{F^s[\![z]\!]\otimes_{F^s} H}, \Lambda), \star^0\big) \xrightarrow{\sim} \big(\mathsf{comod}_{\mathrm{B}_H(\Lambda)}, \otimes_\Lambda\big).$$

It is shown by Mirković–Vilonen [MV07, §11] (see also [BR18, §1.13.2]) that $\mathrm{B}_H(\Lambda)$ is in fact a commutative Hopf algebra, so that

$$\widehat{H}_\Lambda := \mathrm{Spec}(\mathrm{B}_H(\Lambda))$$

is a flat affine group scheme over Λ. Thus, Theorem 6.1 can be restated as an equivalence of monoidal categories

$$\big(\mathsf{Perv}(\mathrm{Hk}_{F^s[\![z]\!]\otimes_{F^s} H}, \Lambda), \star^0\big) \cong \big(\mathsf{Rep}(\widehat{H}_\Lambda), \otimes_\Lambda\big).$$

This construction is compatible with change of scalars in the sense that there are canonical identifications

$$\widehat{H}_\mathbb{K} \cong \mathbb{K} \otimes_\mathbb{O} \widehat{H}_\mathbb{O}, \quad \widehat{H}_\Bbbk \cong \Bbbk \otimes_\mathbb{O} \widehat{H}_\mathbb{O}.$$

Next, in [MV07, §12] (see also [BR18, §1.14]), Mirković–Vilonen construct (using the constant term functor $\mathsf{CT}_{B_H,H}$) a canonical subgroup $\widehat{T}_{H,\Lambda}$ of $\widehat{H}_\Lambda$ canonically isomorphic to the diagonalizable group $\mathrm{D}_\Lambda(\mathbb{X}_*(T_H))$ over Λ associated with the lattice $\mathbb{X}_*(T_H)$ of cocharacters of T_H; this construction is also compatible with extension of scalars in the same sense as above. They then check that $\widehat{H}_\mathbb{O}$ is a split reductive group scheme over $\mathbb{O}$ with maximal torus $\widehat{T}_{H,\mathbb{O}}$, and that the root datum of $(\widehat{H}_\mathbb{O}, \widehat{T}_{H,\mathbb{O}})$ is dual to that of (H, T_H). The theorem will follow from this construction (applied to the group $H = G_{F^s}$, its Borel subgroup B_{F^s} and the maximal torus T_{F^s}) once we explain how to construct a canonical pinning on $\widehat{H}_\mathbb{O}$.

This can be done as follows, following e.g. [FS21, §VI.11].[7] First, one has a canonical Borel subgroup $\widehat{B}_{H,\mathbb{O}}$ (i.e. a choice of a system of positive roots) obtained as the subgroup stabilizing the filtration of the fiber functor F_H on the category $\mathsf{Perv}(\mathrm{Hk}_{F^s[\![z]\!]\otimes_{F^s} H}, \mathbb{O})$ given by

$$(\mathsf{F}_H)^{\geq m}(\mathscr{F}) = \bigoplus_{i \geq m} \mathsf{H}^i(\mathrm{Gr}_{F^s[\![z]\!]\otimes_{F^s} H}, \mathscr{F}).$$

The only remaining structure we have to construct is a basis of each weight space in the Lie algebra of $\widehat{H}_\mathbb{O}$ corresponding to a simple root. Using an appropriate constant term functor one reduces the construction to the case where H is of semisimple rank 1. Then $H' := H/Z(H)$ is isomorphic to PGL_{2,F^s}, hence its affine Grassmannian $\mathrm{Gr}_{F^s[\![z]\!]\otimes_{F^s} H'}$ has a unique 1-dimensional Schubert variety isomorphic to $\mathbb{P}^1_{F^s}$, and $\widehat{H'}_\mathbb{O}$ identifies with the special linear group of the total cohomology of this variety, see [FS21, Comments after Lemma VI.11.2]. The zeroth cohomology of this orbit has a canonical basis, and using the system of ℓ^n-th roots of unity in order to

[7] This construction was known for a long time, but [FS21] provides a convenient and explicit construction.

trivialize Tate twists, so does the second cohomology. We get an identification $\widehat{H'}_{\mathbb{O}} = \mathrm{SL}_{2,\mathbb{O}}$, and in particular a canonical pinning on $\widehat{H'}_{\mathbb{O}}$. Finally, we have

$$(\mathrm{Gr}_{F^s[\![z]\!]\otimes_{F^s} H})_{\mathrm{red}} \cong \pi_1(H) \times_{\pi_1(H')} (\mathrm{Gr}_{F^s[\![z]\!]\otimes_{F^s} H'})_{\mathrm{red}}$$

(where $\pi_1(H)$ is the algebraic fundamental group of H, and similarly for H'), which provides an isomorphism

$$\widehat{H}_{\mathbb{O}} = \mathrm{D}_{\mathbb{O}}(\pi_1(H)) \times^{\mu_{2,\mathbb{O}}} \widehat{H'}_{\mathbb{O}}$$

compatible in the natural way with the canonical maximal torus on each side. (See the proof of Proposition 9.1 below for a detailed version of similar considerations.) We deduce the desired pinning of $\widehat{H}_{\mathbb{O}}$, which finishes the proof. □

7.5. **Galois action.** In §7.4 we have considered the "usual" affine Grassmannian Gr_G of the F^s-group G_{F^s}, an ind-scheme over F^s. In fact one can also consider the affine Grassmannian $\mathrm{Gr}_{F[\![z]\!]\otimes_F G}$ associated with the (reductive, but possibly nonsplit) F-group G (an ind-scheme over F), and we have a canonical identification

$$F^s \otimes_F \mathrm{Gr}_{F[\![z]\!]\otimes_F G} \xrightarrow{\sim} \mathrm{Gr}_G,$$

see (2.2). The Galois group $I = \mathrm{Gal}(F^s/F)$ acts on the left-hand side via its action on $\mathrm{Spec}(F^s)$, which provides an action on Gr_G. (Here I acts by automorphisms as F-ind-scheme, but not as F^s-ind-scheme.)

From the action of the group I on Gr_G we deduce an action on the categories $D^{\mathrm{b}}_{\mathrm{c}}(\mathrm{Hk}_G, \Lambda)$ and $\mathsf{Perv}(\mathrm{Hk}_G, \Lambda)$, which is easily seen to be compatible with the convolution products $\star$ and $\star^0$ respectively. The functor F_G is also invariant under these actions. By Tannakian formalism, and in view of Theorem 7.7, this means that there exists an action of I on $G^\vee_\Lambda$ (by automorphisms of group scheme over Λ) which induces the corresponding action on $\mathsf{Perv}(\mathrm{Hk}_G, \Lambda)$.

Lemma 7.8. *The action of I on $G^\vee_\Lambda$ preserves the canonical pinning constructed in the course of the proof of Theorem 7.7, and factors through an action of a finite quotient.*

Proof. The fact that the canonical maximal torus of $G^\vee_\Lambda$ is preserved by the action of I follows from the fact that this torus identifies with the centralizer of the cocharacter provided by the cohomological grading. (Alternatively, one may prove this independence by noting that the maximal torus does not depend on the choices involved in its construction, see [BR18, §1.5.5].) The induced action on the canonical maximal torus $\mathrm{D}_\Lambda(\mathbb{X}_*(T))$ is via the natural I-action on $\mathbb{X}_*(T)$. Since T splits over a finite extension of F, this implies in particular that there exists a normal subgroup $I' \subset I$ of finite index which acts trivially on this torus.

The canonical Borel subgroup of $G^\vee_\Lambda$ is also manifestly stable under the I-action. The fact that the canonical simple root vectors are permuted is clear from their construction, since I permutes the Levi subgroups of G_{F^s} of semisimple rank 1 according to its permutation of absolute simple roots. Finally, since a pinning-preserving automorphism is determined by the induced action on the root datum, the subgroup I' considered above acts trivially on $G^\vee_\Lambda$, showing that the action of I factors through a finite quotient. □

It is clear that the action of I is compatible with extension-of-scalars, in the sense that the actions on $G^\vee_{\mathbb{K}}$ and $G^\vee_{\mathbb{k}}$ are induced by the action on $G^\vee_{\mathbb{O}}$ via the canonical identifications

$$G^\vee_{\mathbb{K}} = \mathbb{K} \otimes_{\mathbb{O}} G^\vee_{\mathbb{O}}, \quad G^\vee_{\mathbb{k}} = \mathbb{k} \otimes_{\mathbb{O}} G^\vee_{\mathbb{O}}.$$

8. Nearby cycles

8.1. **Beĭlinson–Drinfeld Grassmannians.** We now consider a "Beĭlinson–Drinfeld Grassmannian" relating the ind-schemes $\mathrm{Gr}_{\mathcal{G}}$ and Gr_G. We follow the purely local definition from [Ric21, §0.3]. For any O_F-algebra R we equip $R[\![z]\!]$ with the O_F-algebra structure given by the unique $\mathbb{F}$-algebra map $O_F = \mathbb{F}[\![t]\!] \to R[\![z]\!]$ such that $t \mapsto z + t$.

Set $S := \mathrm{Spec}(O_F)$. The *Beĭlinson–Drinfeld Grassmannian* $\mathrm{Gr}_{\mathcal{G},S}$ associated with $\mathcal{G}$ is the functor from the category of O_F-algebras to sets sending R to the set of isomorphism classes of pairs $(\mathcal{E}, \beta)$ where:

- $\mathcal{E}$ is a $\mathcal{G} \times_S \mathrm{Spec}(R[\![z]\!])$-torsor on $\mathrm{Spec}(R[\![z]\!])$;
- β is a trivialization of $\mathcal{E}$ on $\mathrm{Spec}(R(\!(z)\!))$.

It is proven in [Ric16, Theorem 2.19, using Lemma 3.1] that this functor is represented by an ind-projective ind-scheme over S. Furthermore, it admits a loop uniformization, i.e. it can be written as the étale quotient of the full loop group $\mathrm{L}_S\mathcal{G}$ by the positive loop group $\mathrm{L}_S^+\mathcal{G}$. Here $\mathrm{L}_S\mathcal{G}$ and $\mathrm{L}_S^+\mathcal{G}$ are the group-valued functors on the category of O_F-algebras R given by $\mathcal{G}(R(\!(z)\!))$ and $\mathcal{G}(R[\![z]\!])$, respectively.

Remark 8.1. As explained in [Ric21, §0.3], the ind-scheme $\mathrm{Gr}_{\mathcal{G},S}$ agrees with the base change of the Beĭlinson–Drinfeld Grassmannian constructed using a spreading of $\mathcal{G}$ over some curve as follows. By [Ric16, Lemma 3.1] (or [HR21, §5.1.1]) there exists a smooth affine connected $\mathbb{F}$-curve X with a point $x_0 \in X(\mathbb{F})$, an identification $\hat{\mathcal{O}}_{X,x_0} = O_F$ on completed local rings, and a smooth affine X-group scheme $\underline{\mathcal{G}}$ together with an identification of O_F-group schemes $\underline{\mathcal{G}} \times_X S = \mathcal{G}$. One then has the Beĭlinson–Drinfeld Grassmannian $\mathrm{Gr}_{\underline{\mathcal{G}},X}$ associated with $\underline{\mathcal{G}}$ and X. The base change $\mathrm{Gr}_{\underline{\mathcal{G}},X} \times_X S$ agrees with $\mathrm{Gr}_{\mathcal{G},S}$ as defined above.

8.2. **A nearby cycles setting.** Let us describe the fibers of $\mathrm{Gr}_{\mathcal{G},S}$ in more detail, following [Ric16, §2.2]. Let $s = \mathrm{Spec}(\mathbb{F})$, resp. $\eta = \mathrm{Spec}(F)$, be the special, resp. generic, point of S. As explained in [Ric16, Corollary 2.14], we have a canonical identification

$$s \times_S \mathrm{Gr}_{\mathcal{G},S} \cong \mathrm{Gr}_{\mathcal{G}}. \tag{8.1}$$

On the other hand, $F[\![z]\!] \otimes_{O_F} \mathcal{G}$ is a reductive group scheme over $F[\![z]\!]$, and by [Ric21, Lemma 0.2] there exists an isomorphism

$$F[\![z]\!] \otimes_{O_F} \mathcal{G} \cong F[\![z]\!] \otimes_F G$$

where in the right-hand side we consider the obvious F-algebra morphism $F \to F[\![z]\!]$; we fix once and for all such an isomorphism that maps

$$F[\![z]\!] \otimes_{O_F} \mathcal{A}, \quad \text{resp.} \quad F[\![z]\!] \otimes_{O_F} \mathcal{T}, \quad \text{resp.} \quad F[\![z]\!] \otimes_{O_F} \mathcal{B}$$

into

$$F[\![z]\!] \otimes_F A, \quad \text{resp.} \quad F[\![z]\!] \otimes_F T, \quad \text{resp.} \quad F[\![z]\!] \otimes_F B.$$

Similarly to (8.1), this provides an identification

$$\eta \times_S \mathrm{Gr}_{\mathcal{G},S} \cong \mathrm{Gr}_{F[\![z]\!] \otimes_F G}.$$

Let also $\overline{S}$ be the spectrum of the normalization $\overline{O_F}$ of O_F in F^s, and set

$$\mathrm{Gr}_{\mathcal{G},\overline{S}} := \mathrm{Gr}_{\mathcal{G},S} \times_S \overline{S}.$$

Then $\overline{O_F}$ is a valuation ring with fraction field F^s and residue field $\mathbb{F}$. Let $\overline{\eta} = \mathrm{Spec}(F^s)$ and $\overline{s} = \mathrm{Spec}(\mathbb{F})$ be the generic and special points of $\overline{S}$, respectively. We have

$$\overline{s} \times_{\overline{S}} \mathrm{Gr}_{\mathcal{G},\overline{S}} = s \times_S \mathrm{Gr}_{\mathcal{G},S} = \mathrm{Gr}_{\mathcal{G}}$$

and

$$\overline{\eta} \times_{\overline{S}} \mathrm{Gr}_{\mathcal{G},\overline{S}} = \overline{\eta} \times_{\eta} (\eta \times_S \mathrm{Gr}_{\mathcal{G},S}) = F^s \otimes_F \mathrm{Gr}_{F[\![z]\!] \otimes_F G} \overset{(2.2)}{\cong} \mathrm{Gr}_{F^s[\![z]\!] \otimes_{F^s} G_{F^s}}.$$

8.3. **Actions via cocharacters.** Let $\lambda\colon \mathbb{G}_{\mathrm{m},S} \to \mathcal{A}$ be a cocharacter over S. This cocharacter defines an action of $\mathbb{G}_{\mathrm{m},S}$ on $\mathcal{G}$ over S, hence we can consider the associated fixed points and attractor schemes $\mathcal{M}_\lambda := \mathcal{G}^0$ and $\mathcal{P}_\lambda := \mathcal{G}^+$ as in §3.1. The same considerations as in §8.1 allow to define the S-ind-schemes $\mathrm{Gr}_{\mathcal{M}_\lambda,S}$ and $\mathrm{Gr}_{\mathcal{P}_\lambda,S}$, and the obvious analogues of the statements in §8.2 hold in this case also.

The cocharacter λ induces a $\mathbb{G}_{\mathrm{m},S}$-action on $\mathrm{Gr}_{\mathcal{G},S}$, and we can consider the associated fixed points and attractor functors $(\mathrm{Gr}_{\mathcal{G},S})^0$ and $(\mathrm{Gr}_{\mathcal{G},S})^+$. It follows from [HR21, Lemma 5.3] that these functors are ind-schemes, and by naturality we obtain again morphisms

$$\mathrm{Gr}_{\mathcal{M}_\lambda,S} \to (\mathrm{Gr}_{\mathcal{G},S})^0 \quad \text{and} \quad \mathrm{Gr}_{\mathcal{P}_\lambda,S} \to (\mathrm{Gr}_{\mathcal{G},S})^+. \tag{8.2}$$

Proposition 8.2. *The maps* (8.2) *are isomorphisms.*

Proof. The proof is similar to the fiberwise case (see Proposition 3.2). Namely, we already know that both maps are closed immersions by [HR21, Theorem 5.6]. By compatibility of fixed points and attractors with base change, Proposition 3.2 and its absolute analogue imply that the maps under consideration are isomorphisms over η (see also [HR21, Theorem 5.6]) and over s. In particular, they induce isomorphisms on the underlying reduced sub-ind-schemes after any base change. To show that they are in fact isomorphisms, we use [HLR18, Lemma 8.6 and Remark 8.7] and repeat the proof of Proposition 3.2 with slight modifications adapted to the more general situation. Namely, to conclude the proof it suffices to show that for any O_F-algebra R which is a strictly henselian local artinian ring, the induced maps $\mathrm{Gr}_{\mathcal{M}_\lambda,S}(R) \to (\mathrm{Gr}_{\mathcal{G},S})^0(R)$ and $\mathrm{Gr}_{\mathcal{P}_\lambda,S}(R) \to (\mathrm{Gr}_{\mathcal{G},S})^+(R)$ are bijective. We treat the second case, leaving the first one to the reader.

The proof proceeds by induction on $\mathrm{n}(R)$, where as in the proof of Proposition 3.2 $\mathrm{n}(R)$ is the minimal positive integer n such that $\mathrm{rad}(R)^n = 0$. If $\mathrm{n}(R) = 1$, then R is an algebraically closed field. The map $O_F \to R$ factors either through a map $\mathbb{F} \to R$ or $F \to R$ (depending on whether t maps to zero or not). As the maps in (8.2) are isomorphisms over η and over s, this finishes the case $\mathrm{n}(R) = 1$.

Assume now that $\mathrm{n}(R) > 1$, and let $J \subset R$ be an ideal of square 0 such that $\mathrm{n}(R/J) < \mathrm{n}(R)$. By induction we can assume that the map $\mathrm{Gr}_{\mathcal{P}_\lambda,S}(R/J) \to (\mathrm{Gr}_{\mathcal{G},S})^+(R/J)$ is bijective. Using this and formal smoothness, we are reduced to showing that any $x \in (\mathrm{Gr}_{\mathcal{G},S})^+(R)$ whose image in $(\mathrm{Gr}_{\mathcal{G},S})^+(R/J)$ is the image of the "base point" section $e_S \in \mathrm{Gr}_{\mathcal{G},S}(O_F)$ is in the image of $\mathrm{Gr}_{\mathcal{P}_\lambda,S}(R)$, i.e., x corresponds to an element of $T_{e_S,J}((\mathrm{Gr}_{\mathcal{G},S})^+)$. Here for an S-ind-scheme $X = \mathrm{colim}_{i \in I} X_i$ together with an S-point $e \in X(S)$ we consider the R-module

$$T_{e,J}(X) := \mathrm{colim}_{i \in I} \mathrm{Hom}_{\mathsf{Mod}_R}(\omega_{X_i,e} \otimes_{O_F} R, J), \tag{8.3}$$

where $\omega_{X_i,e} = \Gamma(S, e^*\Omega_{X_i/S})$ is the global sections of the conormal sheaf associated with the immersion e, see also the proof of Proposition 3.2, and Mod_R is the category of R-modules. The R-module $T_{e,J}(X)$ does not depend on the chosen presentation of X as an ind-scheme over S, and its underlying set agrees with $p^{-1}(e_{R/J})$ where $p\colon X(R) \to X(R/J)$ is the obvious map and $e_{R/J}$ denotes the image of e under $X(S) \to X(R/J)$. To finish the proof, it suffices to show that the map

$$T_{e_S,J}(\mathrm{Gr}_{\mathcal{P}_\lambda,S}) \to T_{e_S,J}((\mathrm{Gr}_{\mathcal{G},S})^+) \tag{8.4}$$

is an isomorphism. The proof is completely analogous to the proof of Lemma 3.4 but replacing every occurrence of the tangent space or the Lie algebra by the R-module (8.3), and the loop functors L and L^+ by their relative versions L_S and L^+_S respectively. We recall some key steps. Since R is strictly henselian, one has $\mathrm{Gr}_{\mathcal{G},S}(R) = \mathrm{L}_S\mathcal{G}(R)/\mathrm{L}^+_S\mathcal{G}(R)$ which implies the equality

$$T_{e_S,J}(\mathrm{Gr}_{\mathcal{G},S}) = T_{e_S,J}(\mathrm{L}_S\mathcal{G})/T_{e_S,J}(\mathrm{L}^+_S\mathcal{G}),$$

and similarly for $\mathcal{G}$ replaced by $\mathcal{P}_\lambda$. Next, the $\mathbb{G}_{m,S}$-action induces a $\mathbb{Z}$-grading on $e^*\Omega_{X/S}$ for each $\mathbb{G}_{m,S}$-invariant closed subscheme X in $\mathrm{Gr}_{\mathcal{G},S}$ and $\mathrm{L}_S\mathcal{G}$ respectively, hence a $\mathbb{Z}$-grading on $T_{e_S,J}(\mathrm{Gr}_{\mathcal{G},S})$, $T_{e_S,J}(\mathrm{L}_S\mathcal{G})$ and $T_{e_S,J}(\mathrm{L}^+_S\mathcal{G})$ respectively. We get isomorphisms

$$T_{e_S,J}((\mathrm{Gr}_{\mathcal{G},S})^+) = T_{e_S,J}(\mathrm{Gr}_{\mathcal{G},S})^+ = T_{e_S,J}(\mathrm{L}_S\mathcal{G})^+/T_{e_S,J}(\mathrm{L}^+_S\mathcal{G})^+.$$

So, the claim (8.4) is equivalent to proving the equality

$$T_{e_S,J}(\mathrm{L}_S\mathcal{G})^+/T_{e_S,J}(\mathrm{L}^+_S\mathcal{G})^+ = T_{e_S,J}(\mathrm{L}_S\mathcal{P}_\lambda)/T_{e_S,J}(\mathrm{L}^+_S\mathcal{P}_\lambda).$$

As in (3.7) we use the big cell to prove the equalities $T_{e_S,J}(\mathrm{L}_S\mathcal{G})^+ = T_{e_S,J}(\mathrm{L}_S\mathcal{P}_\lambda)$ and $T_{e_S,J}(\mathrm{L}^+_S\mathcal{G})^+ = T_{e_S,J}(\mathrm{L}^+_S\mathcal{P}_\lambda)$, which finishes the proof. □

Corollary 8.3. *The maps*

$$\mathrm{Gr}_{\mathcal{M}_\lambda,\overline{S}} \to (\mathrm{Gr}_{\mathcal{G},\overline{S}})^0 \quad \textit{and} \quad \mathrm{Gr}_{\mathcal{P}_\lambda,\overline{S}} \to (\mathrm{Gr}_{\mathcal{G},\overline{S}})^+. \tag{8.5}$$

are isomorphisms.

Proof. This follows from Proposition 8.2 and compatibility of fixed points and attractors with base change along $\overline{S} \to S$. □

8.4. **Definition of the functor.** We continue with the geometric setting of §8.2; in particular we have an ind-scheme $\mathrm{Gr}_{\mathcal{G},S} \to S$ where $S = \mathrm{Spec}(O_F)$, whose fiber over the special point $s = \mathrm{Spec}(\mathbb{F})$, resp. over the geometric generic point $\overline{\eta} = \mathrm{Spec}(F^s)$, identifies with $\mathrm{Gr}_{\mathcal{G}}$, resp. with the "traditional" affine Grassmannian

$$\mathrm{Gr}_G := \mathrm{Gr}_{F^s[\![z]\!]\otimes_{F^s} G_{F^s}}$$

considered in Section 7. Consider the natural embeddings

$$\mathrm{Gr}_G \xrightarrow{j} \mathrm{Gr}_{\mathcal{G},\overline{S}} \xleftarrow{i} \mathrm{Gr}_{\mathcal{G}}.$$

The main player in this section will be the t-exact "nearby cycles" functor

$$\Psi_{\mathcal{G}} := i^*j_*\colon D^{\mathrm{b}}_{\mathrm{c}}(\mathrm{Gr}_G, \Lambda) \to D^{\mathrm{b}}_{\mathrm{c}}(\mathrm{Gr}_{\mathcal{G}}, \Lambda)$$

constructed as in §B.1.5.

Remark 8.4. The "true" nearby cycles functor associated with the data above is rather the composition of $\Psi_{\mathcal{G}}$ with the pullback functor under the natural morphism $\mathrm{Gr}_G \to (\mathrm{Gr}_{\mathcal{G},S})_\eta$ where $(\mathrm{Gr}_{\mathcal{G},S})_\eta$ is the fiber of $\mathrm{Gr}_{\mathcal{G},S}$ over η. The results mentioned in §B.1.5 of course have antecedents in the literature; see e.g. [BBDG82, Appendix]. However these statements are usually given for the "true" nearby cycles functor rather than for the version we consider, which justifies our references to [HS23].

As for the affine Grassmannian $\mathrm{Gr}_{\mathcal{G}}$ in §2.4, the ind-scheme Gr_G has a stratification given by the Cartan decomposition, with strata parametrized by $\mathbb{X}_*(T)^+$. Namely, each $\lambda \in \mathbb{X}_*(T)^+$ determines an F^s-point in Gr_G, and the L^+G-orbit of this point is a quasi-projective subscheme Gr_G^λ of Gr_G. Moreover, we have

$$|\mathrm{Gr}_G| = \bigsqcup_{\lambda\in\mathbb{X}_*(T)^+} |\mathrm{Gr}_G^\lambda|.$$

The intersection cohomology complex associated with the constant local system on Gr_G^λ will be denoted $\mathscr{J}^{\mathrm{abs}}_{!*}(\lambda,\Lambda)$. (Here "abs" stands for "absolute.")

Lemma 8.5. *Let $\lambda \in \mathbb{X}_*(T)^+$, and denote by $\overline{\lambda}$ its image in $\mathbb{X}_*(T)_I^+$ (see Lemma 2.6(2)). Then we have*

$$(j^{\overline{\lambda}})^* \Psi_{\mathcal{G}}(\mathscr{J}_{!*}^{\mathrm{abs}}(\lambda, \Lambda)) \cong \underline{\Lambda}_{\mathrm{Gr}_{\mathcal{G}}^{\overline{\lambda}}}[\langle \lambda, 2\rho \rangle].$$

Proof. The proof is the same as that of [Zhu15, Lemma 2.6] (see also [Ric16, p. 3755]). Namely, consider the "global Schubert variety" M_λ of [Ric16, Definition 3.5]. By [Ric16, Corollary 3.14] we have an open subscheme $\mathring{M}_\lambda \subset M_\lambda$ which is smooth over $\overline{S}$ and contains Gr_G^λ, resp. $\mathrm{Gr}_{\mathcal{G}}^{\overline{\lambda}}$, in its generic, resp. special, fiber. The desired claim follows, by compatibility of nearby cycles with smooth pullback. □

Note that in case Λ is a field, Lemma 8.5 implies that $\mathscr{J}_{!*}(\overline{\lambda}, \Lambda)$ is a composition factor of the perverse sheaf $\Psi_{\mathcal{G}}(\mathscr{J}_{!*}^{\mathrm{abs}}(\lambda, \Lambda))$.

8.5. **Compatibility with convolution.** The functor $\Psi_{\mathcal{G}}$ of §8.4 admits an "equivariant version;" more specifically there exists a canonical functor

$$\Psi_{\mathcal{G}} \colon D^{\mathrm{b}}_{\mathrm{c}}(\mathrm{Hk}_G, \Lambda) \to D^{\mathrm{b}}_{\mathrm{c}}(\mathrm{Hk}_{\mathcal{G}}, \Lambda) \tag{8.6}$$

which is related to the functor of §8.4 by the obvious commutative diagram involving pullback functors to sheaves on the respective affine Grassmannians. Moreover this functor is t-exact, hence restricts to an exact functor

$$\mathsf{Perv}(\mathrm{Hk}_G, \Lambda) \to \mathsf{Perv}(\mathrm{Hk}_{\mathcal{G}}, \Lambda). \tag{8.7}$$

Proposition 8.6. *The functor (8.6) admits a canonical monoidal structure with respect to the convolution products $\star$. As a consequence, the functor (8.7) admits a canonical monoidal structure with respect to the convolution products $\star^0$.*

Proof. The proof is similar to that given in [Zhu15, Theorem-Definition 3.1] or [AR, §3.4] (which, itself, essentially goes back to [Gai01]). The idea is to consider a deformation of HkConv_G to $\mathrm{HkConv}_{\mathcal{G}}$; more specifically, given X and $\underline{\mathcal{G}}$ as in Remark 8.1, one considers (the restriction to S of) the stack $\mathrm{HkConv}_{\mathcal{G},X}$ over X defined as follows. Given an $\mathbb{F}$-algebra R and $x \in X(R)$, we denote by $\widehat{\Gamma}_x$ the spectrum of the completion of the ring $R \otimes_{\mathbb{F}} \mathscr{O}(X)$ with respect to the ideal defining the graph $\Gamma_x \subset X \otimes_{\mathbb{F}} R$ of x. Then $\mathrm{HkConv}_{\mathcal{G},X}(R)$ is defined as the category of tuples $(x, \mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3, \alpha, \beta)$ where $x \in X(R)$, $\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3$ are $\underline{\mathcal{G}}$-bundles on $\widehat{\Gamma}_x$, and α, resp. β, is an isomorphism between the restrictions of $\mathcal{E}_1$ and $\mathcal{E}_2$, resp. $\mathcal{E}_2$ and $\mathcal{E}_3$, to $\widehat{\Gamma}_x \smallsetminus \Gamma_x$. As in §8.4, we have natural embeddings

$$\mathrm{HkConv}_G \xrightarrow{\tilde{\jmath}} \mathrm{HkConv}_{\mathcal{G},\overline{S}} \xleftarrow{\tilde{\imath}} \mathrm{HkConv}_{\mathcal{G}}$$

and a nearby cycles functor

$$\widetilde{\Psi}_{\mathcal{G}} := \tilde{\imath}^* \tilde{\jmath}_* \colon D^{\mathrm{b}}_{\mathrm{c}}(\mathrm{HkConv}_G, \Lambda) \to D^{\mathrm{b}}_{\mathrm{c}}(\mathrm{HkConv}_{\mathcal{G}}, \Lambda).$$

By compatibility of nearby cycles with external tensor product, smooth pullback, and proper push-forward, the following diagram commutes up to natural isomorphism:

$$\begin{tikzcd}
D^{\mathrm{b}}_{\mathrm{c}}(\mathrm{Hk}_G, \Lambda) \times D^{\mathrm{b}}_{\mathrm{c}}(\mathrm{Hk}_G, \Lambda) \arrow[r, "\Psi_{\mathcal{G}} \times \Psi_{\mathcal{G}}"] \arrow[d, "p_G^*(- \boxtimes^L_\Lambda -)"'] \arrow[dd, bend right=60, "\star"'] & D^{\mathrm{b}}_{\mathrm{c}}(\mathrm{Hk}_{\mathcal{G}}, \Lambda) \times D^{\mathrm{b}}_{\mathrm{c}}(\mathrm{Hk}_{\mathcal{G}}, \Lambda) \arrow[d, "p_{\mathcal{G}}^*(- \boxtimes^L_\Lambda -)"] \arrow[dd, bend left=60, "\star"] \\
D^{\mathrm{b}}_{\mathrm{c}}(\mathrm{HkConv}_G, \Lambda) \arrow[r, "\widetilde{\Psi}_{\mathcal{G}}"] \arrow[d, "m_{G!}"'] & D^{\mathrm{b}}_{\mathrm{c}}(\mathrm{HkConv}_{\mathcal{G}}, \Lambda) \arrow[d, "m_{\mathcal{G}!}"] \\
D^{\mathrm{b}}_{\mathrm{c}}(\mathrm{Hk}_G, \Lambda) \arrow[r, "\Psi_{\mathcal{G}}"'] & D^{\mathrm{b}}_{\mathrm{c}}(\mathrm{Hk}_{\mathcal{G}}, \Lambda),
\end{tikzcd}$$

and this yields the desired monoidal structure on $\Psi_{\mathcal{G}}$. □

8.6. **Compatibility with constant term functors.** Recall the constant term functors from §4.3, and their "absolute" analogues considered in §7.4. Fix $\lambda \in \mathbb{X}_*(A)$. This cocharacter determines a parabolic subgroup $P_\lambda \subset G$ (containing A) and its Levi factor M_λ, see §3.1. By extension of scalars, it also determines a cocharacter $\lambda_{\mathrm{abs}} \in \mathbb{X}_*(T)$, hence a parabolic subgroup $P_\lambda^{\mathrm{abs}} \subset G_{F^s}$ (containing T_{F^s}) and its Levi factor M_λ^{abs}.

By compatibility of attractors and fixed points with base change, we have

$$P_\lambda^{\mathrm{abs}} = P_\lambda \otimes_F F^s \quad \text{and} \quad M_\lambda^{\mathrm{abs}} = M_\lambda \otimes_F F^s.$$

More specifically, assume that P_λ is standard with respect to B, i.e. that λ is dominant. Then P_λ is the standard parabolic subgroup of G determined by the subset

$$\Phi^{\mathrm{s}}_\lambda := \{\alpha \in \Phi^{\mathrm{s}} \mid \langle \lambda, \alpha \rangle = 0\} \subset \Phi^{\mathrm{s}}$$

(see [BT65, §5.12]), and P_λ^{abs} is the standard parabolic subgroup of G_{F^s} determined by the subset

$$\{\alpha \in \Phi^{\mathrm{s}}_{\mathrm{abs}} \mid \langle \lambda, \alpha \rangle = 0\} \subset \Phi^{\mathrm{s}}_{\mathrm{abs}},$$

i.e. by the inverse image of $\Phi^{\mathrm{s}}_\lambda$ under (2.6).

Let $\mathcal{P}_\lambda$ and $\mathcal{M}_\lambda$ be as in §3.1, and consider the action of $\mathbb{G}_{\mathrm{m},\overline{S}}$ on the ind-scheme $\mathrm{Gr}_{\mathcal{G},\overline{S}}$ via the cocharacter of $\mathcal{A}$ naturally attached to λ (see §8.3). By Corollary 8.3, the fiber over $\overline{s}$, resp. $\overline{\eta}$, of $(\mathrm{Gr}_{\mathcal{G},\overline{S}})^+$ identifies with $\mathrm{Gr}_{\mathcal{P}_\lambda}$, resp. $\mathrm{Gr}_{F^s[\![z]\!]\otimes_{F^s} P_\lambda^{\mathrm{abs}}}$, and the fiber over $\overline{s}$, resp. $\overline{\eta}$, of $(\mathrm{Gr}_{\mathcal{G},\overline{S}})^0$ identifies with $\mathrm{Gr}_{\mathcal{M}_\lambda}$, resp. $\mathrm{Gr}_{F^s[\![z]\!]\otimes_{F^s} M_\lambda^{\mathrm{abs}}}$.

Proposition 8.7. *In the setting above, there exists a canonical isomorphism*

$$\Psi_{\mathcal{M}_\lambda} \circ \mathsf{CT}_{P_\lambda^{\mathrm{abs}},G} \cong \mathsf{CT}_{\mathcal{P}_\lambda,\mathcal{G}} \circ \Psi_{\mathcal{G}} \tag{8.8}$$

of functors from $D^{\mathrm{b}}_{\mathrm{c}}(\mathrm{Hk}_G,\Lambda)$ to $D^{\mathrm{b}}_{\mathrm{c}}(\mathrm{Hk}_{\mathcal{M}_\lambda},\Lambda)$. In case $P_\lambda = B$, the restriction of this isomorphism to $\mathsf{Perv}(\mathrm{Hk}_G,\Lambda)$ is compatible with the monoidal structures on $\mathsf{CT}_{\mathcal{B},\mathcal{G}}$ and $\mathsf{CT}_{B,G}$ from Proposition 5.4, and the monoidal structures on $\Psi_{\mathcal{G}}$ and $\Psi_{\mathcal{T}}$ from Proposition 8.6.

Proof. The isomorphism (8.8) follows from the compatibility of hyperbolic localization with nearby cycles, see [Ric19, Theorem 3.3].

When $P_\lambda = B$, we have the following commutative diagram, in which the front and rear faces come from the proof of Proposition 5.4, and the left and right faces are obtained by applying ${}^{\mathrm{p}}\mathscr{H}^0$ to the diagram in the proof Proposition 8.6. (We omit most subscripts to avoid cumbersome notation, and write P for Perv to save space.)

$$\begin{tikzcd}
\mathsf{P}(\mathrm{Hk}_G,\Lambda)\times\mathsf{P}(\mathrm{Hk}_G,\Lambda) \arrow[rr, "\mathsf{CT}\times\mathsf{CT}"] \arrow[dr, "\Psi\times\Psi"] \arrow[dd, "(-)\star^0(-)"'] & & \mathsf{P}(\mathrm{Hk}_T,\Lambda)\times\mathsf{P}(\mathrm{Hk}_T,\Lambda) \arrow[dr, "\Psi\times\Psi"] \arrow[dd, "(-)\star^0(-)"' near start] & \\
& \mathsf{P}(\mathrm{Hk}_{\mathcal{G}},\Lambda)\times\mathsf{P}(\mathrm{Hk}_{\mathcal{G}},\Lambda) \arrow[rr, "\mathsf{CT}\times\mathsf{CT}" near start] \arrow[dd, "(-)\star^0(-)" near start] & & \mathsf{P}(\mathrm{Hk}_{\mathcal{T}},\Lambda)\times\mathsf{P}(\mathrm{Hk}_{\mathcal{T}},\Lambda) \arrow[dd, "(-)\star^0(-)"] \\
\mathsf{P}(\mathrm{Hk}_G,\Lambda) \arrow[rr, "\mathsf{CT}" near start] \arrow[dr, "\Psi"] & & \mathsf{P}(\mathrm{Hk}_T,\Lambda) \arrow[dr, "\Psi"] & \\
& \mathsf{P}(\mathrm{Hk}_{\mathcal{G}},\Lambda) \arrow[rr, "\mathsf{CT}"] & & \mathsf{P}(\mathrm{Hk}_{\mathcal{T}},\Lambda)
\end{tikzcd}$$

The commutativity of this diagram implies that for $\mathscr{F}, \mathscr{G} \in \mathsf{Perv}(\mathrm{Hk}_G, \Lambda)$ the following diagram commutes, where we write B^{abs} for B_{F^s}:

$$\begin{tikzcd}
\Psi_{\mathcal{T}}(\mathsf{CT}_{B^{\mathrm{abs}},G}(\mathscr{F} \star^0 \mathscr{G})) \arrow[r, "\text{Prop. 5.4}"] \arrow[d, "(8.8)"'] & \Psi_{\mathcal{T}}(\mathsf{CT}_{B^{\mathrm{abs}},G}(\mathscr{F}) \star^0 \mathsf{CT}_{B^{\mathrm{abs}},G}(\mathscr{G})) \arrow[d, "\text{Prop. 8.6}"] \\
\mathsf{CT}_{\mathcal{B},\mathcal{G}}(\Psi_{\mathcal{G}}(\mathscr{F} \star^0 \mathscr{G})) \arrow[d, "\text{Prop. 8.6}"'] & \Psi_{\mathcal{T}}(\mathsf{CT}_{B^{\mathrm{abs}},G}(\mathscr{F})) \star^0 \Psi_{\mathcal{T}}(\mathsf{CT}_{B^{\mathrm{abs}},G}(\mathscr{G})) \arrow[d, "(8.8)"] \\
\mathsf{CT}_{\mathcal{B},\mathcal{G}}(\Psi_{\mathcal{G}}(\mathscr{F}) \star^0 \Psi_{\mathcal{G}}(\mathscr{G})) \arrow[r, "\text{Prop. 5.4}"'] & \mathsf{CT}_{\mathcal{B},\mathcal{G}}(\Psi_{\mathcal{G}}(\mathscr{F})) \star^0 \mathsf{CT}_{\mathcal{B},\mathcal{G}}(\Psi_{\mathcal{G}}(\mathscr{G})).
\end{tikzcd}$$

Thus, in this case, (8.8) is an isomorphism of monoidal functors. ☐

8.7. **Compatibility with fiber functors.** Recall the functor $\mathsf{F}_{\mathcal{G}}$ of §4.4, and its analogue F_G considered in Section 7.

Lemma 8.8. *There exists a canonical isomorphism*

$$\mathsf{F}_{\mathcal{G}} \circ \Psi_{\mathcal{G}} \cong \mathsf{F}_G$$

of monoidal functors from $\mathsf{Perv}(\mathrm{Hk}_G, \Lambda)$ *to* mod_Λ.

Proof. The isomorphism follows from compatibility of nearby cycles with proper pushforward, applied to the proper morphism $\mathrm{Gr}_{\mathcal{G},\overline{S}} \to \overline{S}$. By construction of the monoidal structures on $\mathsf{F}_{\mathcal{G}}$ and F_G (see also the discussion in §7.3), and in view of Proposition 8.7, to prove the compatibility with monoidal structures in general it suffices to prove it in case $\mathcal{G} = \mathcal{T}$, where it is obvious. ☐

8.8. **Application to coalgebras.** Recall from Theorem 6.1, resp. Theorem 7.7, that we have a canonical equivalence of categories

$$\mathsf{Perv}(\mathrm{Hk}_{\mathcal{G}}, \Lambda) \xrightarrow{\sim} \mathsf{comod}_{\mathrm{B}_{\mathcal{G}}(\Lambda)}, \quad \text{resp.} \quad \mathsf{Perv}(\mathrm{Hk}_G, \Lambda) \xrightarrow{\sim} \mathsf{Rep}(G^\vee_\Lambda) = \mathsf{comod}_{\mathscr{O}(G^\vee_\Lambda)}$$

under which the functor $\mathsf{F}_{\mathcal{G}}$, resp. F_G, corresponds to the natural forgetful functor to mod_Λ.

Corollary 8.9. *There exists a canonical morphism of* Λ*-bialgebras*

$$f_{\mathcal{G},\Lambda} \colon \mathscr{O}(G^\vee_\Lambda) \to \mathrm{B}_{\mathcal{G}}(\Lambda)$$

such that the diagram

$$\begin{tikzcd}
\mathsf{Perv}(\mathrm{Hk}_G, \Lambda) \arrow[r, "\mathit{Thm.\ 7.7}", "\sim"'] \arrow[d, "\Psi_{\mathcal{G}}"'] & \mathsf{Rep}(G^\vee_\Lambda) \arrow[d] \\
\mathsf{Perv}(\mathrm{Hk}_{\mathcal{G}}, \Lambda) \arrow[r, "\mathit{Thm.\ 6.1}", "\sim"'] & \mathsf{comod}_{\mathrm{B}_{\mathcal{G}}(\Lambda)}
\end{tikzcd}$$

commutes, where the right vertical arrow is the functor induced by $f_{\mathcal{G},\Lambda}$. *Moreover, this morphism factors through a morphism of* Λ*-bialgebras*

$$\tilde{f}_{\mathcal{G},\Lambda} \colon \mathscr{O}((G^\vee_\Lambda)^I) \to \mathrm{B}_{\mathcal{G}}(\Lambda).$$

Proof. The existence of $f_{\mathcal{G},\Lambda}$ follows from the same considerations as for Proposition 6.7. (In this case the compatibility with products uses the monoidal structure on the functor $\Psi_{\mathcal{G}}$.) To prove that this morphism factors through the invariants $\mathscr{O}((G^\vee_\Lambda)^I)$, as e.g. in [Zhu15, Lemma 4.5] one has to check that for any $\gamma \in I$ and any $\mathscr{F} \in \mathsf{Perv}(\mathrm{Hk}_G, \Lambda)$ we have a canonical isomorphism

$$\Psi_{\mathcal{G}}(\gamma \cdot \mathscr{F}) \cong \Psi_{\mathcal{G}}(\mathscr{F}).$$

Now we have $\mathrm{Gr}_{\mathcal{G},\overline{S}} = \mathrm{Gr}_{\mathcal{G},S} \times_S \overline{S}$, and the I-action on F^s stabilizes $\overline{O_F}$ and induces the trivial action on $\mathbb{F}$. Hence the action of I on $\overline{\eta} \times_{\overline{S}} \mathrm{Gr}_{\mathcal{G},\overline{S}} = \mathrm{Gr}_G$ extends to an action on $\mathrm{Gr}_{\mathcal{G},\overline{S}}$ which restricts to the trivial action on $\overline{s} \times_{\overline{S}} \mathrm{Gr}_{\mathcal{G},\overline{S}}$. The desired property follows. ☐

Let us now consider the setting of §8.6. We can consider the morphism of coalgebras $f_{\mathcal{G},\Lambda}$ from Corollary 8.9, and also the analogous morphism $f_{\mathcal{M}_\lambda,\Lambda}$ for the reductive group M_λ and its parahoric group scheme $\mathcal{M}_\lambda$. On the other hand we have the morphism

$$\mathrm{res}_{\mathcal{P}_\lambda,\mathcal{G}} \colon \mathrm{B}_{\mathcal{G}}(\Lambda) \to \mathrm{B}_{\mathcal{M}_\lambda}(\Lambda)$$

of Proposition 6.7. The same considerations provide a morphism of Hopf algebras

$$\mathrm{res}_{P_\lambda,G} \colon \mathscr{O}(G^\vee_\Lambda) \to \mathscr{O}((M_\lambda)^\vee_\Lambda)$$

which identifies $(M_\lambda)^\vee_\Lambda$ with the Levi subgroup of $G^\vee_\Lambda$ which is Langlands dual to the Levi subgroup $F^s \otimes_F M_\lambda \subset G_{F^s}$; see e.g. [BR18, §1.15.2].

The following claim is a direct consequence of Proposition 8.7.

Corollary 8.10. *In the setting of Proposition 8.7, we have*

$$f_{\mathcal{M}_\lambda,\Lambda} \circ \mathrm{res}_{P_\lambda,G} = \mathrm{res}_{\mathcal{P}_\lambda,\mathcal{G}} \circ f_{\mathcal{G},\Lambda}.$$

In this setting we have actions of I both on $G^\vee_\Lambda$ and on $(M_\lambda)^\vee_\Lambda$ (see §7.5), and it is easily seen that the morphism $\mathrm{res}_{P_\lambda,G}$ is I-equivariant. It therefore induces a morphism

$$\widetilde{\mathrm{res}}_{P_\lambda,G} \colon \mathscr{O}((G^\vee_\Lambda)^I) \to \mathscr{O}(((M_\lambda)^\vee_\Lambda)^I)$$

which satisfies

$$\widetilde{f}_{\mathcal{M}_\lambda,\Lambda} \circ \widetilde{\mathrm{res}}_{P_\lambda,G} = \mathrm{res}_{\mathcal{P}_\lambda,\mathcal{G}} \circ \widetilde{f}_{\mathcal{G},\Lambda}.$$

9. The ramified geometric Satake equivalence

9.1. **Commutativity.** We will now prove that $\mathrm{B}_{\mathcal{G}}(\Lambda)$ is the coordinate ring of a group scheme over Λ.

Proposition 9.1. *The Λ-bialgebra $\mathrm{B}_{\mathcal{G}}(\Lambda)$ is a commutative Hopf algebra. In particular it makes sense to consider the spectrum*

$$\mathcal{G}^\vee_\Lambda := \mathrm{Spec}(\mathrm{B}_{\mathcal{G}}(\Lambda)),$$

and this affine scheme has a canonical structure of flat group scheme over Λ.

Proof. Assume for a moment that G is semisimple of adjoint type (i.e. has trivial center). By Lemma 6.2, the category $\mathsf{Perv}(\mathrm{Hk}_{\mathcal{G}}, \mathbb{K})$ is semisimple; its simple objects are the intersection cohomology complexes $\mathscr{J}_{!*}(\lambda, \mathbb{K})$ for $\lambda \in \mathbb{X}_*(T)^+_I$. Lemma 2.6(2) and Lemma 8.5 imply that any of these simple objects is a subquotient of an object in the image of $\Psi_{\mathcal{G}}$; using [Sch92, Lemma 2.2.13] we deduce that the morphism $f_{\mathcal{G}}$ from Corollary 8.9 is surjective. Since $\mathscr{O}(G^\vee_{\mathbb{K}})$ is commutative, this implies that $\mathrm{B}_{\mathcal{G}}(\mathbb{K})$ is commutative. By flatness of $\mathrm{B}_{\mathcal{G}}(\mathbb{O})$ and the isomorphisms (6.1), we deduce that $\mathrm{B}_{\mathcal{G}}(\Lambda)$ is commutative for any Λ. In particular, we can therefore consider the semigroup scheme

$$\mathcal{G}^\vee_\Lambda := \mathrm{Spec}(\mathrm{B}_{\mathcal{G}}(\Lambda))$$

over Λ. The fact that this semigroup scheme is a group scheme, i.e. that $\mathrm{B}_{\mathcal{G}}(\Lambda)$ admits an antipode, can be checked as in [BR18, Proposition 1.13.4]. Since $\mathrm{B}_{\mathcal{G}}(\Lambda)$ is flat over Λ, this group scheme is flat.

Now we drop the assumption that G is semisimple of adjoint type, and consider the quotient morphism $G \to G_{\mathrm{ad}}$ as in the proof of Lemma 2.6(1). We have an induced equivariant map of buildings $\mathscr{B}(G, F) \to \mathscr{B}(G_{\mathrm{ad}}, F)$ that induces a bijection between facets. Let $\mathfrak{a}_{\mathrm{ad}} \subset \mathscr{B}(G_{\mathrm{ad}}, F)$ be the facet corresponding to $\mathfrak{a}$; then we can consider the parahoric group scheme $\mathcal{G}_{\mathrm{ad}}$ for G_{ad} corresponding to $\mathfrak{a}_{\mathrm{ad}}$. Since parahoric group schemes are smooth, [BT84, Proposition 1.7.6] yields a morphism $\mathcal{G} \to \mathcal{G}_{\mathrm{ad}}$ of group schemes over O_F, which induces a morphism of ind-schemes

$$\mathrm{Gr}_{\mathcal{G}} \to \mathrm{Gr}_{\mathcal{G}_{\mathrm{ad}}}. \tag{9.1}$$

On the other hand, the Kottwitz morphism provides a map

$$\mathrm{Gr}_{\mathcal{G}} \to \underline{\pi_1(G)_I}$$

where $\pi_1(G)_I$ is as in §2.3 and $\underline{\pi_1(G)_I}$ is the associated ind-scheme over $\mathbb{F}$. We also have the similar ind-scheme $\underline{\pi_1(G_{\mathrm{ad}})_I}$ (which in this case is a scheme since $\pi_1(G_{\mathrm{ad}})_I$ is finite), and morphisms

$$\underline{\pi_1(G)_I} \to \underline{\pi_1(G_{\mathrm{ad}})_I} \quad \text{and} \quad \mathrm{Gr}_{\mathcal{G}_{\mathrm{ad}}} \to \underline{\pi_1(G_{\mathrm{ad}})_I}.$$

Together, these morphisms induce a map

$$\mathrm{Gr}_{\mathcal{G}} \to \mathrm{Gr}_{\mathcal{G}_{\mathrm{ad}}} \times_{\underline{\pi_1(G_{\mathrm{ad}})_I}} \underline{\pi_1(G)_I}.$$

The fibers of the morphism $\mathrm{Gr}_{\mathcal{G}_{\mathrm{ad}}} \to \underline{\pi_1(G_{\mathrm{ad}})_I}$ are the connected components of $\mathrm{Gr}_{\mathcal{G}_{\mathrm{ad}}}$; hence any object of $\mathsf{Perv}(\mathrm{Hk}_{\mathcal{G}_{\mathrm{ad}}}, \Lambda)$ admits a canonical grading by $\pi_1(G_{\mathrm{ad}})_I$. By the same considerations as in the proof of Proposition 6.7, we deduce a morphism of Λ-group schemes

$$\mathrm{D}_\Lambda(\pi_1(G_{\mathrm{ad}})_I) \to (\mathcal{G}_{\mathrm{ad}})^\vee_\Lambda.$$

Corollary A.4 implies that the datum of an object in $\mathsf{Perv}(\mathrm{Hk}_{\mathcal{G}}, \Lambda)$ is equivalent to the datum of an object in $\mathsf{Perv}(\mathrm{Hk}_{\mathcal{G}_{\mathrm{ad}}}, \Lambda)$ together with a "lift" of the associated $\pi_1(G_{\mathrm{ad}})_I$-grading to a $\pi_1(G)_I$-grading, or in other words of a representation of the group scheme $(\mathcal{G}_{\mathrm{ad}})^\vee_\Lambda$ together with an "extension" of the action of the diagonalizable group scheme $\mathrm{D}_\Lambda(\pi_1(G_{\mathrm{ad}})_I)$ to an action of the diagonalizable group scheme $\mathrm{D}_\Lambda(\pi_1(G)_I)$. It follows that $\mathrm{B}_{\mathcal{G}}(\Lambda)$ is the structure algebra of the flat Λ-group scheme

$$\mathrm{D}_\Lambda(\pi_1(G)_I) \times^{\mathrm{D}_\Lambda(\pi_1(G_{\mathrm{ad}})_I)}_{\mathrm{Spec}(\Lambda)} (\mathcal{G}_{\mathrm{ad}})^\vee_\Lambda,$$

which finishes the proof. □

Now we consider a parabolic subgroup $P \subset G$ containing A, and the associated morphism $\mathrm{res}_{\mathcal{P},\mathcal{G}}$ from Proposition 6.7.

Proposition 9.2. *For any parabolic subgroup $P \subset G$ containing A, $\mathrm{res}_{\mathcal{P},\mathcal{G}}$ is a morphism of Hopf algebras.*

Proof. The proof is similar to that of Proposition 9.1. Namely, in case G is semisimple of adjoint type the claim follows from the similar claim for $\mathrm{res}_{P,G}$ in case $\Lambda = \mathbb{K}$, which is well known, see §8.8. (Note that in this case any Levi subgroup of G satisfies the conditions in Lemma 2.6(2), so that the associated morphism of Corollary 8.9 is surjective.) The general case can then be reduced to this one. □

9.2. **Identification of the dual group.** Passing to spectra, the morphism $\widetilde{f}_{\mathcal{G}}$ of Corollary 8.9 provides a canonical morphism of Λ-group schemes

$$\varphi_{\mathcal{G},\Lambda} \colon \mathcal{G}^\vee_\Lambda \to (G^\vee_\Lambda)^I.$$

By (6.1), there exist canonical isomorphisms

$$\mathcal{G}^\vee_{\mathbb{K}} \xrightarrow{\sim} \mathbb{K} \otimes_{\mathbb{O}} \mathcal{G}^\vee_{\mathbb{O}}, \quad \mathcal{G}^\vee_{\Bbbk} \xrightarrow{\sim} \Bbbk \otimes_{\mathbb{O}} \mathcal{G}^\vee_{\mathbb{O}}. \tag{9.2}$$

On the other hand, by compatibility of fixed points with base change (see [ALRR23, (2.1)]), we also have canonical isomorphisms

$$(G^\vee_{\mathbb{K}})^I \xrightarrow{\sim} \mathbb{K} \otimes_{\mathbb{O}} (G^\vee_{\mathbb{O}})^I, \quad (G^\vee_{\Bbbk})^I \xrightarrow{\sim} \Bbbk \otimes_{\mathbb{O}} (G^\vee_{\mathbb{O}})^I. \tag{9.3}$$

These identifications are compatible with the corresponding morphisms $\varphi_{\mathcal{G},\Lambda}$ in the obvious way.

The following theorem is the main result of this paper. Its proof will occupy the rest of the section.

Theorem 9.3. *For any Λ, the morphism $\varphi_{\mathcal{G},\Lambda}$ is an isomorphism.*

We start with a reduction of this statement to semisimple groups of adjoint type. Recall from the proof of Proposition 9.1 that we have a canonical integral model $\mathcal{G}_{\mathrm{ad}}$ of the adjoint quotient G_{ad} of G induced by the parahoric model $\mathcal{G}$. Of course, the choice of B determines a Borel subgroup in G_{ad}, and as noted in the proof of Lemma 2.6(1) the choice of A determines a maximal split torus in G_{ad}. Hence we have all the ingredients to run the above constructions for the group $\mathcal{G}_{\mathrm{ad}}$.

Lemma 9.4. *If Theorem 9.3 holds for the group $\mathcal{G}_{\mathrm{ad}}$, then it holds for $\mathcal{G}$.*

Proof. As noted in the course of the proof of Proposition 9.1, we have

$$\mathcal{G}^\vee_\Lambda = \mathrm{D}_\Lambda(\pi_1(G)_I) \times^{\mathrm{D}_\Lambda(\pi_1(G_{\mathrm{ad}})_I)}_{\mathrm{Spec}(\Lambda)} (\mathcal{G}_{\mathrm{ad}})^\vee_\Lambda. \tag{9.4}$$

On the other hand, if $\mathrm{Z}(G^\vee_\Lambda)$ is the scheme-theoretic center of $G^\vee_\Lambda$, then there is a natural identification $\mathrm{Z}(G^\vee_\Lambda) = \mathrm{D}_\Lambda(\pi_1(G))$, which provides an isomorphism

$$(\mathrm{Z}(G^\vee_\Lambda))^I = \mathrm{D}_\Lambda(\pi_1(G)_I),$$

see [ALRR23, Lemma 2.2]. Similarly, for G_{ad} we have

$$(\mathrm{Z}((G_{\mathrm{ad}})^\vee_\Lambda))^I = \mathrm{D}_\Lambda(\pi_1(G_{\mathrm{ad}})_I),$$

and by [ALRR23, Proposition 6.8] the natural morphism

$$(\mathrm{Z}(G^\vee_\Lambda))^I \times^{(\mathrm{Z}((G_{\mathrm{ad}})^\vee_\Lambda))^I} ((G_{\mathrm{ad}})^\vee_\Lambda)^I \to (G^\vee_\Lambda)^I \tag{9.5}$$

is an isomorphism, which finishes the proof. Comparing (9.4) and (9.5) together with the description of fixed points in centers we deduce the claim. ☐

Since the adjoint group of a torus is the trivial group, for which the theorem obviously holds, we deduce in particular from Lemma 9.4 that Theorem 9.3 holds when G is a torus. Once this is known, for general G and $\mathcal{G}$, the morphism $\mathrm{res}_{\mathcal{B},\mathcal{G}}$ of Proposition 6.7 defines a morphism of group schemes $(T^\vee_\Lambda)^I \to \mathcal{G}^\vee_\Lambda$ whose composition with $\varphi_{\mathcal{G},\Lambda}$ is the natural closed immersion $(T^\vee_\Lambda)^I \to (G^\vee_\Lambda)^I$. The former morphism is therefore also a closed immersion.

Now, let us outline the strategy of the proof of Theorem 9.3. By Lemma 9.4 we can assume that G is semisimple of adjoint type. The cases $\Lambda = \mathbb{K}, \Bbbk$ follow from the case $\Lambda = \mathbb{O}$ by base change using (9.2) and (9.3). If $\Lambda = \mathbb{O}$, then we aim to apply the following statement to the map $\tilde{f}_{\mathcal{G},\mathbb{O}} \colon \mathscr{O}((G^\vee_\mathbb{O})^I) \to \mathscr{O}(\mathcal{G}^\vee_\mathbb{O})$.

Lemma 9.5 ([FS21, Lemma VI.11.3])**.** *Let M, N be flat $\mathbb{O}$-modules, and let*

$$f \colon M \to N$$

be a morphism of $\mathbb{O}$-modules such that $\Bbbk \otimes_\mathbb{O} f$ is injective and $\mathbb{K} \otimes_\mathbb{O} f$ is an isomorphism. Then f is an isomorphism.

Both $\mathbb{O}$-modules $\mathscr{O}((G^\vee_\mathbb{O})^I)$ and $\mathscr{O}(\mathcal{G}^\vee_\mathbb{O}) = B_\mathcal{G}(\mathbb{O})$ are flat by [ALRR23, Theorem 5.1(1)] and Theorem 6.1 respectively. By the proof of Proposition 9.1, we know that $\varphi_{\mathcal{G},\mathbb{O}} \otimes_\mathbb{O} \mathbb{K} = \varphi_{\mathcal{G},\mathbb{K}}$ is a closed immersion. Thus, to conclude it suffices to check that $\tilde{f}_{\mathcal{G},\Lambda}$ is injective for both $\Lambda = \mathbb{K}, \Bbbk$, which by [Sta22, Tag 056A] amounts to showing that the scheme-theoretic image of $\varphi_{\mathcal{G},\Bbbk}$ is $(G^\vee_\Lambda)^I$ for these choices of coefficients. This is checked in Subsection 9.3 when G has semisimple F-rank 1. The general case is proven in Subsection 9.4 using constant term functors to construct enough "semisimple-rank-1 Levi subgroups" in $\mathcal{G}^\vee_\Lambda$ to ensure the schematical dominance of $\varphi_{\mathcal{G},\Lambda}$ for both $\Lambda = \mathbb{K}, \Bbbk$. Here the required group theory is supplied by [ALRR23], where the case $\Lambda = \Bbbk$ is particularly interesting in characteristic 2.

9.3. **Groups of semisimple F-rank 1.** In this subsection we prove Theorem 9.3 in case G has semisimple F-rank 1, i.e. $\#\Phi^{\mathrm{s}} = 1$.

The next statement is probably well known, although we could not find a proof in the literature. (It is somehow implicit in [FS21, Lemma VI.11.2].) Here, for any field k, we denote by:

- $\mathrm{T}_{2,k}$, resp. $\mathrm{B}^+_{2,k}$, resp. $\mathrm{B}^-_{2,k}$, the subgroup of $\mathrm{SL}_{2,k}$ consisting of diagonal matrices, resp. upper triangular matrices, resp. lower triangular matrices;
- $\underline{\mathrm{T}}_{2,k}$, resp. $\underline{\mathrm{B}}^+_{2,k}$, resp. $\underline{\mathrm{B}}^-_{2,k}$, the subgroup of $\mathrm{PGL}_{2,k}$ consisting of (images of) diagonal matrices, resp. upper triangular matrices, resp. lower triangular matrices

Lemma 9.6. *Let k be a field.*

1. *If K is a smooth connected closed subgroup of $\mathrm{SL}_{2,k}$ containing $\mathrm{T}_{2,k}$, then K is one of $\mathrm{T}_{2,k}$, $\mathrm{B}^\pm_{2,k}$, or $\mathrm{SL}_{2,k}$;*
2. *If K is a smooth connected closed subgroup of $\mathrm{PGL}_{2,k}$ containing $\underline{\mathrm{T}}_{2,k}$, then K is one of $\underline{\mathrm{T}}_{2,k}$, $\underline{\mathrm{B}}^\pm_{2,k}$, or $\mathrm{PGL}_{2,k}$.*

Proof. We explain the case of $\mathrm{SL}_{2,k}$; that of $\mathrm{PGL}_{2,k}$ can be treated similarly, or deduced using the quotient morphism $\mathrm{SL}_{2,k} \to \mathrm{PGL}_{2,k}$. We denote by $\mathrm{U}^\pm_{2,k}$ the unipotent radical of $\mathrm{B}^\pm_{2,k}$. The multiplication map

$$\mathrm{U}^-_{2,k} \times \mathrm{B}^+_{2,k} \to \mathrm{SL}_{2,k}$$

is an open immersion. Its image $\mathrm{C}_{2,k}$ (the "big cell") intersects K, hence by connectedness $\mathrm{C}_{2,k} \cap K$ is open and dense in K. On the other hand, by [CGP15, Proposition 2.1.8(3)] the multiplication morphism

$$(\mathrm{U}^-_{2,k} \cap K) \times (\mathrm{B}^+_{2,k} \cap K) \to \mathrm{C}_{2,k} \cap K$$

is an isomorphism, and since K contains $\mathrm{T}_{2,k}$ it follows that multiplication induces an isomorphism

$$(\mathrm{U}^-_{2,k} \cap K) \times \mathrm{T}_{2,k} \times (\mathrm{U}^+_{2,k} \cap K) \xrightarrow{\sim} \mathrm{C}_{2,k} \cap K.$$

The other statement in [CGP15, Proposition 2.1.8(3)] guarantees that $\mathrm{U}^\pm_{2,k} \cap K$ is smooth. It is a $\mathbb{G}_{\mathrm{m},k}$-stable subgroup of $\mathrm{U}^\pm_{2,k}$, hence is either trivial or equal to $\mathrm{U}^\pm_{2,k}$. The resulting four possible cases lead to the four cases $\mathrm{T}_{2,k}$, $\mathrm{B}^\pm_{2,k}$, or $\mathrm{SL}_{2,k}$. □

Finally we come to the main result of the subsection.

Proposition 9.7. *Theorem 9.3 holds in case G is semisimple of F-rank 1.*

Proof. By Lemma 9.4 we can (and will) assume that G is furthermore semisimple of adjoint type. Then G_{F^s} is also semisimple of adjoint type, hence a product of simple groups (of adjoint type). Since $\#\Phi^{\mathrm{s}} = 1$, there are two possibilities:

(a) either G_{F^s} is a product of copies of PGL_{2,F^s} and I acts by a transitive permutation of the factors;

(b) or G_{F^s} is a product of copies of PGL_{3,F^s}, I permutes transitively the factors, and the stabilizer of each factor acts non trivially on that factor (via the unique nontrivial diagram automorphism).

In case (a), $G^\vee_\Lambda$ is a product of copies of $\mathrm{SL}_{2,\Lambda}$ permuted transitively by I, so that we have

$$(G^\vee_\Lambda)^I \cong \mathrm{SL}_{2,\Lambda}. \tag{9.6}$$

In case (b), $G^\vee_\Lambda$ is a product of copies of $\mathrm{SL}_{3,\Lambda}$, and we have

$$(G^\vee_\Lambda)^I \cong (\mathrm{SL}_{3,\Lambda})^{\mathbb{Z}/2\mathbb{Z}} \tag{9.7}$$

where in the right-hand side the action is that considered in [ALRR23, §2.3]. (See [ALRR23, §2.3] for details.) Moreover, if 2 is invertible in Λ then the group in (9.7) identifies with $\mathrm{PGL}_{2,\Lambda}$, see [ALRR23, Example 5.9(1)].

Now we treat the case $\Lambda = \mathbb{K}$. As seen in the course of the proof of Proposition 9.1, in this case we know that $\varphi_{\mathcal{G},\mathbb{K}}$ is a closed immersion. Using tannakian formalism one checks as in [BR18, Lemma 1.9.3] that $\mathcal{G}^\vee_{\mathbb{K}}$ is connected (using the fact that $\mathbb{X}_*(T)_I$ is free over $\mathbb{Z}$ under our present assumptions), and as in [BR18, Lemma 1.9.4] (using Lemma 6.2) that it is reductive. This group is not a torus since it admits simple representations whose dimension is at least 2. (This property can e.g. be checked using the results of §6.3.) Hence $\varphi_{\mathcal{G},\mathbb{K}}$ is an isomorphism.

Now we assume that $\Lambda = \Bbbk$, and denote by H the scheme-theoretic image of $\varphi_{\mathcal{G},\Bbbk}$, or in other words the spectrum of the image of $\widetilde{f}_{\mathcal{G},\Bbbk}$ (see the comments at the end of §9.2). Then H is a closed subgroup scheme of $(G^\vee_\Bbbk)^I$. The morphism $\mathscr{O}(H) \to \mathscr{O}(\mathcal{G}^\vee_\Bbbk)$ is injective, hence H is a quotient of $\mathcal{G}^\vee_\Bbbk$; in particular, we have a fully faithful monoidal functor

$$\mathsf{Rep}(H) \to \mathsf{Rep}(\mathcal{G}^\vee_\Bbbk) \cong \mathsf{Perv}(\mathrm{Hk}_{\mathcal{G}}, \Bbbk)$$

whose essential image is stable under subquotients. Consider the reduced subgroup $(G^\vee_\Bbbk)^I_{\mathrm{red}}$. We will now prove that

$$H \supset (G^\vee_\Bbbk)^I_{\mathrm{red}}. \tag{9.8}$$

(Here the right-hand side is isomorphic to $\mathrm{SL}_{2,\Bbbk}$ in case (a) and in case (b) when $\ell = 2$, and to $\mathrm{PGL}_{2,\Bbbk}$ otherwise.)

First, we claim that H is connected. In fact, if it were not then $\mathsf{Rep}(H)$ would contain a subcategory stable under tensor products and containing finitely many isomorphism classes of simple objects (see e.g. [BR18, Proposition 1.2.11(2)]). Hence the same would be true for $\mathsf{Perv}(\mathrm{Hk}_{\mathcal{G}}, \Bbbk)$. As in the "classical" case (see [BR18, Lemma 1.9.3]) this is impossible because $\mathbb{X}_*(T)_I$ is free.

Let us note also that since the morphism $(T^\vee_\Bbbk)^I \to (G^\vee_\Bbbk)^I$ factors through $\mathcal{G}^\vee_\Bbbk$ (see §9.2), H contains a maximal torus of $(G^\vee_\Bbbk)^I_{\mathrm{red}}$. If H does not contain $(G^\vee_\Bbbk)^I_{\mathrm{red}}$, then H_{red} is a strict smooth connected subgroup containing a maximal torus; by Lemma 9.6 it is therefore either equal to the latter subgroup, or to one of the Borel subgroups containing it. In any case, any simple representation of H_{red} is invertible in the monoidal category $\mathsf{Rep}(H_{\mathrm{red}})$. On the other hand, as in the proof of [BR18, Lemma 1.14.6], for $n \gg 0$ the n-th Frobenius morphism Fr^n_H of H can be written as a composition

$$H \xrightarrow{\mathrm{Fr}^{n\prime}_H} (H_{\mathrm{red}})^{(n)} \hookrightarrow H^{(n)}$$

where $\mathrm{Fr}^{n\prime}_H$ is a quotient morphism. Choosing a nontrivial simple representation of $(H_{\mathrm{red}})^{(n)}$, pulling it back to H and then to $\mathcal{G}^\vee_\Bbbk$, we obtain a nontrivial simple representation of $\mathcal{G}^\vee_\Bbbk$ which is invertible in the monoidal category

$$\big(\mathsf{Rep}(\mathcal{G}^\vee_\Bbbk), \otimes\big) \cong \big(\mathsf{Perv}(\mathrm{Hk}_{\mathcal{G}}, \Bbbk), \star^0\big).$$

By the same considerations as in [BR18, Beginning of §1.9] one sees that the invertible objects in $\mathsf{Perv}(\mathrm{Hk}_{\mathcal{G}}, \Bbbk)$ are the objects $\mathscr{J}_{!*}(\lambda, \Bbbk)$ where $\lambda \in \mathbb{X}_*(T)_I$ is W_0-stable. Since G is assumed to be semisimple of adjoint type this implies that $\lambda = 0$; in other words $\mathsf{Perv}(\mathrm{Hk}_{\mathcal{G}}, \Bbbk)$ has no nontrivial simple invertible object, which provides a contradiction.

Now, assume that we are in case (a), or otherwise in case (b) with $\ell \neq 2$. Then the group scheme $(G^\vee_\Bbbk)^I$ is smooth, so that (9.8) implies that $H = (G^\vee_\Bbbk)^I$. Lemma 9.5 applied to the morphism $\tilde{f}_{\mathcal{G},\mathbb{O}}$ implies that this morphism is an isomorphism. It follows that $\tilde{f}_{\mathcal{G},\Bbbk} = \Bbbk \otimes_{\mathbb{O}} \tilde{f}_{\mathcal{G},\mathbb{O}}$ is an isomorphism too, which finishes the proof.

Finally we consider case (b) when $\ell = 2$. If $\mathbb{K} = \mathbb{Q}_2$, i.e. $\mathbb{O} = \mathbb{Z}_2$, then [ALRR23, Proposition 6.9] implies that we in fact have $H = (G^\vee_{\mathbb{F}_2})^I$, which allows us to conclude as above. For a general $\mathbb{O}$, we have a continuous morphism of rings $\mathbb{Z}_2 \to \mathbb{O}$; using the associated change-of-scalars functors (see §B.1.4) we deduce the desired claim from the case of $\mathbb{Z}_2$. □

9.4. **The general case.** We can finally complete the proof of Theorem 9.3.

Proof of Theorem 9.3. By Lemma 9.3 we can (and will) assume that G is semisimple of adjoint type. As explained at the end of §9.2, to conclude it suffices to prove that the scheme-theoretic image of $\varphi_{\mathcal{G},\Lambda}$ is $(G^\vee_\Lambda)^I$ for $\Lambda = \mathbb{K}$ or $\Bbbk$.

So, we finally assume that Λ is either $\mathbb{K}$ or $\Bbbk$, and denote by H the scheme-theoretic image of $\varphi_{\mathcal{G},\Lambda}$. As explained in §8.8, the formalism of constant term functors gives us morphisms $\mathcal{M}^\vee_\Lambda \to \mathcal{G}^\vee_\Lambda$ for every standard Levi subgroup $M \subset G$. (Here $\mathcal{M}$ is the scheme-theoretic closure of M; see (3.1).) Applying this construction to each standard Levi subgroup M_γ attached to a simple relative root γ and using Proposition 9.7, we deduce that H contains each $((M_\gamma)^\vee_\Lambda)^I$. Here, $(M_\gamma)^\vee_\Lambda$ identifies with the Levi subgroup of $G^\vee_\Lambda$ attached to the subset of simple roots whose coroots are the inverse image of γ under (2.6), see §8.8. Therefore, the desired claim follows from [ALRR23, Corollary 6.6]. □

Appendix A. Equivariant versus constructible perverse sheaves

In the setting considered in the body of the paper, let

$$D^{\mathrm{b}}_{(\mathrm{L}^+\mathcal{G})}(\mathrm{Gr}_{\mathcal{G}}, \Lambda)$$

be the full subcategory of $D^{\mathrm{b}}_{\mathrm{c}}(\mathrm{Gr}_{\mathcal{G}}, \Lambda)$ whose objects are the complexes $\mathscr{F}$ such that for any $i \in \mathbb{Z}$ and $\lambda \in \mathbb{X}_*(T)^+_I$ the sheaf $\mathcal{H}^i(\mathscr{F}_{|\mathrm{Gr}^\lambda_{\mathcal{G}}})$ is constant. Lemma A.2 below implies that this subcategory is triangulated. It is easily seen that the perverse t-structure on $D^{\mathrm{b}}_{\mathrm{c}}(\mathrm{Gr}_{\mathcal{G}}, \Lambda)$ restricts to a t-structure on $D^{\mathrm{b}}_{(\mathrm{L}^+\mathcal{G})}(\mathrm{Gr}_{\mathcal{G}}, \Lambda)$, whose heart will be denoted

$$\mathsf{Perv}_{(\mathrm{L}^+\mathcal{G})}(\mathrm{Gr}_{\mathcal{G}}, \Lambda).$$

In case Λ is a field, $\mathsf{Perv}_{(\mathrm{L}^+\mathcal{G})}(\mathrm{Gr}_{\mathcal{G}}, \Lambda)$ is the Serre subcategory of the category of perverse sheaves on $\mathrm{Gr}_{\mathcal{G}}$ generated by the objects $\mathscr{J}_{!*}(\lambda, \Lambda)$ for $\lambda \in \mathbb{X}_*(T)^+_I$. It is clear that the functor h^* (see (4.2)) factors through an exact functor

$$\mathsf{Perv}(\mathrm{Hk}_{\mathcal{G}}, \Lambda) \to \mathsf{Perv}_{(\mathrm{L}^+\mathcal{G})}(\mathrm{Gr}_{\mathcal{G}}, \Lambda). \tag{A.1}$$

Our goal in this appendix is to prove that (A.1) is an equivalence of categories. This will be achieved in Proposition A.3 below. (In the body of the paper, we use this statement through its consequence proved in Corollary A.4.)

Remark A.1. (1) Proposition A.3 is a ramified analogue of [MV07, Proposition 2.1]. Unfortunately, a direct adaptation of the proof of that result (given in [MV07, Appendix A]; see also [BR18, §1.10.2]) presents some technical difficulties. To bypass this problem we will give a slightly different proof of this property (based on similar considerations) which applies in both settings. None of the details introduced in this proof are required elsewhere in the paper.
(2) In case $\Lambda = \mathbb{K}$, the fact that (A.1) is an equivalence can be obtained in a much simpler way, by observing that the same arguments as for Lemma 6.2 show that the category $\mathsf{Perv}_{(\mathrm{L}^+\mathcal{G})}(\mathrm{Gr}_{\mathcal{G}}, \mathbb{K})$ is semisimple.

Lemma A.2. *Let $\lambda \in \mathbb{X}_*(T)^+_I$. The subcategory of the category of sheaves on $\mathrm{Gr}^\lambda_{\mathcal{G}}$ whose objects are the constant local systems is stable under subquotients and extensions.*

Proof. Stability under subquotients is a classical fact, known for any connected scheme of finite type. For stability under extensions, what we need to prove is that for any finitely generated Λ-modules M, N the natural morphism

$$\mathrm{Ext}^1_\Lambda(M,N) \to \mathrm{Hom}_{D^{\mathrm{b}}_{\mathrm{c}}(\mathrm{Gr}^\lambda_{\mathcal{G}},\Lambda)}(\underline{M}_{\mathrm{Gr}^\lambda_{\mathcal{G}}}, \underline{N}_{\mathrm{Gr}^\lambda_{\mathcal{G}}}[1]) \tag{A.2}$$

is an isomorphism.

First, assume that $\Lambda = \mathbb{K}$ or $\Lambda = \Bbbk$. Then it suffices to treat the case $M = N = \Lambda$, and the Ext^1-space vanishes. On the other hand we have

$$\mathrm{Hom}_{D^{\mathrm{b}}_{\mathrm{c}}(\mathrm{Gr}^\lambda_{\mathcal{G}},\Lambda)}(\underline{\Lambda}_{\mathrm{Gr}^\lambda_{\mathcal{G}}}, \underline{\Lambda}_{\mathrm{Gr}^\lambda_{\mathcal{G}}}[1]) = \mathsf{H}^1(\mathrm{Gr}^\lambda_{\mathcal{G}};\Lambda),$$

which vanishes because $\mathrm{Gr}^\lambda_{\mathcal{G}}$ is smooth (so that cohomology is dual to cohomology with compact support up to shift) and admits a paving by affine spaces (see §2.4).

Now we consider the case $\Lambda = \mathbb{O}$. We have

$$\mathrm{Hom}_{D^{\mathrm{b}}_{\mathrm{c}}(\mathrm{Gr}^\lambda_{\mathcal{G}},\mathbb{O})}(\underline{M}_{\mathrm{Gr}^\lambda_{\mathcal{G}}}, \underline{N}_{\mathrm{Gr}^\lambda_{\mathcal{G}}}[1]) = \mathsf{H}^1(R\Gamma(\mathrm{Gr}^\lambda_{\mathcal{G}}, \underline{R\,\mathrm{Hom}_{\mathbb{O}}(M,N)})).$$

Here the complex $R\,\mathrm{Hom}_{\mathbb{O}}(M,N)$ is concentrated in degrees 0 and 1; more explicitly we have a distinguished triangle

$$\mathrm{Hom}_{\mathbb{O}}(M,N) \to R\,\mathrm{Hom}_{\mathbb{O}}(M,N) \to \mathrm{Ext}^1_{\mathbb{O}}(M,N)[-1] \xrightarrow{[1]} .$$

Using the fact that

$$\mathsf{H}^1(R\Gamma(\mathrm{Gr}^\lambda_{\mathcal{G}}, \underline{\mathrm{Hom}_{\mathbb{O}}(M,N)})) = 0$$

(for the same reason as above) and that $\mathrm{Gr}^\lambda_{\mathcal{G}}$ is connected, applying the functor $R\Gamma(\mathrm{Gr}^\lambda_{\mathcal{G}}, \underline{?})$ to this triangle we obtain an embedding

$$\mathsf{H}^1(R\Gamma(\mathrm{Gr}^\lambda_{\mathcal{G}}, \underline{R\,\mathrm{Hom}_{\mathbb{O}}(M,N)})) \hookrightarrow \mathrm{Ext}^1_{\mathbb{O}}(M,N).$$

It is clear that the composition of (A.2) with this morphism is the identity morphism of $\mathrm{Ext}^1_{\mathbb{O}}(M,N)$. Hence both of this morphisms are isomorphisms, which finishes the proof. □

Proposition A.3. *The functor* (A.1) *is an equivalence of categories.*

Proof. For any $\mathrm{L}^+\mathcal{G}$-stable locally closed subscheme $X \subset \mathrm{Gr}_{\mathcal{G}}$ such that $|X|$ has finitely many $\mathrm{L}^+\mathcal{G}$-orbits, we can consider as above the categories

$$\mathsf{Perv}([\mathrm{L}^+\mathcal{G}\backslash X]_{\text{ét}},\Lambda) \quad \text{and} \quad \mathsf{Perv}_{(\mathrm{L}^+\mathcal{G})}(X,\Lambda)$$

of perverse sheaves which are $\mathrm{L}^+\mathcal{G}$-equivariant and constant on each $\mathrm{L}^+\mathcal{G}$-orbit, respectively, and the natural exact "forgetful" functor

$$\mathsf{Perv}([\mathrm{L}^+\mathcal{G}\backslash X]_{\text{ét}},\Lambda) \to \mathsf{Perv}_{(\mathrm{L}^+\mathcal{G})}(X,\Lambda).$$

We will prove that this functor is an equivalence of categories for any X, which will imply the proposition. Note that the general theory of perverse sheaves implies that this functor is fully faithful, and that its essential image is stable under subquotients. What we have to prove is that it is also essentially surjective, and for that it suffices to show that any object in $\mathsf{Perv}_{(\mathrm{L}^+\mathcal{G})}(X,\Lambda)$ is a quotient of an object in the image of $\mathsf{Perv}([\mathrm{L}^+\mathcal{G}\backslash X]_{\text{ét}},\Lambda)$. In case $\Lambda = \mathbb{K}$ or $\Lambda = \Bbbk$, the category $\mathsf{Perv}_{(\mathrm{L}^+\mathcal{G})}(X,\Lambda)$ is a finite-length abelian category, whose simple objects are the perverse sheaves $\mathscr{J}_{!*}(\lambda,\Lambda)_{|X}$ where $\lambda \in \mathbb{X}_*(T)^+_I$ is such that $|\mathrm{Gr}^\lambda_{\mathcal{G}}| \subset |X|$. In case $\Lambda = \mathbb{O}$, the same arguments as in [RSW14, Lemma 2.1.4] show that any object is an extension of objects which are quotients of objects of the form $\mathscr{J}_{!*}(\lambda,\Lambda)_{|X}$ where λ satisfies the same conditions. Hence, in any case, to conclude it suffices to prove that for any such λ the object $\mathscr{J}_{!*}(\lambda,\Lambda)_{|X}$ is a quotient of an object in $\mathsf{Perv}([\mathrm{L}^+\mathcal{G}\backslash X]_{\text{ét}},\Lambda)$ whose image in $\mathsf{Perv}_{(\mathrm{L}^+\mathcal{G})}(X,\Lambda)$ is projective.

We will proceed by induction on the number of orbits contained in $|X|$. Of course, there is nothing to prove in case X is empty. So, we consider a nonempty X as above, and choose some $\mu \in \mathbb{X}_*(T)^+_I$ such that $\mathrm{Gr}^\mu_{\mathcal{G}}$ is closed in X and such that the claim is known for the complement X' of $\mathrm{Gr}^\mu_{\mathcal{G}}$ in X. The open immersion $X' \to X$ will be denoted j. If $\lambda \in \mathbb{X}_*(T)^+_I$ is such that $|\mathrm{Gr}^\lambda_{\mathcal{G}}| \subset |X|$ and $\lambda \neq \mu$ (i.e. $|\mathrm{Gr}^\lambda_{\mathcal{G}}| \subset |X'|$), then as explained in §6.4 there exists a projective object $\mathscr{P}$ in $\mathsf{Perv}([\mathrm{L}^+\mathcal{G}\backslash X']_{\text{ét}}, \Lambda)$ and a surjection

$$\mathscr{P} \twoheadrightarrow \mathscr{J}_{!*}(\lambda, \Lambda)_{|X'} = j^*(\mathscr{J}_{!*}(\lambda, \Lambda)_{|X}).$$

By induction the image of $\mathscr{P}$ in $\mathsf{Perv}_{(\mathrm{L}^+\mathcal{G})}(X', \Lambda)$ is projective, and adjunction provides a morphism

$${}^{\mathrm{p}}\mathscr{H}^0(j_!\mathscr{P}) \to \mathscr{J}_{!*}(\lambda, \Lambda)_{|X}.$$

This morphism can be written as a composition

$${}^{\mathrm{p}}\mathscr{H}^0(j_!\mathscr{P}) \to j_{!*}\mathscr{P} \to j_{!*}\big(\mathscr{J}_{!*}(\lambda, \Lambda)_{|X'}\big) = \mathscr{J}_{!*}(\lambda, \Lambda)_{|X}$$

where the first morphism is surjective by definition, and the second one is also surjective because intermediate extension functors preserve surjections. Since the functor $j^! = j^* \colon \mathsf{Perv}_{(\mathrm{L}^+\mathcal{G})}(X, \Lambda) \to \mathsf{Perv}_{(\mathrm{L}^+\mathcal{G})}(X', \Lambda)$ is exact, by adjunction again the object ${}^{\mathrm{p}}\mathscr{H}^0(j_!\mathscr{P})$ is projective; the problem is therefore solved for these λ's.

It remains to treat the case $\lambda = \mu$. In this case we have

$$\mathscr{J}_{!*}(\mu, \Lambda)_{|X} = \underline{\Lambda}_{\mathrm{Gr}^\mu_{\mathcal{G}}}[\dim(\mathrm{Gr}^\mu_{\mathcal{G}})].$$

Using the notation of §6.4, we claim that there exists a surjective morphism

$$\mathscr{P}_X(\mu, \Lambda) \to \mathscr{J}_{!*}(\mu, \Lambda)_{|X}.$$

In fact, using Lemma 6.4 we see that

$$\mathrm{Hom}(\mathscr{P}_X(\mu, \Lambda), \mathscr{J}_{!*}(\mu, \Lambda)_{|X}) \cong \mathsf{F}^X_{\mathcal{G},\mu}(\mathscr{J}_{!*}(\mu, \Lambda)_{|X}) = \Lambda.$$

The element $1 \in \Lambda$ defines a morphism $\mathscr{P}_X(\mu, \Lambda) \to \mathscr{J}_{!*}(\mu, \Lambda)_{|X}$, which is surjective in case Λ is a field because its codomain is simple, and in case $\Lambda = \mathbb{O}$ because its image under derived tensor product with $\Bbbk$ is surjective (see Lemma 6.5).

We claim that the functor

$$\mathsf{H}^{\langle \mu, 2\rho \rangle}_{\mathrm{T}_\mu \cap X}(X, -) \colon \mathsf{Perv}_{(\mathrm{L}^+\mathcal{G})}(X, \Lambda) \to \mathsf{mod}_\Lambda$$

is exact. In fact, as noted in §6.4, for any $\mathscr{F} \in \mathsf{Perv}([\mathrm{L}^+\mathcal{G}\backslash X]_{\text{ét}}, \Lambda)$ we have

$$\mathsf{H}^k_{\mathrm{T}_\mu \cap X}(X, \mathscr{F}) = 0$$

unless $k = \langle \mu, 2\rho \rangle$. Since any object in $\mathsf{Perv}_{(\mathrm{L}^+\mathcal{G})}(X, \Lambda)$ is an extension of objects in $\mathsf{Perv}([\mathrm{L}^+\mathcal{G}\backslash X]_{\text{ét}}, \Lambda)$, the same property holds for $\mathscr{F} \in \mathsf{Perv}_{(\mathrm{L}^+\mathcal{G})}(X, \Lambda)$, which implies the desired exactness. For simplicity, this functor will also be denoted $\mathsf{F}^X_{\mathcal{G},\mu}$.

We now consider the category $\mathscr{C}$ constructed as in [Vil94, §1] out of the following data:

- the categories are $\mathscr{A} = \mathsf{Perv}_{(\mathrm{L}^+\mathcal{G})}(X', \Lambda) = \mathsf{Perv}([\mathrm{L}^+\mathcal{G}\backslash X']_{\text{ét}}, \Lambda)$ and $\mathscr{B} = \mathsf{mod}_\Lambda$;
- the functors $\mathbf{F}, \mathbf{G} \colon \mathscr{A} \to \mathscr{B}$ are

$$\mathbf{F} = \mathsf{F}^X_{\mathcal{G},\mu} \circ {}^{\mathrm{p}}\mathscr{H}^0 \circ j_! \quad \text{and} \quad \mathbf{G} = \mathsf{F}^X_{\mathcal{G},\mu} \circ {}^{\mathrm{p}}\mathscr{H}^0 \circ j_*;$$

- the morphism $\mathbf{T} \colon \mathbf{F} \to \mathbf{G}$ is the morphism induced by the canonical morphism $j_! \to j_*$.

Explicitly, the objects in $\mathscr{C}$ are the quadruples $(\mathscr{F}, V, m, n)$ where $\mathscr{F}$ is an object in $\mathsf{Perv}_{(\mathrm{L}^+\mathcal{G})}(X', \Lambda) = \mathsf{Perv}([\mathrm{L}^+\mathcal{G}\backslash X']_{\text{ét}}, \Lambda)$, V is an object in mod_Λ, and

$$m \colon \mathsf{F}^X_{\mathcal{G},\mu}({}^{\mathrm{p}}\mathscr{H}^0(j_!(\mathscr{F}))) \to V, \quad n \colon V \to \mathsf{F}^X_{\mathcal{G},\mu}({}^{\mathrm{p}}\mathscr{H}^0(j_*(\mathscr{F})))$$

are morphisms such that nm is induced by our morphism of functors $\mathbf{T}$. The morphisms in this category are pairs consisting of a morphism in $\mathsf{Perv}_{(\mathrm{L}^+\mathcal{G})}(X', \Lambda)$ and a morphism in mod_Λ, which make the obvious diagram commutative. For later use we note that the functor ${}^{\mathrm{p}}\mathscr{H}^0 \circ j_*$ takes values in the subcategory $\mathsf{Perv}([\mathrm{L}^+\mathcal{G}\backslash X]_{\text{ét}}, \Lambda)$ of $\mathsf{Perv}_{(\mathrm{L}^+\mathcal{G})}(X, \Lambda)$; for $\mathscr{F} \in \mathsf{Perv}_{(\mathrm{L}^+\mathcal{G})}(X', \Lambda)$ we therefore have

$$\mathsf{F}^X_{\mathcal{G},\mu} \circ {}^{\mathrm{p}}\mathscr{H}^0 \circ j_*(\mathscr{F}) = \mathrm{Hom}(\mathscr{P}_X(\mu, \Lambda), {}^{\mathrm{p}}\mathscr{H}^0(j_*(\mathscr{F}))) = \mathrm{Hom}(j^*\mathscr{P}_X(\mu, \Lambda), \mathscr{F});$$

in other words, the functor $\mathsf{F}^X_{\mathcal{G},\mu} \circ {}^{\mathrm{p}}\mathscr{H}^0 \circ j_*$ is represented by $j^*\mathscr{P}_X(\mu, \Lambda)$.

By [Vil94, Proposition 1.1(a)], $\mathscr{C}$ is an abelian category and, by [Vil94, Proposition 1.2], the functor

$$E \colon \mathsf{Perv}_{(\mathrm{L}^+\mathcal{G})}(X, \Lambda) \to \mathscr{C}$$

sending $\mathscr{G}$ to the quadruple

$$(j^*\mathscr{G}, \mathsf{F}^X_{\mathcal{G},\mu}(\mathscr{G}), m, n)$$

where m, n are the obvious morphisms (provided by adjunction) is fully faithful and exact. (We apply this proposition with $\widetilde{\mathscr{B}}$ the full subcategory of perverse sheaves supported on $\mathrm{Gr}^\mu_{\mathcal{G}}$, which is equivalent to the category of constant sheaves on $\mathrm{Gr}^\mu_{\mathcal{G}}$, see Lemma A.2.) To prove that the image of $\mathscr{P}_X(\mu, \Lambda)$ in $\mathsf{Perv}_{(\mathrm{L}^+\mathcal{G})}(X, \Lambda)$ is projective, it therefore suffices to prove that its image in $\mathscr{C}$ is projective. What we will show is that this image represents the functor

$$\mathscr{C} \to \mathsf{mod}_\Lambda \tag{A.3}$$

given by $(\mathscr{F}, V, m, n) \to V$. This functor is clearly exact, which will imply projectivity and finish the proof.

In more concrete terms we will prove that

$$E(\mathscr{P}_X(\mu, \Lambda)) = (j^*\mathscr{P}_X(\mu, \Lambda), \mathsf{F}^X_{\mathcal{G},\mu}({}^{\mathrm{p}}\mathscr{H}^0(j_!j^*\mathscr{P}_X(\mu, \Lambda))) \oplus \Lambda, m, n)$$

where m is the obvious embedding and n is the sum of the morphism induced by $\mathbf{T}$ with the morphism

$$\Lambda \to \mathsf{F}^X_{\mathcal{G},\mu}({}^{\mathrm{p}}\mathscr{H}^0(j_*j^*\mathscr{P}_X(\mu, \Lambda))) = \mathrm{End}(j^*\mathscr{P}_X(\mu, \Lambda))$$

sending $1 \in \Lambda$ to the identity morphism. From this description, and using the fact that $j^*\mathscr{P}_X(\mu, \Lambda)$ represents $\mathsf{F}^X_{\mathcal{G},\mu} \circ {}^{\mathrm{p}}\mathscr{H}^0 \circ j_*$, it is not difficult to check that $E(\mathscr{P}_X(\mu, \Lambda))$ indeed represents the functor (A.3). (In case Λ is a field, this object is the projective cover of the simple object $(0, \Lambda, 0, 0)$ described in [Vil94, p. 667], see also [MV87, p. 317].) By definition, the second component in $E(\mathscr{P}_X(\mu, \Lambda))$ is

$$\mathsf{F}^X_{\mathcal{G},\mu}(\mathscr{P}_X(\mu, \Lambda)) = \mathrm{End}(\mathscr{P}_X(\mu, \Lambda)),$$

where we use Lemma 6.4. To prove the above claim it therefore suffices to prove that the natural morphism

$$\begin{aligned}\mathsf{F}^X_{\mathcal{G},\mu}&({}^{\mathrm{p}}\mathscr{H}^0(j_!j^*\mathscr{P}_X(\mu, \Lambda))) \oplus \Lambda \\ &= \mathrm{Hom}(\mathscr{P}_X(\mu, \Lambda), {}^{\mathrm{p}}\mathscr{H}^0(j_!j^*\mathscr{P}_X(\mu, \Lambda))) \oplus \Lambda \to \mathrm{End}(\mathscr{P}_X(\mu, \Lambda))\end{aligned}$$

given by the sum of the morphism induced by the adjunction morphism

$${}^{\mathrm{p}}\mathscr{H}^0(j_!j^*\mathscr{P}_X(\mu, \Lambda)) \to \mathscr{P}_X(\mu, \Lambda)$$

and the morphism $\Lambda \to \mathrm{End}(\mathscr{P}_X(\mu, \Lambda))$ sending $1 \in \Lambda$ to the identity morphism is an isomorphism.

First we consider the case when Λ is a field, i.e. $\Lambda = \mathbb{K}$ or $\Lambda = \Bbbk$. The projective object $\mathscr{P}_X(\mu, \Lambda)$ in the highest weight category $\mathsf{Perv}([\mathrm{L}^+\mathcal{G}\backslash X]_{\text{ét}}, \Lambda)$ (see Remark 6.6) admits a standard filtration; hence there exists an exact sequence

$$\mathscr{F}_1 \hookrightarrow \mathscr{P}_X(\mu, \Lambda) \twoheadrightarrow \mathscr{F}_2 \tag{A.4}$$

where $\mathscr{F}_1$ is an extension of objects $\mathscr{J}_!(\lambda,\Lambda)_{|X}$ where $\lambda \in \mathbb{X}_*(T)_I^+$ is such that $|\mathrm{Gr}_{\mathcal{G}}^\lambda| \subset |X|$ and $\lambda \neq \mu$, and $\mathscr{F}_2$ is a direct sum of copies of $\mathscr{J}_!(\mu,\Lambda)_{|X} = \underline{\Lambda}_{\mathrm{Gr}_{\mathcal{G}}^\mu}[\dim(\mathrm{Gr}_{\mathcal{G}}^\mu)]$. Here we necessarily have

$$\mathscr{F}_1 = {}^{\mathrm{p}}\mathscr{H}^0(j_!j^*\mathscr{P}_X(\mu,\Lambda))$$

and the multiplicity of $\mathscr{J}_!(\mu,\Lambda)_{|X}$ in $\mathscr{F}_2$ is the multiplicity of $\mathscr{J}_!(\mu,\Lambda)$ in $\mathscr{P}_Z(\mu,\Lambda)$ (where $Z \subset \mathrm{Gr}_{\mathcal{G}}$ is any closed subscheme as in §6.4 in which X is open), i.e. the dimension of

$$\mathrm{Hom}(\mathscr{P}_Z(\mu,\Lambda), \mathscr{J}_*(\mu,\Lambda)) \cong \mathsf{F}_{\mathcal{G},\mu}(\mathscr{J}_*(\mu,\Lambda)) = \Lambda.$$

Applying the exact functor $\mathsf{F}^X_{\mathcal{G},\mu}$ to the exact sequence (A.4) we obtain an exact sequence

$$\mathsf{F}^X_{\mathcal{G},\mu}({}^{\mathrm{p}}\mathscr{H}^0(j_!j^*\mathscr{P}_X(\mu,\Lambda))) \hookrightarrow \mathsf{F}^X_{\mathcal{G},\mu}(\mathscr{P}_X(\mu,\Lambda)) \twoheadrightarrow \mathsf{F}^X_{\mathcal{G},\mu}(\mathscr{F}_2),$$

where the rightmost term is 1-dimensional. Since the identity morphism of the perverse sheaf $\mathscr{P}_X(\mu,\Lambda)$ does not factor through ${}^{\mathrm{p}}\mathscr{H}^0(j_!j^*\mathscr{P}_X(\mu,\Lambda))$, we therefore have

$$\mathsf{F}^X_{\mathcal{G},\mu}(\mathscr{P}_X(\mu,\Lambda)) = \mathsf{F}^X_{\mathcal{G},\mu}({}^{\mathrm{p}}\mathscr{H}^0(j_!j^*\mathscr{P}_X(\mu,\Lambda))) \oplus \Lambda$$

where the 1-dimensional factor corresponds to $\mathrm{id} \in \mathrm{End}(\mathscr{P}_X(\mu,\Lambda))$. This completes the proof in this case.

Finally we consider the case $\Lambda = \mathbb{O}$. Using the fact that $\Bbbk \otimes^L_{\mathbb{O}} \mathscr{P}_X(\mu,\mathbb{O}) = \mathscr{P}_X(\mu,\Bbbk)$ is perverse (see Lemma 6.5) and the vanishing property (6.5) one checks that

$$\Bbbk \overset{L}{\otimes}_{\mathbb{O}} \mathsf{F}^X_{\mathcal{G},\mu}(\mathscr{P}_X(\mu,\mathbb{O})) = \mathsf{F}^X_{\mathcal{G},\mu}(\mathscr{P}_X(\mu,\Bbbk)),$$

so that in particular $\mathsf{F}^X_{\mathcal{G},\mu}(\mathscr{P}_X(\mu,\mathbb{O}))$ is free over $\mathbb{O}$. Similarly, we claim that

$$\Bbbk \overset{L}{\otimes}_{\mathbb{O}} {}^{\mathrm{p}}\mathscr{H}^0(j_!j^*\mathscr{P}_X(\mu,\mathbb{O}))$$

is perverse, and hence identifies with ${}^{\mathrm{p}}\mathscr{H}^0(j_!j^*\mathscr{P}_X(\mu,\Bbbk))$. In fact, this follows from the fact that $\mathscr{P}_Z(\mu,\mathbb{O})$ is a direct summand in $\mathscr{P}_Z(\mathbb{O})$ (where $Z \subset \mathrm{Gr}_{\mathcal{G}}$ is as above in the proof), which admits a filtration with subquotients of the form $\mathscr{J}_!(\lambda,\mathbb{O})$ (see §6.5), and Lemma 6.3(2). From this claim we deduce as above that we have

$$\Bbbk \overset{L}{\otimes}_{\mathbb{O}} \mathsf{F}^X_{\mathcal{G},\mu}({}^{\mathrm{p}}\mathscr{H}^0(j_!j^*\mathscr{P}_X(\mu,\mathbb{O}))) = \mathsf{F}^X_{\mathcal{G},\mu}({}^{\mathrm{p}}\mathscr{H}^0(j_!j^*\mathscr{P}_X(\mu,\Bbbk))),$$

hence in particular that $\mathsf{F}^X_{\mathcal{G},\mu}({}^{\mathrm{p}}\mathscr{H}^0(j_!j^*\mathscr{P}_X(\mu,\mathbb{O})))$ is free over $\mathbb{O}$. We now have a morphism of finite free $\mathbb{O}$-modules

$$\mathsf{F}^X_{\mathcal{G},\mu}({}^{\mathrm{p}}\mathscr{H}^0(j_!j^*\mathscr{P}_X(\mu,\mathbb{O}))) \oplus \mathbb{O} \to \mathsf{F}^X_{\mathcal{G},\mu}(\mathscr{P}_X(\mu,\mathbb{O}))$$

whose image under the functor $\Bbbk \otimes_{\mathbb{O}} (-)$ is an isomorphism by the case $\Lambda = \Bbbk$ treated above; it follows that this morphism itself is an isomorphism. $\square$

Let us consider as in the proof of Proposition 9.1 the quotient map $G \to G_{\mathrm{ad}}$ to the adjoint group, the induced morphism $\mathcal{G} \to \mathcal{G}_{\mathrm{ad}}$ on special parahoric group schemes, and finally the associated map $\mathrm{Hk}_{\mathcal{G}} \to \mathrm{Hk}_{\mathcal{G}_{\mathrm{ad}}}$ on Hecke stacks. We also have an induced map $\pi_1(G)_I \to \pi_1(G_{\mathrm{ad}})_I$, denoted $\tau \mapsto \tau_{\mathrm{ad}}$, and for any τ a map between the associated connected components

$$\mathrm{Hk}^\tau_{\mathcal{G}} \to \mathrm{Hk}^{\tau_{\mathrm{ad}}}_{\mathcal{G}_{\mathrm{ad}}}. \tag{A.5}$$

Corollary A.4. *For each $\tau \in \pi_1(G)_I$ the map (A.5) induces an equivalence of categories*

$$\mathsf{Perv}(\mathrm{Hk}^\tau_{\mathcal{G}},\Lambda) \xrightarrow{\sim} \mathsf{Perv}(\mathrm{Hk}^{\tau_{\mathrm{ad}}}_{\mathcal{G}_{\mathrm{ad}}},\Lambda).$$

Proof. The map $\mathrm{Gr}^\tau_{\mathcal{G}} \to \mathrm{Gr}^{\tau_{\mathrm{ad}}}_{\mathcal{G}_{\mathrm{ad}}}$ is a universal homeomorphism by [HR23, Proposition 3.5], so induces an equivalence on categories of sheaves. The corollary is now immediate from Proposition A.3. $\square$

Appendix B. Étale sheaves on stacks

B.1. **Constructible derived categories of stacks.** In this paper we consider constructible derived categories of étale sheaves on some stacks. In this subsection we make a few comments on the definition of such categories, and give appropriate references.

B.1.1. *Finite coefficients.* Let S be a base scheme which is the spectrum of a field, either finite or separably closed. First, assume that Λ is a finite field, of characteristic ℓ which is invertible on S. For any Artin stack X of finite type over S, in [LO08a] the authors define the constructible derived category $D_{\mathrm{c}}(X,\Lambda)$ of étale sheaves of Λ-modules on X, together with their bounded versions $D^+_{\mathrm{c}}(X,\Lambda)$, $D^-_{\mathrm{c}}(X,\Lambda)$, $D^{\mathrm{b}}_{\mathrm{c}}(X,\Lambda)$. They also define bifunctors

$$(-) \overset{L}{\otimes}_{\Lambda} (-) \colon D^-_{\mathrm{c}}(X,\Lambda) \times D^-_{\mathrm{c}}(X,\Lambda) \to D^-_{\mathrm{c}}(X,\Lambda),$$
$$R\mathscr{H}om_{\Lambda}(-,-) \colon D^-_{\mathrm{c}}(X,\Lambda) \times D^+_{\mathrm{c}}(X,\Lambda) \to D^+_{\mathrm{c}}(X,\Lambda)$$

and, for an S-morphism $f \colon X \to Y$ between such stacks, push/pull functors

$$f_* \colon D^+_{\mathrm{c}}(X,\Lambda) \to D^+_{\mathrm{c}}(Y,\Lambda), \quad f_! \colon D^-_{\mathrm{c}}(X,\Lambda) \to D^-_{\mathrm{c}}(Y,\Lambda),$$
$$f^* \colon D_{\mathrm{c}}(Y,\Lambda) \to D_{\mathrm{c}}(X,\Lambda), \quad f^! \colon D_{\mathrm{c}}(Y,\Lambda) \to D_{\mathrm{c}}(X,\Lambda).$$

They prove that these functors satisfy the familiar properties usually gathered under the term "six functors formalism," with the exception of the base change theorem, for which only weaker versions are obtained.

A more general formalism is constructed in [LZ17b], based on the theory of ∞-categories. (This does not require assumptions on S.) Passing to homotopy categories in their construction one recovers the categories of [LO08a] in the setting above, see [LZ17b, §6.5]. Their constructions also provide alternative constructions for the push/pull functors (assuming f is quasi-separated for the functor f_*) and the other functors considered above, and they show that they do satisfy the base change theorem in its usual form.

By construction the (bi)functors f^*, $f^!$, $R\mathscr{H}om_{\Lambda}(-,-)$ and $(-) \otimes^L_{\Lambda} (-)$ restrict to bounded constructible categories. For the functors f_* and $f_!$ this is not always true (e.g. for the natural morphism $\mathrm{pt} \to \mathrm{pt}/\mathbb{G}_{\mathrm{m}}$), but it *is* in case f is obtained from an equivariant morphism of schemes by passing to quotient schemes (with respect to the action of an affine group scheme of finite type). This is the only case we consider in the body of the paper.

In [LO09], the authors explain the construction of the perverse t-structure (associated with the middle perversity) on the category $D^{\mathrm{b}}_{\mathrm{c}}(X,\Lambda)$. See also [LZ17a, §3] for an alternative construction.

B.1.2. *Adic coefficients.* We continue with the geometric setting above, and with our prime number ℓ invertible on S, but take now for Λ the ring of integers in a finite extension of $\mathbb{Q}_\ell$. In [LO08b] the authors explain how to define for any Artin stack of finite type X the constructible derived category $D_{\mathrm{c}}(X,\Lambda)$ of étale sheaves of Λ-modules on X, together with their bounded versions $D^+_{\mathrm{c}}(X,\Lambda)$, $D^-_{\mathrm{c}}(X,\Lambda)$, $D^{\mathrm{b}}_{\mathrm{c}}(X,\Lambda)$. They also define bifunctors $(-) \otimes^L_{\Lambda} (-)$ and $R\mathscr{H}om_{\Lambda}(-,-)$ and, for any S-morphism $f \colon X \to Y$ between such stacks, push/pull functors f_*, $f_!$, f^* and $f^!$ as above. They prove that these functors satisfy all the expected properties, except for the base change theorem.

A more general formalism is constructed in [LZ17a], based on the theory of ∞-categories; see in particular [LZ17a, §2.1 and §2.3]. Passing to homotopy categories in their construction one recovers the categories of [LO08b], see [LZ17a, §2.5]. Their constructions also provide alternative constructions for the push/pull functors (assuming f is quasi-separated for the functor f_*) and the bifunctors considered above, and they show that they do satisfy the base change theorem in its usual form.

The same comments as above apply regarding restrictions to bounded derived categories. In the case X is a scheme, the category $D^{\mathrm{b}}_{\mathrm{c}}(X,\Lambda)$ is equivalent to the version defined by Deligne, as explained in [LO08b, §3.1].

In this setting also we have a perverse t-structure (for the middle perversity), as explained in [LO09] and in [LZ17a, §3].

B.1.3. *Characteristic-0 coefficients.* We consider once again the geometric setting above and our prime number ℓ, and take for Λ a finite extension of $\mathbb{Q}_\ell$. For an Artin stack of finite type X, one can define the derived category $D_{\mathrm{c}}(X,\Lambda)$ following [LO08b, Remark 3.1.7] (see also the discussion in [Zhe15, §6]). Namely, denoting by Λ_0 the ring of integers in Λ, one defines $D_{\mathrm{c}}(X,\Lambda)$ as the Verdier quotient of $D_{\mathrm{c}}(X,\Lambda_0)$ by the triangulated subcategory consisting of complexes all of whose cohomology objects are annihilated by a power of a uniformizer. One can make similar definitions for the bounded versions.

We will denote by

$$\Lambda \otimes_{\Lambda_0} (-) \colon D^{\mathrm{b}}_{\mathrm{c}}(X,\Lambda_0) \to D^{\mathrm{b}}_{\mathrm{c}}(X,\Lambda)$$

the quotient functor; it has the property that for any complexes $\mathscr{F},\mathscr{G} \in D^{\mathrm{b}}_{\mathrm{c}}(X,\Lambda_0)$ we have a canonical identification

$$\Lambda \otimes_{\Lambda_0} \mathrm{Hom}_{D^{\mathrm{b}}_{\mathrm{c}}(X,\Lambda_0)}(\mathscr{F},\mathscr{G}) \xrightarrow{\sim} \mathrm{Hom}_{D^{\mathrm{b}}_{\mathrm{c}}(X,\Lambda)}(\Lambda \otimes_{\Lambda_0} \mathscr{F}, \Lambda \otimes_{\Lambda_0} \mathscr{G}).$$

The six operations for sheaves with coefficients in Λ_0 induce functors between the versions for Λ, which we will denote in a similar way, and these functors satisfy the same properties. Moreover, by construction the functor $\Lambda \otimes_{\Lambda_0} (-)$ commutes with all sheaf operations. Finally we have a perverse t-structure in this case too, such that the functor $\Lambda \otimes_{\Lambda_0} (-)$ is t-exact.

B.1.4. *Change of scalars.* An important role in our constructions is played by some "change of scalars" functors, which are defined as follows. First we consider an extension of finite fields $\Lambda \to \Lambda'$. For such data, and for any Artin stack X as above, as explained in [LZ17b, §6.2][8] there exist natural adjoint functors

$$\Lambda' \otimes_{\Lambda} (-) \colon D^{\mathrm{b}}_{\mathrm{c}}(X,\Lambda) \to D^{\mathrm{b}}_{\mathrm{c}}(X,\Lambda'), \quad D^{\mathrm{b}}_{\mathrm{c}}(X,\Lambda') \to D^{\mathrm{b}}_{\mathrm{c}}(X,\Lambda)$$

which we will call extension and restriction of scalars, respectively. (The second functor will be given no notation.) By construction these functors commute with all sheaf-theoretic functors, and they are t-exact for the perverse t-structures.

Next we consider an extension between finite extensions of $\mathbb{Q}_\ell$, and the induced morphism $\Lambda \to \Lambda'$ between the rings of integers. This morphism induces a morphism between the associated diagrams involved in the $\mathfrak{m}$-adic formalism of [LZ17a, §2], and for any stack X as above we have associated adjoint functors

$$\Lambda' \otimes_{\Lambda} (-) \colon D^{\mathrm{b}}_{\mathrm{c}}(X,\Lambda) \to D^{\mathrm{b}}_{\mathrm{c}}(X,\Lambda'), \quad D^{\mathrm{b}}_{\mathrm{c}}(X,\Lambda') \to D^{\mathrm{b}}_{\mathrm{c}}(X,\Lambda).$$

Once again, these functors commute with all sheaf-theoretic constructions in the obvious way, and they are t-exact for the perverse t-structures. (We also have similar functors for the fraction fields of λ and Λ', but they will not be considered in this paper.)

Finally, we consider the case when Λ is the ring of integers in a finite extension of $\mathbb{Q}_\ell$, and Λ' is its residue field. The definition of $D^{\mathrm{b}}_{\mathrm{c}}(X,\Lambda)$ involves the "ringed diagram" $(\mathbb{N},\Lambda_\bullet)$ in the notation of [LZ17a, §2.1]. There exists a natural morphism from this ringed diagram to the ringed diagram $(\{0\},\Lambda')$, and associated with this diagram we have natural adjoint functors

$$\Lambda' \overset{L}{\otimes}_{\Lambda} (-) \colon D^{\mathrm{b}}_{\mathrm{c}}(X,\Lambda) \to D^{\mathrm{b}}_{\mathrm{c}}(X,\Lambda'), \quad D^{\mathrm{b}}_{\mathrm{c}}(X,\Lambda') \to D^{\mathrm{b}}_{\mathrm{c}}(X,\Lambda).$$

[8] It is not stated explicitly in [LZ17b] that these functors send constructible complexes to constructible complexes. However, by definition it suffices to prove this property in the case of schemes, where it is clear. Similar comments apply for the variants considered below.

Here again, these functors commute with all sheaf-theoretic constructions in the obvious way. The right-hand functor is t-exact for the perverse t-structures, but the functor $\Lambda' \otimes^L_\Lambda (-)$ is only left t-exact.

B.1.5. *Nearby cycles.* The last ingredient from (étale) sheaves theory we use in the paper is an appropriate version of the nearby cycles functor. (See Remark 8.4 for the relation with more standard constructions.) First we assume that Λ is a finite field. Let R be an absolutely integrally closed valuation ring of rank 1, set $S := \mathrm{Spec}(R)$, and denote by s and η the special and generic points in S respectively. (See §8.4 for an example of this setting; in this example the scheme playing the role of S is denoted $\overline{S}$, and the generic point is denoted $\overline{\eta}$.) Given an Artin stack of finite type X over S, we set

$$X_s := X \times_S s, \quad X_\eta := X \times_S \eta,$$

and consider the obvious morphisms

$$i \colon X_s \to X, \quad j \colon X_\eta \to X.$$

We then set

$$\Psi := i^* j_* \colon D(X_\eta, \Lambda) \to D(X_s, \Lambda),$$

where the derived categories of sheaves we consider here are the categories denoted $\mathrm{D}_{\mathrm{cart}}(X_{\mathrm{lis-\acute{e}t}}, \Lambda)$ in [LZ17b]. It follows from [HS23, Corollary 4.2] that this functor restricts to a functor

$$D^{\mathrm{b}}_{\mathrm{c}}(X_\eta, \Lambda) \to D^{\mathrm{b}}_{\mathrm{c}}(X_s, \Lambda).$$

By [HS23, Lemma 6.3] this functor is t-exact for the perverse t-structures. (The authors in [HS23] work with schemes and not with stacks, but the properties we consider here can be tested on schemes.)

The same constructions can be considered for the other choices of coefficients considered in §§B.1.2–B.1.3.

B.2. **A semismallness criterion.** Let k be a separably closed field, and let X, Y be k-schemes of finite type. We assume we are given some finite sets $\mathcal{A}$ and $\mathcal{B}$ any, for any $\alpha \in \mathcal{A}$, resp. $\beta \in \mathcal{B}$, a smooth and irreducible locally closed subscheme $X_\alpha \subset X$, resp. $Y_\beta \subset Y$, such that

$$|X| = \bigsqcup_{\alpha \in \mathcal{A}} |X_\alpha|, \quad \text{resp.} \quad |Y| = \bigsqcup_{\beta \in \mathcal{B}} |Y_\beta|,$$

and such that the closure of each stratum is a union of strata. We will denote by $j_\alpha \colon X_\alpha \to X$ and $j_\beta \colon Y_\beta \to Y$ the inclusions. Recall that a morphism of schemes $f \colon X \to Y$ is said to be:

- *stratified locally trivial* if for any $\alpha \in \mathcal{A}$ the subset $f(X_\alpha)$ is a union of strata in Y and if moreover for any $\alpha \in \mathcal{A}$ and $\beta \in \mathcal{B}$ such that $Y_\beta \subset f(X_\alpha)$ the restriction of f to a morphism $f^{-1}(Y_\beta) \cap X_\alpha \to Y_\beta$ is an étale locally trivial fibration;
- *stratified semismall* if for any $\alpha \in \mathcal{A}$, $\beta \in \mathcal{B}$ and $y \in Y_\beta$ we have

$$\dim(f^{-1}(y) \cap X_\alpha) \leq \tfrac{1}{2}(\dim(X_\alpha) - \dim(Y_\beta)). \tag{B.1}$$

It is well known that if f is proper, stratified locally trivial and stratified semi-small, then the push-forward functor f_* (for any ring of coefficients as considered in the paper) sends perverse sheaves on X that are constructible with respect to the given stratification of X to perverse sheaves on Y that are constructible with respect to the given stratification on Y. (See, for instance, [MV07, Lemma 4.3] or [BR18, Proposition 1.6.1].)

It turns out that this statement has a partial converse, which we will now explain. Let ℓ be a prime number that is invertible in k, and denote by $\mathbb{K}$ a finite extension of $\mathbb{Q}_\ell$. Assume that the following conditions are satisfied:

- for any $\alpha, \alpha' \in \mathcal{A}$ and any $n \in \mathbb{Z}$ the sheaf
$$\mathscr{H}^n((j_{\alpha'})^*(j_\alpha)_*\underline{\mathbb{K}}_{X_\alpha})$$
is constant.
- for any $\alpha \in \mathcal{A}$ we have $\mathsf{H}^1(X_\alpha; \mathbb{K}) = 0$.

Then as in [BBDG82, §§2.2.9–2.2.18] one can consider the full triangulated subcategory $D^{\mathrm{b}}_{\mathcal{A}}(X, \mathbb{K})$ of $D^{\mathrm{b}}_{\mathrm{c}}(X, \mathbb{K})$ whose objects are the complexes $\mathscr{F}$ such that for any $n \in \mathbb{Z}$ and $\alpha \in \mathcal{A}$ the sheaf $\mathscr{H}^n((j_\alpha)^*\mathscr{F})$ is constant. The perverse t-structure on $D^{\mathrm{b}}_{\mathrm{c}}(X, \mathbb{K})$ restricts to a t-structure on $D^{\mathrm{b}}_{\mathcal{A}}(X, \mathbb{K})$; in particular, for any $\alpha \in \mathcal{A}$ we have the intersection cohomology complex $\mathscr{IC}(X_\alpha, \underline{\mathbb{K}}) \in D^{\mathrm{b}}_{\mathcal{A}}(X, \mathbb{K})$ associated with the constant local system on X_α, which is a simple perverse sheaf. Similarly, under the analgous assumptions on Y and its stratification, one can consider the category $D^{\mathrm{b}}_{\mathcal{B}}(Y, \mathbb{K})$ and its perverse t-structure.

Let us assume that both of these conditions are satisfied, and consider a morphism $f : X \to Y$.

Lemma B.1. *Let $f : X \to Y$ be a morphism of k-schemes, and assume that the following conditions are satisfied:*

- *for any $\alpha \in \mathcal{A}$, $\beta \in \mathcal{B}$ and $n \in \mathbb{Z}$ the sheaf*
$$\mathscr{H}^n((j_\beta)^* f_! \mathscr{IC}(X_\alpha, \underline{\mathbb{K}}))$$
is constant (in other words, for any $\alpha \in \mathcal{A}$ the complex $f_!\mathscr{IC}(X_\alpha, \underline{\mathbb{K}})$ belongs to $D^{\mathrm{b}}_{\mathcal{B}}(Y, \mathbb{K})$);
- *for any $\alpha \in \mathcal{A}$ the complex $f_!\mathscr{IC}(X_\alpha, \underline{\mathbb{K}})$ is a perverse sheaf.*

Then f is stratified semismall.

Proof. We need to prove the inequality (B.1) for any $\alpha \in \mathcal{A}$, $\beta \in \mathcal{B}$ and $y \in Y_\beta$. We proceed by induction on α, with respect to the order given by inclusions of closures of strata. Let $\partial X_\alpha := \overline{X_\alpha} \smallsetminus X_\alpha$, so that we can assume the claim is known for any α' such that $X_{\alpha'} \subset \partial X_\alpha$. Fix also $\beta \in \mathcal{B}$ and $y \in Y_\beta$. The assumption that $f_!\mathscr{IC}(X_\alpha, \underline{\mathbb{K}})$ is perverse implies that the $\mathbb{K}$-vector space
$$\mathsf{H}^j\big((f_!\mathscr{IC}(X_\alpha, \underline{\mathbb{K}}))_y\big) = \mathsf{H}^j_c(f^{-1}(y), \mathscr{IC}(X_\alpha, \underline{\mathbb{K}})_{|f^{-1}(y)})$$
vanishes unless $j \leq -\dim(Y_\beta)$. On the other hand, $\mathscr{IC}(X_\alpha, \underline{\mathbb{K}})$ is supported on $\overline{X_\alpha}$, and its restriction to X_α is $\underline{\mathbb{K}}_{X_\alpha}[\dim(X_\alpha)]$. We deduce a long exact sequence
$$\cdots \to \mathsf{H}^{j-1}_c(f^{-1}(y) \cap \partial X_\alpha, \mathscr{IC}(X_\alpha, \underline{\mathbb{K}})) \to \mathsf{H}^{j+\dim(X_\alpha)}_c(f^{-1}(y) \cap X_\alpha; \mathbb{K})$$
$$\to \mathsf{H}^j_c(f^{-1}(y), \mathscr{IC}(X_\alpha, \underline{\mathbb{K}})) \to \mathsf{H}^j_c(f^{-1}(y) \cap \partial X_\alpha, \mathscr{IC}(X_\alpha, \underline{\mathbb{K}})) \to \cdots$$
For any $\alpha' \in \mathcal{A}$ such that $X_{\alpha'} \subset \partial X_\alpha$, the complex $(j_{\alpha'})^*\mathscr{IC}(X_\alpha, \underline{\mathbb{K}})$ is concentrated in degrees $\leq -\dim(X_{\alpha'}) - 1$, and by induction we have
$$\dim(f^{-1}(y) \cap X_{\alpha'}) \leq \tfrac{1}{2}(\dim(X_{\alpha'}) - \dim(Y_\beta)).$$
It follows that the space $\mathsf{H}^j_c(f^{-1}(y) \cap \partial X_\alpha, \mathscr{IC}(X_\alpha, \underline{\mathbb{K}}))$ vanishes unless
$$j \leq \dim(X_{\alpha'}) - \dim(Y_\beta) - \dim(X_{\alpha'}) - 1,$$
i.e. unless
$$j \leq -\dim(Y_\beta) - 1.$$
(Here we use the fact that cohomology with compact supports of a scheme of finite type is concentrated in degrees at most twice the dimension of the scheme.)

Now, assume for a contradiction that
$$d := \dim(f^{-1}(y) \cap X_\alpha) > \tfrac{1}{2}(\dim(X_\alpha) - \dim(Y_\beta)).$$
Then we have an injection
$$\mathsf{H}^{2d}_c(f^{-1}(y) \cap X_\alpha; \mathbb{K}) \hookrightarrow \mathsf{H}^{2d-\dim(X_\alpha)}_c(f^{-1}(y), \mathscr{IC}(X_\alpha, \underline{\mathbb{K}}))$$

since $\mathsf{H}_c^{2d-\dim(X_\alpha)-1}(f^{-1}(y) \cap \partial X_\alpha, \mathscr{I}\mathscr{C}(X_\alpha, \underline{\mathbb{K}})) = 0$. The left-hand side is nonzero, and hence so is $\mathsf{H}_c^{2d-\dim(X_\alpha)}(f^{-1}(y), \mathscr{I}\mathscr{C}(X_\alpha, \underline{\mathbb{K}}))$, which implies that

$$2d - \dim(X_\alpha) \leq -\dim(Y_\beta),$$

a contradiction. □

Remark B.2. The principle expressed in Lemma B.1 is mentioned in [MV07, Remark 4.5], and was explained to the fourth named author by I. Mirković a long time ago. It justifies [BR18, Remark 1.6.5(2)], which explains that the stratified semismallness claim needed in the "classical" geometric Satake equivalence can be deduced from a claim about convolution of intersection cohomology complexes over $\mathbb{Q}_\ell$, which itself can be deduced from the results of [Lus83].

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MOD p SHEAVES ON WITT FLAGS

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Abstract. We characterize Cohen–Macaulay and φ-rational perfect schemes in terms of their perverse étale $\mathbb{F}_p$-sheaves. Using inversion of adjunction, we prove that sufficiently small Schubert varieties in the Witt affine flag variety are perfections of globally $+$-regular varieties, and hence they are φ-rational. Our methods apply uniformly to all affine Schubert varieties in equicharacteristic, as well as classical Schubert varieties, thereby answering a question of Bhatt. As a corollary, we deduce that scheme-theoretic local models always have φ-split special fiber.

Contents

1. Introduction

Hecke categories, i.e., categories of sheaves on local Hecke stacks $\mathrm{Hk}_{\mathcal{G}}$, play a major role in geometric representation theory and in geometric approaches to the Langlands program. Here $\mathcal{G}$ is a parahoric model of a connected reductive group G over a local field F with residue field k of characteristic p. If one considers $\mathbb{F}_\ell$ or $\mathbb{Q}_\ell$-étale sheaves for some prime $\ell \neq p$, then these categories are well-studied for F of equal or mixed characteristic. For example, at hyperspecial level one has the geometric Satake equivalence, e.g. [MV07, Zhu17], and at other parahoric levels there is at least a collection of central sheaves, e.g. [Gai01, ALWY23]. Recently, it has even become possible to talk about $\mathbb{Q}$- and $\mathbb{Z}$-linear motivic sheaves on Hecke stacks, e.g. [RS21, CvdHS22, CvdHS24, vdH24].

The situation changes drastically when one considers étale sheaves for $\ell = p$. There is still a perverse t-structure due to Gabber [Gab04], but its behavior can be quite strange, as half of the six functors do not preserve constructibility. When F has characteristic p, the first author constructed a geometric Satake equivalence in [Cas22], where the dual object is a monoid instead of a group, and also a central functor in [Cas21]. The purpose of this paper, as the title suggests, is to launch an investigation of these properties when F has characteristic 0. In this case, the local Hecke stack $\mathrm{Hk}_{\mathcal{G}}$ and the corresponding affine flag variety $\mathrm{Fl}_{\mathcal{G}}$ exist only canonically as functors on perfect k-algebras.

Let us note that in [CX25] the mod p Hecke category for F of characteristic 0 was already studied, but with no concern for the perverse t-structure. Thus, our first order of business is to investigate IC sheaves in the Hecke category. Toward this direction, we prove the following general result.

Theorem 1.1. *Let k be a perfect field of characteristic p and let X be a connected perfectly finitely presented k-scheme. Then X is Cohen–Macaulay (resp., φ-rational) if and only if the shifted constant sheaf $\mathbb{F}_p[\dim X]$ is perverse (resp., perverse and simple).*

In [Cas22] it was shown that the above commutative-algebraic properties of a finite-type k-scheme imply the corresponding properties of perverse $\mathbb{F}_p$-sheaves, but the fact that the converse holds after passing to the perfection lies much deeper. As it turns out, Theorem 1.1 was known to experts in the φ-singularities community and appeared in [BBL+23], after we already found an argument independently. We have included our argument for the benefit of readers unfamiliar with the literature on φ-singularities, and because it differs significantly from the one in [BBL+23] in that we perform most of the key arguments on the coherent as opposed to topological side. Our notions of Cohen–Macaulayness and φ-rationality for a perfect scheme X, which are in fact properties of the local rings of X, require the following notions from commutative algebra.

Recall that a noetherian local ring $(R, \mathfrak{m})$ is Cohen–Macaulay if the local cohomology groups $H^i_{\mathfrak{m}}(R)$ vanish for $i < \dim R$. If R has characteristic p, the absolute Frobenius φ gives $H^i_{\mathfrak{m}}(R)$ a module structure over the non-commutative polynomial ring $R[\varphi]$. When R is also excellent and φ-finite, one says that R is φ-rational if it is Cohen–Macaulay and $H^{\dim R}_{\mathfrak{m}}(R)$ is a simple $R[\varphi]$-module. In the perfect case, we define Cohen–Macaulayness and φ-rationality in exactly the same way. Since the local cohomology of the perfection R^{perf} is the perfection of $H^i_{\mathfrak{m}}(R)$ with respect to φ, these notions capture phenomena from the noetherian case up to elements annihilated by some iterate of φ. We prove in Lemma 2.21 that R^{perf} is φ-rational if and only if R is φ-nilpotent, where the latter property is a topic of active research in commutative algebra.

The proof of the converse direction for both properties in Theorem 1.1 involves a noetherian induction and passage to the strict henselization. To prove Cohen–Macaulayness we apply an argument already present in [Bha20], which allows us to prove that the lower local cohomomlogy groups are supported on a geometric point and are holonomic in the sense of Bhatt–Lurie's Riemann–Hilbert correspondence [BL19]. In particular, their vanishing can be checked after passing to φ-invariants, where it is guaranteed by the perversity of $\mathbb{F}_p[\dim X]$. To prove φ-rationality we also need to invoke a strong finiteness result for simple φ-submodules of the top local cohomology due to Lyubeznik [Lyu97]. Both arguments utilize Matlis duality in an essential way.

We now return to the affine flag variety $\mathrm{Fl}_{\mathcal{G}}$. The Bruhat decomposition yields Schubert subvarieties $\mathrm{Fl}_{\mathcal{G},\leq w}$ indexed by double cosets w of the Iwahori–Weyl group. At this point, it is natural to formulate the following expectation.

Conjecture 1.2. *The perfect Schubert schemes $\mathrm{Fl}_{\mathcal{G},\leq w}$ are φ-rational.*

Let us present some evidence for this conjecture. If F has characteristic p, this result follows from [Cas22] for split G and [FHLR22] for general G. If one had an analogue of the Grauert–Riemenschneider theorem in perfect geometry (which is known for finite-type φ-split varieties), then Cohen–Macaulayness would be a consequence of the triviality of the higher direct images of the structure sheaf along Demazure resolutions, which was proved in [CX25]. We do not know of such a result, but for a certain class of sufficiently small w, we are able to show using inversion of adjunction that there is a certain deperfection $\mathrm{Fl}_{\mathcal{G},\leq w,1}$ which is globally $+$-regular in the sense of [BMP+23]. This property, which we explain next, is stronger than φ-rationality.

The property of φ-rationality is of a local nature, and in particular, it does not descend along proper covers. In order to get proper descent, one has to define a global variant of φ-rationality, but it is unclear how to proceed in the perfect setting. For classical schemes, this is well understood via the property of strong φ-regularity of Hochster–Huneke [HH89] and its global variant [Smi00]. In this paper, we prefer to use the closely related property of global $+$-regularity, as it carries the advantage of making every $\mathbb{Q}$-divisor integral up to passing to a cyclic cover. However, new ideas are still required because the proof strategy in [Cas22, FHLR22] relies on the criterion of Mehta–Ramanathan [MR85]. This pressuposes the existence of certain theta divisors that Faltings [Fal03] constructs in equicharacteristic on the natural deperfection of the whole $\mathrm{Fl}_{\mathcal{G}}$, but such a deperfection does not exist in mixed characteristic. Let us first state our result, and then we will explain the various notations and hypotheses.

Theorem 1.3. *Assume $s_\bullet$ is a reduced word for w, and $q_\bullet = 1$ is $s_\bullet$-permissible. Then $(\mathrm{Fl}_{\mathcal{G},\leq w,1}, \Delta)$ is globally $+$-regular for any $\mathbb{Q}$-divisor $\Delta \leq \partial_{w,1}$ with $\lfloor \Delta \rfloor = 0$.*

When F has characteristic p each $\mathrm{Fl}_{\mathcal{G},\leq w}$ is canonically isomorphic to the perfection of a projective k-scheme $\mathrm{Fl}_{\mathcal{G},\leq w,1}$ (the seminormalization of the affine Schubert variety in [PR08], see also [HLR24, FHLR22] for the necessity of this functor). In this case, the hypothesis that $q_\bullet = 1$ is $s_\bullet$-permissible is automatically satisfied for every reduced word for w. The divisor $\partial_{w,1}$ is the sum of the Iwahori–Schubert subvarieties in codimension one for a fixed choice of Iwahori $\mathcal{I}$ mapping to $\mathcal{G}$. Moreover, whenever the hypotheses of Theorem 1.3 are satisfied (even when F has characteristic 0), we deduce in Proposition 4.11 that $\mathrm{Fl}_{\mathcal{G},\leq w,1}$ is globally φ-regular, and compatibility φ-split with all of its Schubert subvarieties. Thus, when F has characteristic p we obtain a new proof of the global φ-regularity of affine Schubert varieties, first shown in [Cas22, FHLR22], but which avoids the the Mehta–Ramanathan criterion. Moreover, applying this criterion to wildly ramified groups in [FHLR22] required extra casework, whereas our new proof is uniform across all groups.

Remark 1.4. Bhatt [Bha12] proved that Schubert varieties in the classical finite flag variety of GL_n in positive characteristic are derived splinters (an alternative name for globally $+$-regular), using inversion of adjunction. Bhatt asked in [Bha12, Remarks 7.8 and 7.10] if his methods could be generalized to general groups, and our proof of Theorem 1.3 answers this question positively. Indeed, all classical Schubert varieties arise as particular affine Schubert varieties for F of characteristic p. In fact, our proof also applies in the context of Kac–Moody groups, again without any conditions on w. We note here that global φ-regularity of classical Schubert varieties was known much earlier by [LRPT06], which used the Mehta–Ramanathan criterion

Let us now explain the permissibility hypothesis in Theorem 1.3, which cannot be avoided when F has characteristic 0. The first step to prove global $+$-regularity is to replace $\mathrm{Fl}_{\mathcal{G},\leq w}$ by its proper modification $\mathrm{Fl}_{\mathcal{I},\leq s_\bullet}$ for some reduced word $s_\bullet$ for w. Here $\mathcal{I} \subset \mathcal{G}$ is an Iwahori subgroup, and $\mathrm{Fl}_{\mathcal{I},\leq s_\bullet}$ is the Demazure resolution. We then construct a class of deperfections $\mathrm{Fl}_{\mathcal{I},\leq s_\bullet,q_\bullet}$, where $q_\bullet$ is a certain $s_\bullet$-permissible sequence of powers of p. This notion of permissibility is defined by induction on the length of the sequence. The factor $\mathrm{Fl}_{\mathcal{I},\leq s_i,q_i}$ in the twisted product $\mathrm{Fl}_{\mathcal{I},\leq s_\bullet,q_\bullet}$ is the φ_{q_i}-twist of the canonical Iwahori-equivariant smooth deperfection $\mathrm{Fl}_{\mathcal{I},\leq s_i,1} \simeq \mathbb{P}^1_k$ of $\mathrm{Fl}_{\mathcal{I},\leq s_i}$. In order for the twisted product to exist when F has characteristic 0, we are forced to twist every new factor to the right by a nonnegative power of p, which we have no control over. Thus, $q_\bullet$ is is a non-decreasing (and not so rarely increasing) sequence, which ultimately hinders proving global $+$-regularity.

If the constant sequence $q_\bullet = 1$ is $s_\bullet$-permissible, then we can carry out inversion of adjunction. Indeed, a calculation for the boundary pair $(\mathrm{Fl}_{\mathcal{I},\leq s_\bullet,q_\bullet}, \partial_{s_\bullet,q_\bullet})$ reveals that its anti-canonical divisor is semi-ample (and in this case also big) precisely when $q_\bullet$ is non-increasing, so constant sequences are the optimal scenario. This assumption holds for all w if F has characteristic p and for all w in the μ-admissible set of Kottwitz–Rapoport [KR00] associated with some minuscule conjugacy class of geometric coweights μ. The idea for applying the inversion of adjunction criterion for global $+$-regularity of pairs of [BMP+23] is then to slightly perturb the coefficients of the boundary $\partial_{s_\bullet,1}$ in such a way that the anti-canonical divisor of the pair becomes ample.

Finally, we give an application to local models. Recall that [AGLR22, GL24] prove the existence and uniqueness of normal flat O_E-schemes $M_{\mathcal{G},\mu}$ with reduced special fiber representing a certain closed v-subsheaf of the Beilinson–Drinfeld Grassmannian $\mathrm{Gr}_{\mathcal{G}}$, provided either μ is minuscule or F has characteristic p. In [FHLR22] it was proved for all groups except wild odd unitary ones that the special fiber is moreover φ-split. Now, we can generalize this to all groups and prove it uniformly. This finishes the problem of determining the special fiber of $M_{\mathcal{G},\mu}$ in full generality and thus Cohen–Macaulayness is the only property remaining in the above mentioned series of papers for which the infamous hypothesis "$p > 2$ or Φ_G is reduced" is still needed.

Corollary 1.5. *Assume F has characteristic p or μ is minuscule. Then, the special fiber of $M_{\mathcal{G},\mu}$ equals the canonical deperfection $A_{\mathcal{G},\mu,1}$ in the sense of* [AGLR22] *of the μ-admissible locus. Moreover, $A_{\mathcal{G},\mu,1}$ is φ-split compatibly with every $\mathcal{G}(O)$-stable closed subscheme.*

The idea goes as follows: once we know that Schubert varieties in the μ-admissible locus have globally $+$-regular deperfections $\mathrm{Fl}_{\mathcal{G},\leq w,1}$, we can construct a φ-split canonical deperfection $A_{\mathcal{G},\mu,1}$ of the admissible locus $A_{\mathcal{G},\mu}$. Then, it suffices to prove the coherence conjecture of [PR08], i.e., we need to compute global sections of certain line bundles for the previous deperfection and for the generic fiber of $M_{\mathcal{G},\mu}$. The φ-splitness yields higher vanishing of cohomology for ample line bundles, so we get an inclusion-exclusion type formula in terms of Schubert subvarieties for the global sections dimension. But the latter can be computed by the Demazure character formula, so it does not change if we replace G by another group with the same combinatorics. Therefore, we can reduce to tame G and equicharacteristic F, already handled by Zhu [Zhu14]. We also note that the corollary above was used in [Lou23] when F has equicharacteristic and $\pi_1(G)$ is p-torsion free to finish the proof for all G of normality of Schubert varieties embedded in the usual scheme-theoretic affine flag varieties (i.e., before taking seminormalizations).

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2. Perverse $\mathbb{F}_p$-sheaves and φ-singularities

Fix a prime number p. For a scheme X over $\mathbb{F}_p$, let φ be the absolute Frobenius morphism. We will often be concerned with noetherian schemes which are φ-finite, meaning that $\varphi_*\mathcal{O}_X$ is a finite $\mathcal{O}_X$-module. By Kunz's theorem [Kun76, Theorem 2.5], a φ-finite noetherian ring is excellent. Additionally, a noetherian φ-finite scheme admits a coherent dualizing complex [Gab04, Remark 13.6]. The proof in loc. cit. only applies when X is affine, which is the only case we will use. Recall also that the perfection of a scheme X is $X^{\mathrm{perf}} = \lim(\cdots \xrightarrow{\varphi} X \xrightarrow{\varphi} X)$. A deperfection of a perfect scheme X is a scheme X_0 equipped with an isomorphism $X_0^{\mathrm{perf}} \cong X$.

2.1. **Cartier modules.** Let R be a ring over $\mathbb{F}_p$ and let φ_*R be the R-module associated to $\varphi_*\mathcal{O}_{\mathrm{Spec}(R)}$. Recall from [BB11] that a Cartier module over R consists of an R-module M with a map $\varphi_*M \to M$. Homomorphisms between Cartier modules must respect this map. A Cartier module M is said to be nilpotent if $\varphi^e_*M \to M$ is zero for some $e \geq 0$. Furthermore, a Cartier module is said to be coherent if its underlying R-module is finite. We have the following decisive structure theorem for Cartier modules.

Theorem 2.1 (Blickle–Böckle)**.** *Let R be a noetherian φ-finite ring and let M be a coherent Cartier module.*

(1) *There exists a finite composition series* $0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$ *by coherent Cartier submodules such that each* M_i/M_{i+1} *is either nilpotent, or non-nilpotent and simple.*
(2) *If* M *is a simple coherent Cartier module then* M *has a unique associated prime* $\mathfrak{p} \in \operatorname{Spec}(R)$. *Furthermore,* $M \subset M_{\mathfrak{p}}$, *and the latter is a finite-dimensional vector space over* $R/\mathfrak{p}$.

Proof. Part (1) is [BB11, Proposition 4.23], and part (2) is proved in [BB11, Propositions 4.14, 4.15]. □

Important examples of coherent Cartier modules include the cohomology sheaves $\mathcal{H}^i(\omega_R^\bullet)$ of dualizing complexes on φ-finite noetherian rings. Here the map $\varphi_*\mathcal{H}^i(\omega_R^\bullet) \to \mathcal{H}^i(\omega_R^\bullet)$ is obtained from exactness of φ_* and the adjoint of the canonical isomorphism $\omega_R^\bullet \to \varphi^!\omega_R^\bullet$ from Grothendieck duality.

2.2. **φ-modules.** Let R be an $\mathbb{F}_p$-algebra, and let $R[\varphi]$ be the non-commutative polynomial ring over R in one variable, also denoted φ, subject to the relation $\varphi a = a^p\varphi$ for all $a \in R$. A left $R[\varphi]$-module is the same as an R-module M with an R-linear map $M \to \varphi_*M$; note that the map goes in the direction opposite to that of Cartier modules.

We recall a decisive structure result for $R[\varphi]$-modules closely related to Theorem 2.1. As in the case of Cartier modules, we say that an $R[\varphi]$-module M is nilpotent if $M \to \varphi_*^e M$ is zero for some $e \geq 0$. Similarly, an $R[\varphi]$-module M is co-finite if it is Artinian as an R-module. Important examples of co-finite $R[\varphi]$ modules include the local cohomology groups $H^i_{\mathfrak{m}}(R)$ of noetherian local $\mathbb{F}_p$-algebras $(R, \mathfrak{m})$ [BS13b, Theorem 7.1.3].

Theorem 2.2 (Lyubeznik). *Let* $(R, \mathfrak{m})$ *be a noetherian local* $\mathbb{F}_p$*-algebra and let* M *be a co-finite* $R[\varphi]$*-module.*

(1) M *admits a finite composition series* $0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$ *by co-finite* $R[\varphi]$*-submodules such that each* M_i/M_{i+1} *is either nilpotent, or non-nilpotent and simple.*
(2) *The collection of non-nilpotent simple subquotients of* M *is independent of the composition series.*

Proof. See [Lyu97, Theorem 4.7]. □

It is worth mentioning the following special case of Lyubeznik's theorem, which has been proved via different means by Hartshorne–Speiser, Lyubeznik, Gabber, and Bhatt–Blickle–Lyubeznik–Singh–Zhang.

Corollary 2.3. *Let* $(R, \mathfrak{m})$ *be a noetherian local* $\mathbb{F}_p$*-algebra and let* M *be a co-finite* $R[\varphi]$*-module. Then some power of* φ *annihilates*

$$\{a \in H^i_{\mathfrak{m}}(R) \,:\, \varphi^e(a) = 0 \text{ for some } e > 0\}.$$

Proof. This follows from Theorem 2.2; see also [HS77, Proposition 1.11], [Lyu97, Proposition 4.4], [Gab04, Lemma 13.1] or [BBL+23, Corollary 4.24]. □

We conclude by explaining a precise relation between Cartier modules and $R[\varphi]$-modules. Suppose that $(R, \mathfrak{m})$ is a complete, local, noetherian and φ-finite $\mathbb{F}_p$-algebra. Following [Sta23, Tag 0A82], we normalize the dualizing complex $\omega_R^\bullet$ so that $R\Gamma_{\mathfrak{m}}(\omega_R^\bullet) = E[0]$ lies in degree 0, in which case E is an injective hull of $R/\mathfrak{m}$. Recall that Matlis duality $M \mapsto \operatorname{Hom}_R(M, E)$ gives an anti-equivalence between coherent and Artinian R-modules. Then Matlis duality also induces an anti-equivalence between coherent Cartier modules and co-finite $R[\varphi]$-modules [BB11, Proposition 5.2]. The integer $d := \dim R$ is the largest integer such that $\mathcal{H}^{-d}(\omega_R^\bullet) \neq 0$ [Sta23, 0AWN]; the cohomology sheaf $\omega_R := \mathcal{H}^{-d}(\omega_R^\bullet)$ is called the dualizing sheaf. For each i there is a canonical isomorphism $\mathrm{Hom}_R(\mathcal{H}^{-i}(\omega_R^\bullet), E) \cong H^i_{\mathfrak{m}}(R)$ even if R is not complete, e.g. see [BS13a, 10.2.19], [Sta23, Tag 0AAK].

2.3. **Perverse $\mathbb{F}_p$-sheaves.** For a scheme X over $\mathbb{F}_p$ let $D(X, \mathbb{F}_p)$ be the derived category of étale $\mathbb{F}_p$-sheaves on X, and let $D^b_c(X, \mathbb{F}_p)$ be the bounded constructible subcategory. In this subsection we fix a perfect field k of characteristic p. Every scheme of finite type over k is automatically φ-finite.

Definition 2.4. Let X be a k-scheme of finite type. For each point $x \in X$, fix a strict henselization $\mathcal{O}_x^{\mathrm{sh}}$ of the local ring at x, and let $i_x \colon \overline{x} \to \mathrm{Spec}(\mathcal{O}_x^{\mathrm{sh}})$ be the inclusion of the closed point. We define the full subcategory ${}^pD^{\leq 0}(X, \mathbb{F}_p)$ (resp. ${}^pD^{\geq 0}(X, \mathbb{F}_p)$) of $D(X, \mathbb{F}_p)$ consisting of $\mathcal{F}^\bullet \in D(X, \mathbb{F}_p)$ such that $\mathcal{H}^n(i_x^*\mathcal{F}^\bullet) = 0$ for all $x \in X$ and $n > -\dim \overline{\{x\}}$ (resp. $\mathcal{F}^\bullet$ has bounded below cohomology sheaves and $\mathcal{H}^n(Ri_x^!\mathcal{F}^\bullet) = 0$ for all $x \in X$ and $n < -\dim \overline{\{x\}}$).

The following special case of a theorem of Gabber implies that the subcategories above give a t-strucutre on $D(X, \mathbb{F}_p)$, cf. [EK04b, Theorem 11.5.4]. We call objects in the heart perverse $\mathbb{F}_p$-sheaves.

Theorem 2.5 (Gabber)**.** *Let X be a k-scheme of finite type.*

(1) *The pair $({}^pD^{\leq 0}(X, \mathbb{F}_p), {}^pD^{\geq 0}(X, \mathbb{F}_p))$ gives rise to a t-structure on $D(X, \mathbb{F}_p)$.*
(2) *The t-structure above restricts to a t-structure on $D^b_c(X, \mathbb{F}_p)$.*
(3) *Every perverse subquotient of a constructible perverse $\mathbb{F}_p$-sheaf is constructible, i.e. lies in $D^b_c(X, \mathbb{F}_p)$.*
(4) *Every constructible perverse $\mathbb{F}_p$-sheaf has finite length.*

Proof. See [Gab04, Theorem 10.4, Corollary 12.4]. □

The t-structure on $D^b_c(X, \mathbb{F}_p)$ has also been studied in [Cas22, BBL+23]. By the topological invariance of the small étale site [Sta23, Tag 04DY], we have a canonical equivalence $D^b_c(X, \mathbb{F}_p) \cong D^b_c(X^{\mathrm{perf}}, \mathbb{F}_p)$, so we also get a t-structure for perfections of k-schemes of finite type.

We now recall the notion of intermediate extension for perverse $\mathbb{F}_p$-sheaves. Let $\mathcal{F}^\bullet$ be a constructible perverse $\mathbb{F}_p$-sheaf on U, and let $j \colon U \to X$ be an open immersion into a k-scheme of finite type. By taking perverse truncations of Rj_* and $Rj_!$, we may define

$$j_{!*}\mathcal{F}^\bullet := \mathrm{Im}\,({}^pj_!\mathcal{F}^\bullet \to {}^pj_*\mathcal{F}^\bullet).$$

Note that while ${}^pj_*\mathcal{F}^\bullet$ may not be constructible, both ${}^pj_!\mathcal{F}^\bullet$ and $j_{!*}\mathcal{F}^\bullet$ are constructible. The intermediate extension $j_{!*}\mathcal{F}^\bullet$ is characterized as the unique perverse extension of $\mathcal{F}^\bullet$ with no quotients or subobjects supported on $X \setminus U$. If $i \colon X \setminus U \to X$ is the inclusion (with the reduced scheme structure), the latter conditions are equivalent to $i^*\mathcal{F}^\bullet \in {}^pD^{\leq -1}(X \setminus U, \mathbb{F}_p)$ and $Ri^!(\mathcal{F}^\bullet) \in {}^pD^{\geq 1}(X \setminus U, \mathbb{F}_p)$, respectively [Cas22, lemma 2.7].

2.4. **The Riemann–Hilbert correspondence.** We now recall the Riemann–Hilbert correspondence of Bhatt–Lurie [BL19]. Let R be an $\mathbb{F}_p$-algebra and let (R, φ) be the ring R regarded as an $R[\varphi]$-module via the Frobenius. For an R-algebra S, extension of scalars provides a functor from $R[\varphi]$-modules to $S[\varphi]$-modules, which is used implicitly in the following definition taken from [BL19, Construction 2.3.1].

Definition 2.6. Let $D(R[\varphi])$ be the derived category of $R[\varphi]$-modules. Define the functor

$$\mathrm{Sol}(-) := \underline{\mathrm{RHom}}_{D(R[\varphi])}((R, \varphi), -) \colon D(R[\varphi]) \to D(\mathrm{Spec}(R), \mathbb{F}_p).$$

Informally, Sol can be thought of as the derived functor of φ-invariants. The functor Sol is not an equivalence of categories because $D(R[\varphi])$ is too large. To solve this issue in the constructible case, Bhatt–Lurie define a notion of holonomicity [BL19, Definition 4.1.1.]. A holonomic $R[\varphi]$-module is an $R[\varphi]$-module isomorphic to one of the form

$$M^{\mathrm{perf}} := \mathrm{colim}(M \to \varphi_* M \to \varphi_*^2 M \to \cdots)$$

for an $R[\varphi]$-module M which is finitely presented as an R-module. Note that a holonomic $R[\varphi]$-module is in particular perfect, meaning that $M \to \varphi_* M$ is an isomorphism. Restriction of scalars along $R \to R^{\mathrm{perf}}$ identifies the categories of perfect $R[\varphi]$-modules and perfect $R^{\mathrm{perf}}[\varphi]$-modules [BL19, Proposition 3.4.3], which by the following theorem is closely related to the topological invariance of the small étale site.

Theorem 2.7 (Bhatt–Lurie)**.** *Let $D_{\mathrm{hol}}(R[\varphi]) \subset D(R[\varphi])$ be the full subcategory of complexes with holonomic cohomology sheaves. Then Sol restricts to an equivalence of categories $D_{\mathrm{hol}}(R[\varphi]) \cong D_c^b(\mathrm{Spec}(R), \mathbb{F}_p)$ which is t-exact for the standard t-structures on the source and target*

Proof. See [BL19, Theorem 7.4.1, Corollary 12.1.7]. □

By t-exactness, if M is a holonomic $R[\varphi]$-module then $\mathrm{Sol}(M)$ is the étale sheaf on $\mathrm{Spec}(R)$ whose value on an étale R-algebra S is

$$\mathrm{Sol}(M)(S) = \{x \in M \otimes_R S \;:\; \varphi(x) = x\}.$$

Remark 2.8. In [BBL+23] the authors use a different definition of the perverse t-structure on $D_c^b(\mathrm{Spec}(R), \mathbb{F}_p)$, in terms of the Riemann–Hilbert correspondence and a perverse t-structure on coherent sheaves, but the two agree by [BBL+23, Theorem 4.43]. Correspondingly, our proofs of Theorem 2.17 and Theorem 2.25 below are quite different from their analogues [BBL+23, Remark 4.39, Corollary 5.15]. Our Theorem 2.25 also differs from [BBL+23, Corollary 5.15] in that we allow a non-complete base and hence have to eliminate the possibility of branching behavior (with the help of [DMP23]).

2.5. **Cohen–Macaulayness.** There are numerous equivalent definitions of Cohen–Macaulayness for a noetherian local ring, for example involving regular sequences, local cohomology, or a dualizing complex. While there is no standard definition of Cohen–Macaulayness in the non-noetherian setting, the one in [Bha20, Definition 2.1] will be useful here (see also [Bha20, Remark 2.4]).

Definition 2.9. Let X be a topologically noetherian scheme. We say that X is Cohen–Macaulay if for every local ring $(R, \mathfrak{m})$ on X, the (Zariski) local cohomology groups $H^i_{\mathfrak{m}}(R) := R^i\Gamma_{\{\mathfrak{m}\}}(\mathcal{O}_{\mathrm{Spec}(R)})$ vanish for $i < \dim R$.

If R is a noetherian local $\mathbb{F}_p$-algebra then $H^i_{\mathfrak{m}}(R)$ has a canonical $R[\varphi]$-module structure as the cohomology of a Koszul complex of $R[\varphi]$-modules [Sta23, Tag 0956]. In this case, $H^i_{\mathfrak{m}}(R)$ is finitely generated as an $R[\varphi]$-module, and since it is Artinian as an R-module it is even a module over the completion $\hat{R}$. Furthermore, if M is an R-module (R is still noetherian) we have $H^i_{\mathfrak{m}}(M) = \mathrm{colim}_n \mathrm{Ext}^i_R(R/\mathfrak{m}^n, M)$ [Sta23, Tag 0955].

Lemma 2.10. *Let $(R, \mathfrak{m})$ be the perfection of a noetherian local $\mathbb{F}_p$-algebra $(R_0, \mathfrak{m}_0)$ of dimension d with normalized dualizing complex $\omega^\bullet_{R_0}$.*

(1) *$H^i_{\mathfrak{m}}(R) = 0$ if and only if $H^i_{\mathfrak{m}_0}(R_0)$ is nilpotent.*
(2) *$H^d_{\mathfrak{m}}(R) \neq 0$.*
(3) *If $H^i_{\mathfrak{m}}(R) = 0$ for all $i < d$ then R is equidimensional.*
(4) *If R_0 is φ-finite then $H^i_{\mathfrak{m}_0}(R_0)$ is nilpotent if and only if $\mathcal{H}^{-i}(\omega^\bullet_{R_0})$ is nilpotent.*
(5) *If R_0 is φ-finite and $H^i_{\mathfrak{m}}(R) = 0$ for all $i < d$, then $\mathrm{Spec}(R)$ is Cohen–Macaulay in the sense of Definition 2.9.*

Remark 2.11. A noetherian local $\mathbb{F}_p$-algebra $(R_0, \mathfrak{m}_0)$ such that $H^i_{\mathfrak{m}_0}(R_0)$ is nilpotent for $i < \dim R_0$ is called weakly φ-nilpotent [Mad19], see also [PQ19, Quy19].

Proof. Since R is an R_0-module and $\operatorname{Spec}(R) \cong \operatorname{Spec}(R_0)$ then $H^i_{\mathfrak{m}}(R) = H^i_{\mathfrak{m}_0}(R)$. Next, $H^i_{\mathfrak{m}_0}(R) = \operatorname{colim}_n \operatorname{Ext}^i_{R_0}(R_0/\mathfrak{m}_0^n, \operatorname{colim}_e \varphi^e_* R_0)$. By taking a resolution of $R_0/\mathfrak{m}_0^n$ by finite free R_0-modules and using that filtered colimits are exact [Sta23, Tag 00DB], the inner colimit over e commutes with $\operatorname{Ext}^i_{R_0}(R_0/\mathfrak{m}_0^n, -)$. Then by exchanging the colimits and using exactness of φ_* to commute the latter with $H^i_{\mathfrak{m}_0}(-)$, we get

$$H^i_{\mathfrak{m}}(R) = \operatorname*{colim}_e \varphi^e_* H^i_{\mathfrak{m}_0}(R_0) = H^i_{\mathfrak{m}_0}(R_0)^{\mathrm{perf}}. \tag{2.1}$$

Thus, $H^i_{\mathfrak{m}}(R) = 0$ if and only if every element of $H^i_{\mathfrak{m}_0}(R_0)$ is annihilated by some power of φ. Now (1) follows from Corollary 2.3.

For (2), we consider two cases. If $d = 0$ then $H^0_{\mathfrak{m}}(R) = R$ is nonzero. On the other hand, if $d > 0$, then by (1) we need only show that $H^d_{\mathfrak{m}_0}(R_0)$ is not nilpotent. But if $H^d_{\mathfrak{m}_0}(R_0)$ were nilpotent, then since it is finitely generated as an $R_0[\varphi]$-module, it would also be finitely generated as an R_0-module, which is impossible [BS13b, Corollary 7.3.3].

Part (3) follows from the proof of [PQ19, Proposition 2.8(3)]; we reproduce the argument here for completeness. If R is not equidimensional, let $\mathfrak{p} \subset R_0$ be a minimal prime such that $n := \dim R_0/\mathfrak{p} < d = \dim R_0$, and let I be the intersection of the other minimal primes. Then we have an exact sequence of $R_0[\varphi]$-modules

$$0 \to R_0 \to R_0/\mathfrak{p} \oplus R_0/I \to R_0/(\mathfrak{p} + I) \to 0$$

where $\dim R_0/(\mathfrak{p} + I) < n$. Applying $R\Gamma_{\{\mathfrak{m}_0\}}$ gives a surjection $H^n_{\mathfrak{m}_0}(R_0) \to H^n_{\mathfrak{m}_0}(R_0/\mathfrak{p})$ by [Sta23, Tag 0DXC]. This implies that $H^n_{\mathfrak{m}_0}(R_0/\mathfrak{p})$ is nilpotent, which contradicts part (2).

Next, we note that if R_0 is complete then (4) follows immediately from Matlis duality. Remarkably this statement is true even if R_0 is not complete, as was observed in [ST17, Lemma 2.3]. The argument is similar to [Sch09, Proposition 4.3], using that the double Matlis duality functor is isomorphic to $(-) \otimes_{R_0} \hat{R}_0$ on finite R_0-modules, and faithful flatness of $R_0 \to \hat{R}_0$; we refer to loc. cit for more details.

For each prime $\mathfrak{p} \subset R_0$, the localization $(\omega^\bullet_{R_0})_{\mathfrak{p}}$ is a dualizing complex for $(R_0)_{\mathfrak{p}}$, and since R_0 is equidimensional, $(\omega_{R_0})_{\mathfrak{p}}$ is a dualizing sheaf for $(R_0)_{\mathfrak{p}}$ [Smi93, Proposition 2.3.2]. Thus, (5) follows from (4). □

Since $H^i_{\mathfrak{m}}(R) = H^i_{\mathfrak{m}_0}(R)$, then $H^i_{\mathfrak{m}}(R)$ has a canonical $R[\varphi]$-module structure as the cohomology of a Koszul complex, constructed from finitely many generators of $\mathfrak{m}$ up to radical, independent of the chosen deperfection. The canonicity of the $R[\varphi]$-module structure can also be deduced by noting that the action of φ comes from applying $H^i_{\mathfrak{m}}(-)$ to the Frobenius map $R \to \varphi_* R$. Furthermore, $H^i_{\mathfrak{m}}(R)$ is a perfect $R[\varphi]$-module in the sense of [BL19, Definition 3.2.1].

We now introduce some notation for working with perfections. If R is an $\mathbb{F}_p$-algebra and $r \in R$, we denote by $r^{1/p^e} \in R^{\mathrm{perf}}$ the p^eth root. If M is an $R[\varphi]$-module and $m \in M$, we denote by $\varphi^{-e}(m) \in M^{\mathrm{perf}}$ the image of m under the map $\varphi^e_* M \to M^{\mathrm{perf}}$. We use this notation in the proof of the following structure result for perfect φ-modules.

Proposition 2.12. *Let $(R, \mathfrak{m})$ be the perfection of a noetherian local $\mathbb{F}_p$-algebra $(R_0, \mathfrak{m}_0)$.*

(1) *The functor $M_0 \mapsto M_0^{\mathrm{perf}}$ from $R_0[\varphi]$-modules to $R[\varphi]$-modules is exact.*
(2) *If M_0 is a co-finite $R_0[\varphi]$-module then M_0^{perf} has finite length as an $R[\varphi]$-module.*
(3) *If M_0 is a co-finite, non-nilpotent and simple $R_0[\varphi]$-module then M_0^{perf} is a simple, nonzero $R[\varphi]$-module.*

Proof. Part (1) follows from the exactness of φ_*. For (2), we take the perfection of a composition series for M_0 as in Theorem 2.2. This kills the nilpotent subqoutients, so (2) will then follow from (3). For (3), let $M := M_0^{\mathrm{perf}}$, and let $m \in M$ be nonzero. We need to show that $R[\varphi] \cdot m = M$. Write $m = \varphi^{-e}(m')$ for some $m' \in M_0$ and $e \geq 0$. It suffices to show that for all $n \in M_0$ and

$f \geq 0$, we have $\varphi^{-e-f}(n) \in R[\varphi] \cdot m$. By simplicity $R_0[\varphi] \cdot \varphi^f(m') = M_0$, so there exist $r_i \in R_0$ such that $\sum_i r_i \cdot \varphi^{i+f}(m') = n$. Then we conclude since $\sum_i r_i^{1/p^{e+f}} \cdot \varphi^i(m) = \varphi^{-e-f}(n)$. $\square$

The following result is the key input in the proof of Theorem 2.17 below.

Proposition 2.13. *Let $(R, \mathfrak{m})$ be the perfection of a φ-finite noetherian local $\mathbb{F}_p$-algebra $(R_0, \mathfrak{m}_0)$. Suppose that R is equidimensional, the punctured spectrum of $(R, \mathfrak{m})$ is Cohen–Macaulay, and $R/\mathfrak{m}$ is algebraically closed. Then* $\mathrm{Spec}(R)$ *is Cohen–Macaulay if and only if for all $i < \dim R$, there does not exist a nonzero element $x \in H^i_{\mathfrak{m}}(R)$ such that $\varphi(x) = x$.*

Proof. The necessity of the condition on φ-fixed elements is clear. For sufficiency, by Lemma 2.10 we may suppose for contradiction that that $\mathcal{H}^{-i}(\omega^\bullet_{R_0})$ has a non-nilpotent simple Cartier subquotient M for some $i \neq \dim R$. By Theorem 2.1 and our assumption on the punctured spectrum, the unique associated prime of M must be $\mathfrak{m}_0$, so M has finite length as an R_0-module. Since M was arbitrary, $\mathcal{H}^{-i}(\omega^\bullet_{R_0})$ has finite length as an R_0-module up to nilpotents. Let E be an injective hull of $R_0/\mathfrak{m}_0$. Then $\mathrm{Hom}_{R_0}(\mathcal{H}^{-i}(\omega^\bullet_{R_0}), E) \cong H^i_{\mathfrak{m}_0}(R_0)$, even as $R_0[\varphi]$-modules [BB11, Lemma 5.1], so $H^i_{\mathfrak{m}_0}(R_0)$ has finite length as an R_0-module up to nilpotents.

Now by Proposition 2.12, $H^i_{\mathfrak{m}}(R)$ has a finite composition series with holonomic subqoutients in the sense of [BL19, Definition 4.1.1]. Since the abelian category of holonomic $R_0[\varphi]$-modules is closed under extensions [BL19, Corollary 4.3.3, Remark 3.2.2], then $H^i_{\mathfrak{m}}(R)$ is a holonomic R_0-module which is moreover set-theoretically supported on $\mathrm{Spec}(R/\mathfrak{m})$ (holonomicity can also be deduced from [Bha20, Lemma 2.16]). By construction the functor Sol is compatible with pullback, so that $\mathrm{Sol}(H^i_{\mathfrak{m}}(R))$ is an étale sheaf supported on $\mathrm{Spec}(R/\mathfrak{m})$. Since $R/\mathfrak{m}$ is algebraically closed, an étale sheaf vanishes if and only if its global sections vanish. Now we conclude using that Sol is an equivalence when restricted to holonomic modules by Theorem 2.7. $\square$

Lemma 2.14. *Let $(R, \mathfrak{m})$ be a noetherian local ring which is equidimensional and universally catenary. Let $R \to S$ be an étale ring map and let $\mathfrak{q}$ be a prime of S lying above $\mathfrak{m}$. Then the localization $S_{\mathfrak{q}}$ is equidimensional and $\dim S_{\mathfrak{q}} = \dim R$.*

Proof. Let $\mathfrak{q}_0$ be a minimal prime of S contained in $\mathfrak{q}$. Then $\mathfrak{p}_0 := R \cap \mathfrak{q}_0$ is minimal by flatness, and $R/\mathfrak{p}_0$ is universally catenary by [Sta23, Tag 00NK]. By the dimension formula [Sta23, Tag 02IJ] applied to the ring extension $R/\mathfrak{p}_0 \subset S/\mathfrak{q}_0$, we have $\mathrm{ht}(\mathfrak{m}/\mathfrak{p}_0) = \mathrm{ht}(\mathfrak{q}/\mathfrak{q}_0)$. $\square$

Lemma 2.15. *Let $(R, \mathfrak{m})$ be a local $\mathbb{F}_p$-algebra. Then if R satisfies any of the following three properties, so does the strict henselization $(R^{\mathrm{sh}}, \mathfrak{m}^{\mathrm{sh}})$.*

(1) *R is noetherian.*
(2) *R is φ-finite.*
(3) *R is excellent and equidimensional.*

Proof. Property (1) is part of [Gro67, Proposition 18.8.8]. Property (2) is in the proof of [BCRG+19, Theorem 4.1], the point being that R^{sh} is an ind-étale R algebra, so $\varphi_* R \otimes_R R^{\mathrm{sh}} = \varphi_* R^{\mathrm{sh}}$. For property (3), excellence is preserved by the last remark in [FK88, Ch. 1 §1], and it remains to prove equidimensionality.

Let $\mathfrak{q}_0$ be a minimal prime of R^{sh}. Then $\mathfrak{p}_0 := R \cap \mathfrak{q}_0$ is minimal by flatness. As in [Sta23, Tag 06LK], write $R^{\mathrm{sh}} = \mathrm{colim}_i R_i$ as a direct limit of local rings R_i which are localizations of étale R-algebras faithfully flat over R. Then $R_i \to R^{\mathrm{sh}}$ is faithfully flat for all i by [Sta23, Tag 00U7, Tag 05UT]. We have the minimal primes $\mathfrak{p}_i := R_i \cap \mathfrak{q}_0$ in the R_i. By prime avoidance, for all large enough i, $\mathfrak{q}_0$ is the only minimal prime of R^{sh} lying over $\mathfrak{p}_i$. Fix such an i and let $C = (\mathfrak{Q}_0 \subset \cdots \subset \mathfrak{Q}_n)$ be a maximal chain of primes in $R^{\mathrm{sh}}/\mathfrak{p}_i R^{\mathrm{sh}}$. By faithful flatness of $R_i/\mathfrak{p}_i \to R^{\mathrm{sh}}/\mathfrak{p}_i$ and the going down property, we have $n \geq \dim R_i/\mathfrak{p}_i$. But by Lemma 2.14, $\dim R_i/\mathfrak{p}_i = \dim R$. Since $\dim R = \dim R^{\mathrm{sh}}$ [Sta23, Tag 06LK], this implies that the preimage

$C \cap R^{\mathrm{sh}}$ also a maximal chain of primes in R^{sh}. By our choice of i, the minimal prime of $C \cap R^{\mathrm{sh}}$ is $\mathfrak{q}_0$. Thus, for an arbitrary minimal prime $\mathfrak{q}_0$ we have exhibited a chain C of primes starting at $\mathfrak{q}_0$ and of length $\dim R^{\mathrm{sh}}$, so R^{sh} is equidimensional. □

Lemma 2.16. *Let $(R, \mathfrak{m})$ be the perfection of a noetherian local $\mathbb{F}_p$-algebra, and let $(R^{\mathrm{sh}}, \mathfrak{m}^{\mathrm{sh}})$ be the strict henselization.*

(1) $H^i_{\mathfrak{m}^{\mathrm{sh}}}(R^{\mathrm{sh}}) = H^i_{\mathfrak{m}}(R) \otimes_R R^{\mathrm{sh}}$.
(2) $H^i_{\mathfrak{m}}(R) = 0$ *if and only if* $H^i_{\mathfrak{m}^{\mathrm{sh}}}(R^{\mathrm{sh}}) = 0$.
(3) *If the punctured spectrum of $(R, \mathfrak{m})$ is Cohen–Macaulay, so is the punctured spectrum of $(R^{\mathrm{sh}}, \mathfrak{m}^{\mathrm{sh}})$.*

Proof. It follows from topological invariance of the small étale site [Sta23, Tag 04DY] that perfection commutes with strict henselization, so that R^{sh} is topologically noetherian. Since $\mathfrak{m}^{\mathrm{sh}} = \mathfrak{m}R^{\mathrm{sh}}$ then $H^i_{\mathfrak{m}^{\mathrm{sh}}}(R^{\mathrm{sh}}) = H^i_{\mathfrak{m}}(R^{\mathrm{sh}})$. Via the description of $H^i_{\mathfrak{m}}(R)$ as the cohomology of a Koszul complex of R-modules, (1) and (2) follow from faithful flatness of $R \to R^{\mathrm{sh}}$ [Sta23, Tag 07QM]. For (3), let $\mathfrak{q}$ be a non-maximal prime of R^{sh}, and let $\mathfrak{p}$ be its preimage in R. We claim that $\mathfrak{q}R^{\mathrm{sh}}_{\mathfrak{q}} = \mathfrak{p}R^{\mathrm{sh}}_{\mathfrak{q}}$ and $\dim R^{\mathrm{sh}}_{\mathfrak{q}} = \dim R_{\mathfrak{p}}$. Granting these claims, $H^i_{\mathfrak{q}}(R^{\mathrm{sh}}_{\mathfrak{q}}) = H^i_{\mathfrak{p}}(R_{\mathfrak{p}}) \otimes R^{\mathrm{sh}}_{\mathfrak{q}}$ and (3) follows.

To prove the claims, use [Sta23, Tag 06LK] to write $R^{\mathrm{sh}} = \mathrm{colim}_i R_i$ as a direct limit of local rings R_i which are localizations of étale R-algebras. If $\mathfrak{p}_i$ is the preimage of $\mathfrak{q}$ in R_i, then $R^{\mathrm{sh}}_{\mathfrak{q}} = \mathrm{colim}_i (R_i)_{\mathfrak{p}_i}$. We have $\mathfrak{p}_i(R_i)_{\mathfrak{p}_i} = \mathfrak{p}(R_i)_{\mathfrak{p}_i}$ by [Sta23, Tag 00U4], so the first claim follows. For the claim about dimensions, note that $\dim R^{\mathrm{sh}}_{\mathfrak{q}} \geq \dim R_{\mathfrak{p}}$ by faithful flatness and the going down property. For the other direction, if $\mathfrak{q}_0 \subset \mathfrak{q}_1 \subset \cdots \subset \mathfrak{q}_n = \mathfrak{q}$ is a chain of primes in R^{sh}, then for some i this restricts to a chain of primes in R_i of the same length. Now we conclude since $\dim(R_i)_{\mathfrak{p}_i} = \dim R_{\mathfrak{p}}$ by [Sta23, Tag 07QP]. □

We now come to the main result of this subsection. In the proof, we use the Artin–Schreier sequence

$$0 \to \mathbb{F}_p \to \mathcal{O}_{\mathrm{Spec}(R)} \xrightarrow{\varphi - 1} \mathcal{O}_{\mathrm{Spec}(R)} \to 0 \tag{2.2}$$

to translate between quasi-coherent cohomology and étale cohomology. This sequence is exact in the étale topology on $\mathrm{Spec}(R)$ for any $\mathbb{F}_p$-algebra R. If $(R, \mathfrak{m})$ is the perfection of a local $\mathbb{F}_p$-algebra R_0 then $\mathcal{O}_{\mathrm{Spec}(R)}$ may be viewed as a quasi-coherent sheaf on $\mathrm{Spec}(R_0)$, and then (2.2) is also exact in the étale topology on $\mathrm{Spec}(R_0)$. In particular, if R_0 is noetherian then $R^i\Gamma_{\{\mathfrak{m}\}}(\mathcal{O}_{\mathrm{Spec}(R)}) = H^i_{\mathfrak{m}_0}(R)$ is the same in both the Zariski and étale topology by [Sta23, Tag 04DY] and descent for quasi-coherent sheaves.

Theorem 2.17. *Let k be a perfect field of characteristic p and let X be a scheme isomorphic to the perfection of a connected finite-type k-scheme. Then the following are equivalent.*

(1) *X is Cohen–Macaulay in the sense of Definition 2.9.*
(2) *The shifted constant sheaf $\mathbb{F}_p[\dim X] \in D^b_c(X, \mathbb{F}_p)$ is perverse.*

Furthermore, X is equidimensional in both cases.

Proof. We first show that (1) implies (2). By Lemma 2.10, X is equidimensional. It is immediate that $\mathbb{F}_p[\dim X] \in {}^pD^{\leq 0}(X, \mathbb{F}_p)$. To prove that $\mathbb{F}_p[\dim X] \in {}^pD^{\geq 0}(X, \mathbb{F}_p)$, fix a point $x \in X$. Let $(R, \mathfrak{m})$ be the strict henselization of the corresponding perfect local ring. Let $d := \dim R$, which also agrees with the dimension before strict henselization [Sta23, Tag 06LK]. By Lemma 2.16, (1) is equivalent to the statement that $H^i_{\mathfrak{m}}(R) = 0$ for all $i < d$ and points $x \in X$. On the other hand, as X is equidimensional, (2) is equivalent to the statement that for all points x, we have $\mathcal{H}^i(Ri^!_x \mathbb{F}_p) = 0$ for $i < d$, where $i_x \colon \overline{x} \to \mathrm{Spec}(R)$ is the inclusion of the closed point. Since $\overline{x}$

is a geometric point, $\mathcal{H}^i(Ri_x^!\mathbb{F}_p) = R^i\Gamma_{\{\mathfrak{m}\}}(\mathbb{F}_p)$. Now the fact that (1) implies (2) follows from applying $R\Gamma_{\{\mathfrak{m}\}}$ to the Artin–Schreier sequence (2.2).

Next, we show that (2) implies X is equidimensional. A straightforward argument using the Artin–Schreier sequence as above shows that if Y is Cohen–Macaulay, irreducible, and of finite-type over k, then $\mathbb{F}_p[\dim Y] \in D_c^b(Y, \mathbb{F}_p)$ is perverse. Now let X_0 be a deperfection of X by a finite-type k-scheme, which we may assume is reduced. Since X_0 is of finite type over the perfect field k, we may let Y be a smooth dense open subscheme of X_0, so the perversity of $\mathbb{F}_p[\dim Y]$ implies that X_0 is equidimensional.

To show that (2) implies (1), we proceed by descending induction on $\dim \overline{\{x\}}$. By Lemma 2.16, it suffices to show the strict henselization $(R, \mathfrak{m})$ of the local ring at x satisfies $H^i_{\mathfrak{m}}(R) = 0$ for $i < d := \dim R = \dim X - \dim \overline{\{x\}}$, and we may assume the punctured spectrum of $(R, \mathfrak{m})$ is Cohen–Macaulay. By Lemma 2.15 the hypotheses of Proposition 2.13 are satisfied, so we are reduced to checking that $H^i_{\mathfrak{m}}(R)$ has trivial φ-invariants for $i < d$. But this condition follows from induction and the long exact sequence obtained from applying $R\Gamma_{\{\mathfrak{m}\}}$ to the Artin–Schreier sequence (2.2), together with the perversity of $\mathbb{F}_p[\dim X]$. □

2.6. **φ-rationality.** Next we discuss a perfect notion of φ-rationality. Recall that by a theorem of Smith [Smi97], an excellent local $\mathbb{F}_p$-algebra $(R_0, \mathfrak{m}_0)$ is φ-rational if and only if it is Cohen–Macaulay and $H^{\dim R_0}_{\mathfrak{m}_0}(R_0)$ is a simple $R_0[\varphi]$-module (this differs from the original definition in terms of tight closure, cf. [FW89, HH94]). One of the present authors showed in [Cas22, Theorem 1.7] that if X is an irreducible scheme of finite type over an algebraically closed field, all of whose local rings are φ-rational, then $\mathbb{F}_p[\dim X]$ is simple as a perverse sheaf. In the opposite direction, we will show that if $\mathbb{F}_p[\dim X]$ is simple, then X^{perf} is Cohen–Macaulay and $H^{\dim R_0}_{\mathfrak{m}_0}(R_0)^{\mathrm{perf}}$ is simple. The latter properties are encapsulated by the following definition.

Definition 2.18. Let X be the perfection of a noetherian φ-finite $\mathbb{F}_p$-scheme. We say that X is φ-rational if it is Cohen–Macaulay in the sense of Definition 2.9, and, for every local ring $(R, \mathfrak{m})$ on X, the top local cohomology group $H^{\dim R}_{\mathfrak{m}}(R)$ is a simple $R[\varphi]$-module.

It will be useful to characterize φ-rationality of a perfect scheme in terms of a property of one (equivalently, every) deperfection. The latter property turns out to be φ-nilpotence, first introduced by Blickle–Bondu [BB05] under the name close to F-rational, and further studied e.g. in [ST17, PQ19, DMP23, KMPS23].

Definition 2.19. Let $(R, \mathfrak{m})$ be a φ-finite noetherian local $\mathbb{F}_p$-algebra of dimension d.

(1) The tight closure of the zero submodule in $H^d_{\mathfrak{m}}(R)$, denoted $0^*_{H^d_{\mathfrak{m}}(R)}$, is the $R[\varphi]$-submodule consisting of all elements $x \in H^d_{\mathfrak{m}}(R)$ such that there exists some $c \in R$ not contained in any minimal prime with the property that $c\varphi^e(x) = 0$ for all $e \gg 0$.
(2) The ring R is said to be φ-nilpotent if each of the the $R[\varphi]$-modules $H^0_{\mathfrak{m}}(R), \ldots, H^{d-1}_{\mathfrak{m}}(R), 0^*_{H^d_{\mathfrak{m}}(R)}$ is nilpotent.

By [PQ19, Proposition 2.8 (2)], R is φ-nilpotent if and only if its reduction is φ-nilpotent, so we will usually assume R is reduced. To relate this notion to coherent objects, note that Matlis duality gives a canonical pairing $f \colon H^d_{\mathfrak{m}}(R) \otimes_{\hat{R}} \omega_{\hat{R}} \to E$. The parameter test module $\tau(\omega_R) \subset \omega_R$ is the Cartier submodule consisting of all $\eta \in \omega_R$ such that $f(x \otimes \eta) = 0$ for all $x \in 0^*_{H^d_{\mathfrak{m}}(R)}$. The parameter test module is well-behaved under localization [HT04, Proposition 3.1] (cf. [Bli13, Proposition 3.2 (e)]), completion [HT04, Proposition 3.2], and more generally under flat base change when the residue field extension is separable [ST17, Lemma 1.5]. By construction, $0^*_{H^d_{\mathfrak{m}}(R)}$ is the Matlis dual of $\omega_R/\tau(\omega_R)$, even if R is not complete. When combined with Lemma 2.10, the following gives a characterization of φ-nilpotence in terms of $\omega_R^\bullet$.

Lemma 2.20. *Let $(R, \mathfrak{m})$ be a reduced, φ-finite noetherian local $\mathbb{F}_p$-algebra of dimension d. Then $0^*_{H^d_{\mathfrak{m}}(R)}$ is nilpotent if and only if $\omega_R/\tau(\omega_R)$ is nilpotent.*

Proof. It is observed [ST17, Lemma 2.3] that this follows from an argument similar to [HT04, Lemma 2.1]. □

Lemma 2.21. *Let $(R, \mathfrak{m})$ be the perfection of a φ-finite noetherian local $\mathbb{F}_p$-algebra $(R_0, \mathfrak{m}_0)$ of dimension d. Then the following are equivalent.*

(1) $\mathrm{Spec}(R)$ *is φ-rational in the sense of Definition 2.19.*
(2) $\mathrm{Spec}(R^{\mathrm{sh}})$ *is φ-rational in the sense of Definition 2.19.*
(3) R_0 *is φ-nilpotent.*
(4) R_0^{sh} *is φ-nilpotent.*

Furthermore, if any of the above conditions is satisfied then R is geometrically unibranch.

Proof. By [PQ19, Proposition 2.8] we may assume that R_0 is reduced. The equivalence of (3) and (4) then follows from Lemma 2.16 and Lemma 2.20, together with the compatibility of $\tau(\omega_{R_0})$ and its structure map as a Cartier module under faithfully flat base change to R_0^{sh} as in [ST17, Proposition 2.4 (4)], cf. [KMPS23, Theorem 4.4]. Once we prove the equivalence of (1) and (3), the equivalence with (2) will then follow.

First suppose that R_0 is φ-nilpotent. Then the completion $\hat{R}_0$ is also φ-nilpotent by [PQ19, Proposition 2.8 (4)]. Thus $\hat{R}_0$ is a domain by [DMP23, Theorem 3.1], and after the equivalence of (1)-(4) is established, loc. cit. will also imply the final claim about geometric unibranchedness. By [Smi93, Theorem 3.1.4], $0^*_{H^d_{\mathfrak{m}_0}(R_0)}$ is the unique maximal proper $R_0[\varphi]$-submodule of $H^d_{\mathfrak{m}_0}(R_0)$. Since this submodule is nilpotent, then Proposition 2.12 implies that $H^d_{\mathfrak{m}}(R)$ is a simple $R[\varphi]$-module. Furthermore, $H^i_{\mathfrak{m}}(R) = 0$ for $i < d$ by Lemma 2.10. For every prime $\mathfrak{p} \subset R_0$ the localization $(R_0)_{\mathfrak{p}}$ is φ-nilpotent by [ST17, Proposition 2.4 (3)] or [PQ19, Corollary 5.17], so the same arguments apply to $(R_0)_{\mathfrak{p}}$ and hence $\mathrm{Spec}(R)$ is φ-rational.

Now suppose that $\mathrm{Spec}(R)$ is φ-rational. Then $H^i_{\mathfrak{m}_0}(R_0)$ is nilpotent for $i < d$ and R_0 is equidimensional by Lemma 2.10. By [Bli04, Corollary 3.9] (and the surrounding discussion if R_0 is not complete), $0^*_{H^d_{\mathfrak{m}_0}(R_0)}$ is the intersection of the maximal proper $R_0[\varphi]$-submodules of $H^d_{\mathfrak{m}_0}(R_0)$. Furthermore, the quotient of $H^d_{\mathfrak{m}_0}(R_0)$ by each of these maximal proper $R_0[\varphi]$-submodules is non-nilpotent by [Bli04, Theorem 3.8]. Since $H^d_{\mathfrak{m}_0}(R_0)$ is simple up to nilpotents then $0^*_{H^d_{\mathfrak{m}_0}(R_0)}$ must be nilpotent, so R_0 is φ-nilpotent. □

Lemma 2.22. *Let $(R, \mathfrak{m})$ be the perfection of a φ-finite noetherian local $\mathbb{F}_p$-algebra $(R_0, \mathfrak{m}_0)$. If the punctured spectrum of $(R, \mathfrak{m})$ is φ-rational, then so is the punctured spectrum of $(R^{\mathrm{sh}}, \mathfrak{m}^{\mathrm{sh}})$.*

Proof. By Lemma 2.21 the punctured spectrum of $(R_0, \mathfrak{m}_0)$ is φ-nilpotent, and it suffices to show the same is true of $(R_0^{\mathrm{sh}}, \mathfrak{m}_0^{\mathrm{sh}})$. We can assume R_0 is reduced. If $\mathfrak{q} \subset R_0^{\mathrm{sh}}$ is a non-maximal prime then it lies over some non-maximal prime $\mathfrak{p} \in R_0$. The map of local rings $((R_0)_{\mathfrak{p}}, \mathfrak{p}(R_0)_{\mathfrak{p}}) \to ((R_0^{\mathrm{sh}})_{\mathfrak{q}}, \mathfrak{q}(R_0^{\mathrm{sh}})_{\mathfrak{q}})$ is faithfully flat and the residue field extension is separable (for separability, use that R_0^{sh} is a filtered colimit of étale R_0-algebras [Sta23, Tag 04GW] and apply [Sta23, Tag 00U4]). Since $(R_0)_{\mathfrak{p}}$ is φ-nilpotent, so is $(R_0^{\mathrm{sh}})_{\mathfrak{q}}$ by [ST17, Proposition 2.4 (4)]. □

The following results generalizes [ST17, Proposition 2.5] to the case where the deperfected punctured spectrum of $(R_0, \mathfrak{m}_0)$ is φ-nilpotent instead of φ-rational.

Proposition 2.23. *Let $(R, \mathfrak{m})$ be the perfection of a φ-finite noetherian local $\mathbb{F}_p$-algebra $(R_0, \mathfrak{m}_0)$ of dimension $d > 0$. Suppose that R is equidimensional, the punctured spectrum of $(R, \mathfrak{m})$ is φ-rational, and $R/\mathfrak{m}$ is algebraically closed. Then $\mathrm{Spec}(R)$ is φ-rational if and only if for all i, there does not exist a nonzero element $x \in H^i_{\mathfrak{m}}(R)$ such that $\varphi(x) = x$.*

Proof. First suppose that $\mathrm{Spec}(R)$ is φ-rational. We may assume that R_0 is reduced. By Lemma 2.13 we only need to deal with the conditions on $H^d_{\mathfrak{m}}(R)$. As in the proof of Lemma 2.21, R_0 is a domain and $H^d_{\mathfrak{m}_0}(R_0)$ has a unique simple $R_0[\varphi]$-module quotient M_0, which is also non-nilpotent. We claim that $\mathrm{Ann}_{R_0}(M_0) = (0)$. This follows from $\mathrm{Ann}_{\hat{R}_0}(M_0) = (0)$, which in turn follows from the fact that Matlis duality preserves annihilators [BS13a, 10.2.14] and torsion-freeness of the dualizing sheaf $\omega_{\hat{R}_0}$ [Sta23, Tag 0AWK]. Since R is φ-rational then $H^d_{\mathfrak{m}}(R) = M_0^{\mathrm{perf}}$, and it follows that $\mathrm{Ann}_R(H^d_{\mathfrak{m}}(R)) = (0)$. Now if there exists a nonzero $x \in H^d_{\mathfrak{m}}(R)$ with $\varphi(x) = x$, then by simplicity x generates $H^d_{\mathfrak{m}}(R)$ as an R-module. But every element of $H^d_{\mathfrak{m}}(R)$ is also annihilated by some collection of elements which generate $\mathfrak{m}$ up to radical, so $\mathfrak{m} = (0)$, a contradiction since $d > 0$.

Now suppose the punctured spectrum of $(R_0, \mathfrak{m}_0)$ is φ-nilpotent. Again, we only need to deal with the conditions on $H^d_{\mathfrak{m}}(R)$, and R_0 is equidimensional by Lemma 2.10. For contradiction we may assume that R_0 is reduced and $\omega_{R_0}/\tau(\omega_{R_0})$ is non-nilpotent (Lemma 2.20). Let M be a simple non-nilpotent Cartier subqoutient of $\omega_{R_0}/\tau(\omega_{R_0})$, and let $\mathfrak{p} \subset R_0$ be its unique associated prime (Theorem 2.1). By the compatibility of $\tau(\omega_{R_0})$ with localization [HT04, Proposition 3.1], our assumption on the punctured spectrum implies $\mathfrak{p} = \mathfrak{m}_0$. Now we conclude as in the proof of Proposition 2.13. Briefly, the Matlis dual of $\omega_{R_0}/\tau(\omega_{R_0})$ is $0^*_{H^d_{\mathfrak{m}_0}(R_0)}$, which therefore has finite length as an R_0-module up to nilpotents. Thus, the perfection of $0^*_{H^d_{\mathfrak{m}_0}(R_0)}$ inside $H^d_{\mathfrak{m}}(R)$ is a holonomic R_0-module whose image under Sol is an étale sheaf supported on $\mathrm{Spec}(R/\mathfrak{m})$. Since $R/\mathfrak{m}$ is algebraically closed, the condition on φ-fixed elements implies that $0^*_{H^d_{\mathfrak{m}_0}(R_0)}$ is nilpotent. □

Lemma 2.24 (Emerton–Kisin)**.** *Let X be a smooth irreducible scheme of finite type over a perfect field k. Let $\mathcal{L}$ be an étale local system of $\mathbb{F}_p$-vector spaces on X. Then $\mathcal{L}[\dim X]$ is simple as a perverse sheaf if and only if it is simple as a local system.*

Proof. The property of being perverse is étale-local so that Theorem 2.17 implies $\mathcal{L}[\dim X]$ is perverse. The part about simplicity follows from the claim that every perverse subsheaf of $\mathcal{L}[\dim X]$ is again a shifted local system. Indeed, Gabber's result [Gab04, Corollary 12.4] implies every perverse subsheaf is constructible, and then there are multiple ways to proceed; here we sketch the argument of Emerton–Kisin in [EK04a, Corollary 4.3.3]. Via their Riemann–Hilbert correspondence, $\mathcal{L}[\dim X]$ corresponds to a unit φ-crystal, i.e., an $\mathcal{O}_X[\varphi]$-module M, locally free of finite rank over $\mathcal{O}_X$, where the unit condition means that the adjoint map $\varphi^*M \to M$ is an isomorphism. Their correspondence is a perverse t-exact anti-equivalence, so that perverse subsheaves of $\mathcal{L}[\dim X]$ correspond to unit $\mathcal{O}_X[\varphi]$-module qoutients of M. Then the key input is that any such quotient is is locally free [EK04a, Proposition 1.2.3], so that its Riemann–Hilbert partner is a shifted local system. □

Our main result in this subsection characterizes those schemes for which a simple local system corresponds to a simple perverse sheaf.

Theorem 2.25. *Let k be a perfect field of characteristic p and let X be a connected scheme isomorphic to the perfection of a finite-type k-scheme. Then the following are equivalent.*

1. *X is φ-rational in the sense of Definition 2.19.*
2. *The shifted constant sheaf $\mathbb{F}_p[\dim X]$ is a simple perverse sheaf.*

Proof. First suppose that X is φ-rational. Then $\mathbb{F}_p[\dim X]$ is perverse by Theorem 2.17. Furthermore, X is irreducible by Lemma 2.21 and [DMP23, Theorem 3.1]. Let $U \subset X$ be a nonempty open subscheme isomorphic to the perfection of a smooth finite-type k-scheme, which exists by [Sta23, Tag 056V]. Let $i\colon X \setminus U \to X$ be a complementary closed immersion. Then $\mathbb{F}_p[\dim X]_U$ is perverse and simple on U by Lemma 2.24, so it suffices to show the intermediate extension to X is $\mathbb{F}_p[\dim X]$. Clearly $i^*\mathbb{F}_p[\dim X] \in {}^pD^{\leq -1}(X \setminus U, \mathbb{F}_p)$, and it remains to verify that $Ri^!\mathbb{F}_p[\dim X] \in {}^pD^{\geq 1}(X \setminus U, \mathbb{F}_p)$. Let $(R, \mathfrak{m})$ be a strict henselization of the local ring at a point in $X \setminus U$. By Lemma 2.21 the hypothesis of Lemma 2.23 are satisfied, and in particular R is a domain. We must therefore verify that $R^i\Gamma_{\{\mathfrak{m}\}}(\mathbb{F}_p) = 0$ for $i \leq \dim R$, where $\mathbb{F}_p$ is viewed as an étale sheaf on $\operatorname{Spec}(R)$. When $i < \dim R$ this vanishing follows from perversity, and furthermore $H^i_{\mathfrak{m}}(R) = 0$ by Cohen–Macaulayness. The case $i = \dim R$ then follows from Proposition 2.23, by applying $R\Gamma_{\{\mathfrak{m}\}}$ the Artin–Schreier sequence (2.2).

For the other direction, let $j\colon U \to X$ be an irreducible open subscheme isomorphic to the perfection of a smooth finite-type k-scheme. By simplicity we must have $\mathbb{F}_p[\dim X] \cong j_{!*}(\mathbb{F}_p[\dim X]_U)$. On the other hand, $j_{!*}(\mathbb{F}_p[\dim X])$ is supported on the closure of U, so X is irreducible. We now show by descending induction on $\dim \overline{\{x\}}$ that the local ring at $x \in X$ is φ-rational. If $x \in U$ this follows since regular local rings φ-rational even before passing to the perfection [HH89, Theorem 2.1 a)]. If $x \in X \setminus U$ we may assume the punctured spectrum of the local ring at x is φ-rational, so the same is true of the strict henselization $(R, \mathfrak{m})$ by Lemma 2.22. Since X is irreducible, the condition $j_{!*}(\mathbb{F}_p[\dim X]_U) = \mathbb{F}_p[\dim X]$ implies that $R^i\Gamma_{\{\mathfrak{m}\}}(\mathbb{F}_p) = 0$ for $i \leq \dim R$, where $\mathbb{F}_p$ is viewed as an étale sheaf on $\operatorname{Spec}(R)$. Perversity of $\mathbb{F}_p[\dim X]$ implies that $H^i_{\mathfrak{m}}(R) = 0$ for $i < \dim R$ (Theorem 2.17 and Lemma 2.16). Then by the Artin–Schreier sequence (2.2), $H^{\dim R}_{\mathfrak{m}}(R)$ has trivial φ-invariants. The remaining hypotheses of Proposition 2.23, equidimensionality in particular, are satisfied for R by Lemma 2.15. Thus, R is φ-rational, and hence so is the local ring at x Lemma 2.21. $\square$

A priori, the fact that the irreducibility of a scheme is not an étale-local property could prevent the simplicity of $\mathbb{F}_p[\dim X]$ from being an étale-local property. However, the relation with φ-nilpotence shows that this is not the case.

Corollary 2.26. *Let k be a perfect field of characteristic p and let X_0 be a finite-type k-scheme. If shifted constant sheaf $\mathbb{F}_p[\dim X_0]$ is a simple perverse sheaf, then X_0 is geometrically unibranch.*

Proof. By Theorem 2.25 and Lemma 2.21, the local rings of X_0 are φ-nilpotent, so the result follows from [DMP23, Theorem 3.1, Remark 3.2]. $\square$

Remark 2.27. Let X be a normal irreducible scheme of finite type over a perfect field k. Two important theorems in commutative algebra assert that the absolute integral closure X^+ of X in its field of fractions is Cohen–Macaulay (due to Hochster–Huneke [HH92]) and φ-rational (due to Smith [Smi94]) in an appropriate sense. This is true more generally over a φ-finite base. As observed in [BBL+23, §5.6], these theorems can be recovered from results such as Theorem 2.25. Informally, the idea is to show that $\mathbb{F}_p[\dim X]_{X^+}$ is a simple perverse sheaf on X^+. To prove this, one must compute $i^*\mathbb{F}_p[\dim X]_{X^+}$ and $Ri^!\mathbb{F}_p[\dim X]_{X^+}$, where $i\colon Z \to X^+$ is a proper closed subscheme. But the $*$-pullback is constant, and the !-pullback vanishes since $\mathbb{F}_p[\dim X]_{X^+}$ is the $*$-extension of its restriction to any open subset [Bha20, Proposition 3.10]. We refer to [BBL+23, §5.6] for more details.

3. Global +-regularity and inversion of adjunction

In this section, we review some of the material from [BMP+23] on globally +-regular varieties, explain the proof of their criterion for inversion of adjunction, and adapt it to a certain asymptotic analogue. This is going to be applied later to certain Demazure varieties.

Remark 3.1. In positive characteristic, ideas such as these have been known for at least a decade in advance. For instance, Das [Das15] proved inversion of adjunction for strong φ-regularity in characteristic p, but we would rather avoid his treatment, because it circumvents the Kawamata–Viehweg $+$-vanishing of [Bha12] and because it forces us to work with $\mathbb{Z}_{(p)}$-divisors everywhere instead of $\mathbb{Q}$-divisors.

3.1. **Global $+$-regularity.** Let k be a perfect field and X be a finite type connected normal k-scheme. It is helpful to consider the notion of boundary and subboundary $\mathbb{Q}$-divisors, as they constitute a very flexible tool in studying singularities.

Definition 3.2. Let $\Delta = \sum_i r_i D_i$ be a $\mathbb{Q}$-divisor on X, i.e., a finite rational linear combination of prime divisors on X. We say that Δ is a boundary (resp. a subboundary) if $0 \le r_i \le 1$ for all i (resp. if $0 \le r_i < 1$). We refer to (X, Δ) as a boundary (resp. subboundary) pair.

Let us recall the notion of global $+$-regularity following [BMP$^+$23, Definition 6.1].

Definition 3.3. We say that the pair (X, Δ) is globally $+$-regular if for every finite cover $f\colon Y \to X$ with Y connected normal, the natural map $\mathcal{O}_X \to f_*\mathcal{O}_Y(\lfloor f^*\Delta \rfloor)$ splits in the category of $\mathcal{O}_X$-modules.

If $\Delta = 0$, then we simply say that X is globally $+$-regular. Note that this condition only has to be verified for a cofinal family of finite covers f. By the cyclic covering trick, we may even assume that $f^*\Delta$ is integral, compare with [BMP$^+$23, Remark 6.2]. Let us start with the first basic stability property.

Lemma 3.4. *If the boundary pair (X, Δ) is globally $+$-regular, the same holds true for (X, Δ') for any boundary $\Delta' \le \Delta$.*

Proof. Compose the inclusion $f_*\mathcal{O}_Y(\lfloor f^*\Delta' \rfloor) \to f_*\mathcal{O}_Y(\lfloor f^*\Delta \rfloor)$ with the given section of $\mathcal{O}_X \to f_*\mathcal{O}_Y(\lfloor f^*\Delta \rfloor)$. □

In particular, (X, Δ) being globally $+$-regular implies that $(X, \epsilon\Delta)$ also is for every $0 \le \epsilon \le 1$. More importantly, global $+$-regularity satisfies proper descent:

Proposition 3.5. *Let $f\colon X \to Y$ be a proper birational map of normal connected k-schemes of finite type. If (X, Δ) is globally $+$-regular, then so is $(Y, f_*\Delta)$.*

Proof. This is [BMP$^+$23, Proposition 6.19]. Note that pushforwards and pullbacks of $\mathbb{Q}$-divisors along alterations of normal connected finite type are defined locally in codimension 1 on principal divisors via the norm map and the inclusion map, respectively, see [Sta23, Tag 02RS], and then extended by normality to the entire space. Let $g\colon Z \to Y$ be a finite cover by a normal integral k-scheme such that $g^*f_*\Delta$ is an integral divisor. Let W be the normalization of $X \times_Y Z$ with base maps $g'\colon W \to X$ and $f'\colon W \to Z$. Then, we know that $\mathcal{O}_X \to g'_*\mathcal{O}_W(g'^*\Delta)$ splits in $\mathcal{O}_X$-modules. The same holds therefore for $\mathcal{O}_Y \to g_*\mathcal{O}_Z(f'_*g'^*\Delta)$ in $\mathcal{O}_Y$-modules, because there is a natural map $g_*\mathcal{O}_Z(f'_*g'^*\Delta) \to f'_*g'_*\mathcal{O}_W(g'^*\Delta)$. Noticing that $g^*f_*\Delta = f'_*g'^*\Delta$, we deduce our desired splitting. □

Next, we translate the notion of globally $+$-regularity in terms of trace maps by applying Grothendieck–Serre duality. Recall that there is a 6-functor formalism on the category of quasi-coherent sheaves and the canonical sheaves ω_X arise as the $H^{-\dim(X)}$ of the complex $Rp^!k$, where $p\colon X \to \mathrm{Spec}(k)$ denotes the structure morphism. Since we assume X to be normal and connected, it turns out that ω_X is a reflexive sheaf and we usually fix an arbitrary canonical divisor K_X such that $\omega_X \simeq \mathcal{O}_X(K_X)$.

Proposition 3.6. *Assume $K_X + \Delta$ is $\mathbb{Q}$-Cartier. Then, the boundary pair (X, Δ) is globally $+$-regular if and only if the trace map*

$$H^0(Y, \mathcal{O}_Y(K_Y - \lfloor f^*(K_X + \Delta)\rfloor)) \to H^0(X, \mathcal{O}_X) \tag{3.1}$$

is surjective for all connected normal finite covers $f\colon Y \to X$.

Proof. This is a particular case of [BMP+23, Proposition 6.8]. By duality, the natural map $\mathcal{O}_X \to f_*\mathcal{O}_Y(\lfloor f^*\Delta\rfloor)$ of $\mathcal{O}_X$-modules is a split injection if and only if the map

$$\mathcal{H}\mathrm{om}_{\mathcal{O}_X}(f_*\mathcal{O}_Y(\lfloor f^*\Delta\rfloor), \mathcal{O}_X) \to \mathcal{O}_X \tag{3.2}$$

of $\mathcal{O}_X$-modules is a split surjection. By Grothendieck–Serre duality, the left side identifies with $f_*\mathcal{H}\mathrm{om}_{\mathcal{O}_Y}(\mathcal{O}_Y(\lfloor f^*\Delta\rfloor), f^!\mathcal{O}_X)$ where $f^!$ is the abelian truncation of the shriek pullback $Rf^!$. Note that this is reflexive, so to compute it we are allowed to restrict to the smooth locus of Y. Over there, $\lfloor f^*\Delta\rfloor$ becomes an actual Cartier divisor, so, in particular, we can write the left side as $f_*f^!\mathcal{O}_X(-\lfloor f^*\Delta\rfloor)$ by pulling the divisor across the Hom. On the other hand, by definition of the canonical divisor, we have $f^!\mathcal{O}_X(K_X) = \mathcal{O}_Y(K_Y)$, and since we are over the smooth locus of Y, we get the identity $f^!\mathcal{O}_X = \mathcal{O}_Y(K_Y - K_X)$ and our sheaf identifies with $f_*\mathcal{O}_Y(K_Y - \lfloor f^*(K_X + \Delta)\rfloor)$, just like in the statement of the proposition. Now, since $\mathcal{O}_X$ is free, the surjectivity of the trace map can be tested at the level of global sections. □

Motivated by the previous proposition, one has the k-module of $+$-stable sections $B^0(X, \Delta; \mathcal{O}_X)$ in [BMP+23, Definition 4.2] given by the intersection across all normal finite covers $f\colon Y \to X$ of the images of the trace maps (3.1) appearing in the statement of Proposition 3.6. In particular, global $+$-regularity amounts to demanding an equality $B^0(X, \Delta, \mathcal{O}_X) = H^0(X, \mathcal{O}_X)$. We finish this subsection with the following quite non-standard notion

Definition 3.7. We say that a boundary pair (X, Δ) is $\mathbb{Q}$-Fano if the $\mathbb{Q}$-divisor $K_X + \Delta$ is $\mathbb{Q}$-Cartier and anti-ample.

Eventually, we will want to provide an inductive criterion for lifting global $+$-regularity along closed subschemes and this positivity condition will play a significant role.

3.2. **Pure variant and inversion of adjunction.** Our next topic consists of a variant of global $+$-regularity defined along a prime divisor $S \subset X$. The following notion is a simplification of [BMP+23, Definition 6.24]. Let (X, Δ) be a pair consisting of a finite type k-scheme and an effective $\mathbb{Q}$-divisor Δ on X. Assume $\Delta = S + B$ where S is a prime divisor and B an effective $\mathbb{Q}$-divisor on X with irreducible components different from S.

Definition 3.8. We say that $(X, S + B)$ is purely globally $+$-regular along S if the map of $\mathcal{O}_X$-modules $\mathcal{O}_X \to f_*\mathcal{O}_Y(-S_Y + \lfloor f^*(S + B)\rfloor)$ splits for every finite cover $f\colon Y \to X$ with Y connected normal, where the $S_Y \subset Y$ form a compatible family of prime divisors lying over $S \subset X$.

Using the Galois action on X^+ over X, we can show that the previous definition is independent of the choice of the prime divisors $S_Y \subset Y$ (equivalently, of an absolute integral closure $S^+ \subset X^+$). There is a close relationship between pure global $+$-regularity and global $+$-regularity after slightly tweaking the divisors.

Lemma 3.9. *If $(X, S+B)$ is purely globally $+$-regular along S, then $(X, (1-\epsilon)S+B)$ is globally $+$-regular for every rational number $0 < \epsilon \leq 1$.*

Proof. This is [BMP+23, Lemma 4.26]. We just have to notice that $f^*(\epsilon S+B) \leq -S_Y + f^*(S+B)$ for sufficiently large normal finite covers $f\colon Y \to X$, so that the pure $+$-splitting of $(X, S + B)$ along S factors over a $+$-splitting for the pair $(X, (1 - \epsilon)S + B)$. □

Proposition 3.10. *The pair $(X, S+B)$ is purely globally $+$-regular along S if and only if the trace map*

$$H^0(Y, \mathcal{O}_Y(K_Y + S_Y - \lfloor f^*(K_X + S + B)\rfloor)) \to H^0(X, \mathcal{O}_X) \tag{3.3}$$

is surjective for all normal finite covers $f\colon Y \to X$.

Proof. The proof is the same as the non-pure along S case, requiring us to check that the $\mathcal{O}_X$-module dual of $f_*\mathcal{O}_Y(-S_Y + \lfloor f^*(S+B)\rfloor)$ equals $f_*\mathcal{O}_Y(K_Y + S_Y - \lfloor f^*(K_X + S + B)\rfloor)$. □

Again, there exists a module $B^0_S(X, S+B; \mathcal{O}_X)$ of pure $+$-stable sections along S, see [BMP+23, Definition 4.21], and pure global $+$-regularity along S translates into an equality $B^0_S(X, S+B; \mathcal{O}_X) = H^0(X, \mathcal{O}_X)$. The pure along S variant of global $+$-regularity was setup in this way, precisely because we want to study how to lift global $+$-regularity from a prime divisor S to the whole k-variety X – this is known as inversion of adjunction.

Theorem 3.11 ([BMP+23])**.** *Let X be a connected normal proper k-scheme, $S \subset X$ a normal prime divisor, and B a subboundary with components different from S. If $(X, S+B)$ is $\mathbb{Q}$-Fano, then $(X, S+B)$ is purely globally $+$-regular along S if and only if $(S, B|_S)$ is globally $+$-regular.*

Proof. This is a particular case of [BMP+23, Theorem 7.2], see also [BMP+23, Corollary 7.5], and we give a sketch of the argument. During the proof, we use the shorthand $\Delta = S + B$. By Serre duality, we can identify the trace maps with the natural maps

$$H^d(X, \mathcal{O}_X(K_X)) \to H^d(Y, \mathcal{O}_Y(\lfloor f^*(K_X + \Delta)\rfloor)) \tag{3.4}$$

induced by pullback along f^* and multiplication by the divisor $\lfloor f^*\Delta\rfloor$, and similarly

$$H^d(X, \mathcal{O}_X(K_X)) \to H^d(Y, \mathcal{O}_Y(-S_Y + \lfloor f^*(K_X + \Delta)\rfloor)) \tag{3.5}$$

in the pure along S case. Note that $\mathcal{O}_X(K_X + S)$ pulls back to $\mathcal{O}_S(K_S)$ because this holds away from codimension 2, and then we apply Hartogs' theorem by normality of S and X. In particular, the associated long exact sequence yields a connecting homomorphism

$$H^{d-1}(S, \mathcal{O}_S(K_S)) \to H^d(X, \mathcal{O}_X(K_X)) \tag{3.6}$$

which is surjective by normality and connectedness: indeed, it arises by dualising on k-modules the non-zero ring map $H^0(X, \mathcal{O}_X) \to H^0(S, \mathcal{O}_S)$ between finite field extensions of k. Similarly, we can connect the right sides of the pullback maps via the following map

$$H^{d-1}(S^+, \mathcal{O}^+_S(\nu^*_S(K_S + B|_S))) \to H^d(X^+, \mathcal{O}^+_X(-S^+ + \nu^*_X(K_X + \Delta))) \tag{3.7}$$

where we let the $+$-notation denote the colimit with respect to a family of connected normal finite covers $f\colon Y \to X$, and ν is the structure map of the absolute integral closures. The kernel of the connecting homomorphism at the $+$-level is given by the image of $H^{d-1}(X^+, \mathcal{O}^+_X(\nu^*_X(K_X + \Delta)))$. The latter vanishes by anti-ampleness of the $\mathbb{Q}$-Cartier divisor $K_X + \Delta$ and the Kodaira $+$-vanishing theorem of [Bha12]. A diagram chase reveals that injectivity for S gives rise to injectivity for X, and vice-versa. □

3.3. An assymptotic variant.

Theorem 3.11 provides a criterion for inversion of adjunction of global $+$-regularity but has the somewhat unpleasant feature that it lifts global $+$-regularity to at most pure global $+$-regularity. At the same time, the latter comes pretty close to global $+$-regularity itself by Lemma 3.9. This leads us to formulate a variant that treats boundary pairs asymptotically and increases the clarity of our exposition when applying the criterion to Demazure varieties.

Definition 3.12. Given a boundary decomposition $\Delta = S + B$ with S prime and $B \geq 0$ with no common components with S, we similarly say that the boundary pair (X, Δ) is asymptotically purely $\mathbb{Q}$-Fano along S if there exist arbitrarily close subboundaries $B' < B$ such that $K_X + S + B'$ is an anti-ample Cartier divisor.

The definition above is again quite non-standard, but it fits well within our paper. The next step is to define the asymptotic analogue of global $+$-regularity.

Definition 3.13. We say that the boundary pair (X, Δ) is asymptotically globally $+$-regular if for all subboundaries $\Delta' < \Delta$, the pair (X, Δ') is globally $+$-regular in the usual sense.

Note that we do not here the condition applies to all smaller subboundaries, because global $+$-regularity is stable under parallelipepides, unlike ampleness. We can safely ignore a corresponding asymptotic notion of pure global $+$-regularity along a prime divisor, as the criterion for inversion of adjunction now takes the following form.

Corollary 3.14. *Let X be a connected normal proper k-scheme, $S \subset X$ a normal prime divisor, and B a boundary with components different from S. If $(X, S+B)$ is asymptotically purely $\mathbb{Q}$-Fano along S and $(S, B|_S)$ is asymptotically globally $+$-regular, then $(X, S+B)$ is asymptotically globally $+$-regular.*

Proof. Let $B' < B$ be a subboundary such that the corresponding pair (X, Δ') with $\Delta' = S + B'$ is Fano. Now, since we know that $(S, B'|_S)$ is globally $+$-regular, we may apply Theorem 3.11 to get that (X, Δ') is purely globally $+$-regular along S. But then $(X, (1-\epsilon)S + B')$ is actually globally $+$-regular for any $\epsilon > 0$. Letting ϵ go to 0 and B' to B, we get arbitrarily close to the original boundary Δ, so it induces a asymptotically globally $+$-regular pair. $\square$

Remark 3.15. There is a corresponding version of the corollary which is an equivalence between the behavior of the pairs (X, Δ) and $(S, B|_S)$, but it requires defining asymptotic pure global $+$-regularity along a divisor. The only thing happening here is that the asymptotic pure version for (X, Δ) implies the asymptotic non-pure one for the same pair without tampering with the divisor, precisely because of the asymptoticity.

4. Affine flag varieties

4.1. **Affine Schubert varieties.** Throughout this section k denotes an algebraically closed field of characteristic $p > 0$. Let F be a complete discretely valued field with ring of integers $\mathcal{O}$ and residue field k. Fix a connected reductive group G over F and a parahoric O-model $\mathcal{G}$ in the sense of Bruhat–Tits [BT84]. We introduce the affine Schubert schemes following [AGLR22, §3.2], with some simplifications since we assume k is algebraically closed.

Let $\mathrm{Alg}_k^{\mathrm{perf}}$ denote the category of perfect k-algebras. For $R \in \mathrm{Alg}_k^{\mathrm{perf}}$ let $W(R)$ be the ring of p-typical Witt vectors over R. The ring of $\mathcal{O}$-Witt vectors over R is defined as

$$W_{\mathcal{O}}(R) = \begin{cases} W(R) \otimes_{W(k)} \mathcal{O}, & \mathrm{char}(F) = 0 \\ R \,\widehat{\otimes}_k\, \mathcal{O}, & \mathrm{char}(F) = p. \end{cases}$$

Note that if $\mathrm{char}(F) = p$ and $t \in \mathcal{O}$ is a uniformizer, then $\mathcal{O} \cong k[\![t]\!]$ and $W_{\mathcal{O}}(R) \cong R[\![t]\!]$.

We define the following two functors $\mathrm{Alg}_k^{\mathrm{perf}} \to \mathrm{Grp}$,

$$LG(R) = G(W_{\mathcal{O}}(R) \otimes_{\mathcal{O}} F), \quad L^+G(R) = \mathcal{G}(W_{\mathcal{O}}(R)).$$

The affine flag variety for $\mathcal{G}$ is the étale quotient

$$\mathrm{Fl}_{\mathcal{G}} = LG/L^+\mathcal{G}.$$

The functor $\mathrm{Fl}_{\mathcal{G}}$ is represented by an increasing union of perfections of projective k-schemes. Indeed, if $\mathrm{char}(F) = p$ then $\mathrm{Fl}_{\mathcal{G}}$ is the perfection of the affine flag variety in the sense of [PR08] (which admits a natural moduli problem for all k-algebras), and if $\mathrm{char}(F) = 0$ we obtain the affine flag variety in the sense of [Zhu17] whose representability was proved in [BS17, Corollary 9.6].

The Schubert varieties for the parahoric group scheme $\mathcal{G}$ arise as the $L^+\mathcal{G}$-orbit closures inside $\mathrm{Fl}_{\mathcal{G}}$. As we explain now, these are enumerated via double cosets of the Iwahori–Weyl group and this combinatorics captures their closure relations. Let $\mathbf{f}$ be the unique facet in the Bruhat–Tits building $\mathscr{B}(G,F)$ whose connected stabilizer is $\mathcal{G}(O)$. Let $S\subset G$ be a maximal F-split torus whose apartment contains $\mathbf{f}$. The centralizer $T=Z_G(S)$ is a maximal F-torus and we let $\mathcal{T}$ be its connected Néron O-model. The Iwahori–Weyl group associated to S is $\tilde{W}:=N(F)/\mathcal{T}(\mathcal{O})$. The choice of an alcove $\mathbf{a}$ in the apartment of S gives rise to a split exact sequence

$$1\to W_{\mathrm{af}}\to\tilde{W}\to\pi_1(G)_I\to 1 \tag{4.1}$$

where $\pi_1(G)$ is the algebraic fundamental group and I is the inertia group of F. The affine Weyl group W_{af} is the Coxeter group generated by the reflections in the walls of $\mathbf{a}$. By declaring elements of $\pi_1(G)_I$ to have length zero, $\tilde{W}$ is a quasi-Coxeter group. Let $W_{\mathcal{G}}\subset\tilde{W}$ be the subgroup generated by reflections in the walls of $\mathbf{f}$. Then we have the Bruhat decomposition

$$L^+\mathcal{G}(k)\backslash LG(k)/L^+\mathcal{G}(k)=W_{\mathcal{G}}\backslash\tilde{W}/W_{\mathcal{G}} \tag{4.2}$$

describing the k-valued points of the Hecke stack $\mathrm{Hk}_{\mathcal{G}}:=[L^+\mathcal{G}\backslash\mathrm{Fl}_{\mathcal{G}}]$. Since these capture the entirety of the $L^+\mathcal{G}$-orbits, we can now give the formal definition of Schubert varieties.

Definition 4.1. Let $w\in W_{\mathcal{G}}\backslash\tilde{W}/W_{\mathcal{G}}$. The affine Schubert variety $\mathrm{Fl}_{\mathcal{G},\le w}\subset\mathrm{Fl}_{\mathcal{G}}$ is the closure of the $\mathcal{G}$-orbit of any choice of lift of w to $\mathrm{Fl}_{\mathcal{G}}(k)$.

The affine Schubert variety $\mathrm{Fl}_{\mathcal{G},\le w}$ is isomorphic to the perfection of a projective k-scheme. The notation $\mathrm{Fl}_{\mathcal{G},\le w}$ reflects the fact that $\mathrm{Fl}_{\mathcal{G},\le w}$ is set-theoretically a disjoint union of the finitely many $L^+\mathcal{G}$-orbits for the $v\in W_{\mathcal{G}}\backslash\tilde{W}/W_{\mathcal{G}}$ bounded by w in the Bruhat order $\le$. There is a refinement of this collection of closed subschemes obtained as $L^+\mathcal{I}$-orbit closures $\mathrm{Fl}_{(\mathcal{I},\mathcal{G}),\le w}$ inside $\mathrm{Fl}_{\mathcal{G}}$, called Iwahori–Schubert varieties and indexed by any $w\in\tilde{W}/W_{\mathcal{G}}$.

We will also need convolution Schubert varieties, in order to have access to Demazure resolutions. Let $w_\bullet=(w_1,\dots,w_n)$ be a sequence of elements in $\tilde{W}$. We define the convoluted Schubert variety

$$\mathrm{Fl}_{\mathcal{G},\le w_\bullet}:=(LG)_{\le w_1}\times^{L^+\mathcal{G}}\dots\times^{L^+\mathcal{G}}\mathrm{Fl}_{\mathcal{G},\le w_n}, \tag{4.3}$$

where $(LG)_{\le w}\subset LG$ is the pullback of the Schubert variety $\mathrm{Fl}_{\mathcal{G},\le w}\subset\mathrm{Fl}_{\mathcal{G}}$ along the natural projection $LG\to\mathrm{Fl}_{\mathcal{G}}$ and the notation $\times^{L^+\mathcal{G}}$ stands for the étale quotient by the diagonal $L^+\mathcal{G}$-action on the adjacent factors. If $\mathcal{G}=\mathcal{I}$ is a Iwahori and all the $w_i=:s_i$ have length 1, then $(LG)_{\le s_i}$ identifies with the jet group $L^+\mathcal{G}_{s_i}$ of the unique parahoric O-model of G such that $\mathcal{G}_{s_i}(O)$ stabilizes the codimension 1 subfacet $\mathbf{f}_i\subset\bar{\mathbf{a}}$ fixed under s_i. Thus, we can write

$$\mathrm{Fl}_{\mathcal{I},\le s_\bullet}=L^+\mathcal{G}_{s_1}\times^{L^+\mathcal{I}}\dots\times^{L^+\mathcal{I}}L^+\mathcal{G}_{s_n}/L^+\mathcal{I}. \tag{4.4}$$

and call this a Demazure variety. This is a perfectly smooth variety of dimension n and any convolution Schubert variety admits a proper birational cover given by a Demazure variety under the natural multiplication map. It is costumary to demand that the s_i are simple reflections in W_{aff}, but this forces one to explicitly deal with translations.

Let us compute the Picard group at Iwahori level.

Proposition 4.2. *Suppose $\mathcal{G}=\mathcal{I}$ is an Iwahori model. Then there is an isomorphism* $\deg\colon\mathrm{Pic}(\mathrm{Fl}_{\mathcal{I},\le w_\bullet})\xrightarrow{\sim}\prod_{i,s\le w_i}\mathbb{Z}[1/p]^n$ *given by the degree of the restriction to $\mathrm{Fl}_{\mathcal{I},\le s}$, where s runs through length 1 words s bounded by w_i for each i.*

Proof. See [FHLR22, Lemma 4.8] when F has characteristic p and [AGLR22, Theorem 3.8] when F has characteristic 0. While the degree isomorphism $\mathrm{Pic}(\mathbb{P}^{1,\mathrm{pf}}_k)\simeq\mathbb{Z}[1/p]$ implicitly uses the choice of a deperfection, there is no ambiguity when it comes to $\mathrm{Fl}_{\mathcal{I},\le s}$, as we can take the natural smooth deperfection $\mathrm{Fl}_{\mathcal{I},\leq s,1}$ that comes with a $\mathcal{I}_k$-action and smooth stabilizers, see [AGLR22, Definition 3.14] and the next section on Demazure deperfections. □

In order to apply our criterion on inversion of adjunction for asymptotic global +-regularity, we shall need to have strong control over positivity of line bundles on Demazure varieties.

Lemma 4.3. *Suppose $\mathcal{G} = \mathcal{I}$ is an Iwahori model. Then, a line bundle $\mathcal{L}$ on $\mathrm{Pic}(\mathrm{Fl}_{\mathcal{I},\leq w_\bullet})$ is ample (resp. semi-ample) if and only if $\deg(\mathcal{L})$ is a sequence of positive (resp. non-negative) rationals and the subsequence indexed by any s is strictly decreasing (resp. decreasing).*

Proof. This essentially follows from [HZ20, Theorem 3.1], but we give a self-contained proof. For the forward direction, notice that ampleness (resp. semi-ampleness) is preserved under pullback along a closed immersion (resp. an arbitrary map). By restricting to $\mathrm{Fl}_{\mathcal{I},\leq s}$, it follows that $\deg(\mathcal{L})$ consists of positive (resp. non-negative) rationals if $\mathcal{L}$ is ample (resp. semi-ample). In order to obtain the monotonicity condition, we restrict $\mathcal{L}$ to the convolution $\mathrm{Fl}_{\mathcal{I},\leq (s,s)}$, and identify it with the usual product $\mathrm{Fl}^2_{\mathcal{I},\leq s}$ via the first projection and multiplication. Observe that this isomorphism maps $\mathrm{Fl}_{\mathcal{I},\leq(s,1)} \subset \mathrm{Fl}_{\mathcal{I},\leq(s,s)}$ (resp. $\mathrm{Fl}_{\mathcal{I},\leq(1,s)} \subset \mathrm{Fl}_{\mathcal{I},\leq(s,s)}$) to the diagonal (resp. second factor) of the untwisted product $\mathrm{Fl}^2_{\mathcal{I},\leq s}$, so the claim is clear.

For the converse, we can reduce to the case where $w_\bullet = w$ by considering the natural embedding $\mathrm{Fl}_{\mathcal{I},\leq w_\bullet} \subset \mathrm{Fl}^n_{\mathcal{I}}$ whose i-th coordinate is the projection $\mathrm{Fl}_{\mathcal{I},\leq w_\bullet} \to \mathrm{Fl}_{\mathcal{I},\leq w_{\bullet\leq i}}$ post-composed with the multiplication $\mathrm{Fl}_{\mathcal{I},\leq w_{\bullet\leq i}} \to \mathrm{Fl}_{\mathcal{I}}$. Indeed, we can extend $\mathcal{L}$ to the right side preserving the positivity condition on degrees. We may also pass to the adjoint quotient of G and then to each of its simple F-factors, and hence assume that G is an almost simple F-group. Now, we consider the closed embedding $\mathrm{Fl}_{\mathcal{I}} \to \prod_s \mathrm{Fl}_{\mathcal{G}^s}$, where s runs through all simple reflections and $\mathcal{G}^s$ is the unique maximal parahoric O-model such that $\mathcal{I}(O) \subset \mathcal{G}^s(O)$ but $s \notin \mathcal{G}^s(O)$. Since $\mathrm{Pic}(\mathrm{Fl}_{\mathcal{G}^s}) = \mathbb{Z}[1/p]$, positivity equals ampleness for these partial flag varieties and the result is clear by pullback. □

4.2. **Central extension.** In this section, we discuss a special central extension of the loop group building on [FHLR22, §4.1.3]. This relates to equivariance of line bundles on $\mathrm{Fl}_{\mathcal{I}}$ with respect to the loop group LG. We assume from now on that G is simply connected and almost simple. One can show by reduction to maximal parahorics $\mathcal{G}^s$ as in the previous lemma and then to GL_n and its determinant line bundle, see [BS17, Theorem 8.8], that every line bundle in $\mathrm{Pic}(\mathrm{Fl}_{\mathcal{I}})$ is $LG(k)$-equivariant. However, it is not true that $LG(k)$-equivariance implies LG-equivariance. Indeed, we have

$$\mathrm{Pic}([LG\backslash \mathrm{Fl}_{\mathcal{I}}]) \simeq \mathrm{Pic}([*/L^+\mathcal{I}]) \simeq X^*(S) \tag{4.5}$$

and we can describe the map towards $\mathrm{Pic}(\mathrm{Fl}_{\mathcal{I}})$ as follows, see also [FHLR22, Lemma 4.10]. Let $\nu \in X^*(S)$ and denote by $\mathcal{O}(\nu)$ the associated line bundle. Given a simple reflection $s \in W_{\mathrm{af}}$ with associated affine root α_s, we can check that the degree of $\mathcal{L}(\nu)$ over $\mathrm{Fl}_{\mathcal{I},\leq s}$ equals $\langle a_s^\vee, \nu\rangle$, where a is the euclidean root underlying α_s, compare with [FHLR22, Lemma 4.12]. The map $X^*(S) \to \mathrm{Pic}(\mathrm{Fl}_{\mathcal{I}})$ is a split injection, but cannot possibly be surjective, as the image has rank equal to that of W (the *euclidean rank* of G), whereas $\mathrm{Pic}(\mathrm{Fl}_{\mathcal{I}})$ has rank equal to that of W_{af} (the *affine rank* of G). The cokernel of $X^*(S) \to \mathrm{Pic}(\mathrm{Fl}_{\mathcal{I}})$ is a free $\mathbb{Z}[1/p]$-module of rank 1, again by the same proof as in [FHLR22, Lemma 4.12]. Employing the same ordering as in [FHLR22, Lemma 4.13] after decomposing G into simple factors, we can select a certain semi-ample line bundle $\mathcal{L}$ to define a direct summand $\mathbb{Z}[1/p]$ inside $\mathrm{Pic}(\mathrm{Fl}_{\mathcal{I}})$ complementary to $X^*(S)$. This leads to the central charge map

$$c\colon \mathrm{Pic}(\mathrm{Fl}_{\mathcal{I}}) \to \mathbb{Z}[1/p] \tag{4.6}$$

with kernel equal to $X^*(S)$. We define the central extension

$$1 \to \mathbb{G}^{\mathrm{pf}}_{m,k} \to \widehat{LG} \to LG \to 1 \tag{4.7}$$

that classifies isomorphisms between $\mathcal{L}$ and $g^*\mathcal{L}$ for all the i. This is independent of the choice of $\mathcal{L}$ by the proof of [FHLR22, Lemma 4.27]. Observe now that we have

$$\mathrm{Pic}([\widehat{LG}\backslash \mathrm{Fl}_{\mathcal{I}}]) \simeq \mathrm{Pic}([*/\widehat{L^+\mathcal{I}}]) \simeq X^*(\widehat{S}) \tag{4.8}$$

and the forgetful map $X^*(\widehat{S}) \to \mathrm{Pic}(\mathrm{Fl}_{\mathcal{I}})$ is an isomorphism by construction. Therefore, every line bundle on $\mathrm{Fl}_{\mathcal{I}}$ has become $\widehat{LG}$-equivariant. Using this, we can define affine coroots.

Definition 4.4. Let s be a simple reflection and $\mathcal{G}_s$ be the associated parahoric group scheme. The affine coroot $\alpha_s^\vee \in X_*(\widehat{S})$ is defined as the unique lift of $a_s^\vee \in X_*(S)$ such that the associated map $\mathbb{G}_{m,k}^{\mathrm{pf}} \to \widehat{LG}$ lands in the subgroup generated by $L^+\mathcal{U}_{\pm\alpha_s}$.

In the definition, we are using the fact that the pullback of $LU_a \subset LG$ along the central extension splits canonically, which is a consequence of the group U_a being unipotent. The dual weight ω_s corresponds under our isomorphism to the line bundle $\mathcal{L}_s$ with degree 1 on $\mathrm{Fl}_{\mathcal{I},\leq s}$ and 0 on every other $L^+\mathcal{I}$-stable $\mathbb{P}_k^{1,\mathrm{pf}}$. The sum of the dual weights $\rho = \sum_s \omega_s$ corresponds to the critical line bundle with degree 1 on every $L^+\mathcal{I}$-stable $\mathbb{P}_k^{1,\mathrm{pf}}$, compare with the terminology in [FHLR22, Lemma 4.17].

Remark 4.5. It is slightly confusing that there are central weights of $\widehat{LG}$ giving rise to non-trivial line bundles by Iwahori induction, but the difference for this very large ind-group is that the center and the cocenter are not isogenous. Indeed, one checks that $\widehat{LG}$ is equal to its own derived subgroup. If there were a rotation $\mathbb{G}_m$-action on LG (e.g., in equicharacteristic and for tame G), we could produce affine roots and their dual coweights as well. In general, however, there does not seem to be a loop interpretation for the affine roots for p-adic G.

4.3. **Equivariant $q_\bullet$-twisted deperfections.** If F has characteristic p, then the moduli space that the perfect flag variety $\mathrm{Fl}_{\mathcal{G}}$ underlies extends naturally to arbitrary k-algebras and is represented by an ind-scheme. As in the perfect case, one defines Schubert varieties as L^+G-orbit closures, where now L^+G is also functor on arbitrary k-algebras. However, since these can fail to be normal in certain cases by [HLR24], one is led to consider their seminormalizations in [FHLR22]. Similarly, one can form convolution products via the moduli interpretation of $\mathrm{Fl}_{\mathcal{G}}$ and we set $\mathrm{Fl}_{\mathcal{G},\leq w_\bullet}$ for the seminormalization (which is again normal).

If F has characteristic 0, then a finite type deperfection $\mathrm{Fl}_{\mathcal{G},\leq w}^{\mathrm{can}}$ was proposed in [AGLR22, Definition 3.14], but it is not clear how well behaved it is beyond low dimensional cases. For us, it is actually preferable to deperfect Demazure varieties. Recall that the loop group $L^+\mathcal{G}$ associated with a parahoric model admits a deperfection $\mathrm{Res}_{O/k}\mathcal{G}$ given by the Greenberg realization. Let $s_\bullet = (s_1, \dots, s_n)$ be a not necessarily reduced sequence of simple reflections in the Iwahori–Weyl group. Then, we get a stacky deperfection of $\mathrm{Fl}_{\mathcal{I},\leq s_\bullet}$ as follows:

$$\mathrm{Fl}_{\mathcal{I},\leq s_\bullet}^{\mathrm{stk}} := \mathrm{Res}_{O/k}\mathcal{G}_{s_1} \times^{\mathrm{Res}_{O/k}\mathcal{I}} \times \cdots \times^{\mathrm{Res}_{O/k}\mathcal{I}} \mathrm{Res}_{O/k}\mathcal{G}_{s_n}/\mathrm{Res}_{O/k}\mathcal{I} \tag{4.9}$$

where $\mathcal{G}_{s_i}$ is the unique parahoric model such that $\mathcal{G}_{s_i}(O) = \mathcal{I}(O) \cup \mathcal{I}(O)s_i\mathcal{I}(O)$. This is never a scheme because the maps $\mathrm{Res}_{O/k}\mathcal{I} \to \mathrm{Res}_{O/k}\mathcal{G}_{s_i}$ are never injective due to p-torsion in the Witt rings of imperfect rings. Getting actual schemes requires twisting by the Frobenius φ as follows.

We define a certain k-smooth deperfection $\mathrm{Fl}_{\mathcal{I},\leq s_\bullet,q_\bullet}$ by induction on the length of the sequence $s_\bullet$. Here, $q_\bullet$ is going to be an increasing sequence of powers of p defined also in an inductive manner, which we call $s_\bullet$-permissible. Suppose we have constructed the k-smooth variety $\mathrm{Fl}_{\mathcal{I},\leq t_\bullet,r_\bullet}$, where $s_\bullet = (s_1, t_\bullet)$ and $r_\bullet$ is $t_\bullet$-permissible. Suppose that the congruence subgroup $L^+\mathcal{I} \cap L^{\geq n}\mathcal{G}_{s_1}$ acts trivially on the perfect scheme $\mathrm{Fl}_{\mathcal{I},\leq t_\bullet}$. Then, the smooth k-group $\mathrm{Res}_{O_n/k}(\mathcal{I}, \mathcal{G}_{s_1}) := \mathrm{im}(\mathrm{Res}_{O_n/k}\mathcal{I} \to \mathrm{Res}_{O_n/k}\mathcal{G}_{s_1})$ necessarily acts on $(\mathrm{Fl}_{\mathcal{I},\leq t_\bullet,r_\bullet})^q$ for some sufficiently large power q of p. By rescaling $r_\bullet$ with this same power of q, we may assume that there was an action from the start. We then define

$$\mathrm{Fl}_{\mathcal{I},\leq s_\bullet,q_\bullet} := \mathrm{Res}_{O_n/k}\mathcal{G}_{s_1} \times^{\mathrm{Res}_{O_n/k}(\mathcal{I},\mathcal{G}_{s_1})} \mathrm{Fl}_{\mathcal{I},\leq t_\bullet,r_\bullet} \tag{4.10}$$

with $q_\bullet := (1, r_\bullet)$. This is a smooth k-scheme by induction and the fact that $\mathrm{Res}_{O_n/k}\mathcal{G}_{s_1} \to \mathrm{Fl}_{\mathcal{I},\leq s_1}$ has Zariski local sections.

Definition 4.6. For any $s_\bullet$-permissible sequence, we define the equivariant $q_\bullet$-twisted deperfection of $\mathrm{Fl}_{\mathcal{I},\leq s_\bullet}$ as the smooth k-variety $\mathrm{Fl}_{\mathcal{I},\leq s_\bullet,q_\bullet}$ constructed above by induction.

Technically, we are abusing notation by not including the corresponding sequence of integers $n_\bullet$ used to truncate the deperfected jets, as enlarging $n_\bullet$ could force $q_\bullet$ to become large for the twisted product to be well-defined. However, $\mathrm{Fl}_{\mathcal{I},\leq s_\bullet,q_\bullet}$ is still independent of $n_\bullet$ in the following sense: if $q_\bullet$ works for both $n_\bullet$ and $m_\bullet$, then the resulting smooth deperfections are isomorphic.

Next, we compute the canonical sheaf of the deperfection $\mathrm{Fl}_{\mathcal{I},\leq s_\bullet,q_\bullet}$. Note that this variety carries a natural effective divisor $\partial_{s_\bullet,q_\bullet}$ regarded as its boundary, and given by the sum of all the prime divisors $\mathrm{Fl}_{\mathcal{I},\leq s_{\bullet\setminus i},q_{\bullet\setminus i}}$ for all $1 \leq i \leq n$.

Proposition 4.7. *There is an isomorphism*

$$\omega^{-1}_{s_\bullet,q_\bullet} \simeq \mathcal{O}(\partial_{s_\bullet,q_\bullet}) \otimes_{\mathcal{O}} \mathcal{O}(q_\bullet) \tag{4.11}$$

of line bundles on $\mathrm{Fl}_{\mathcal{I},\leq s_\bullet,q_\bullet}$.

Proof. Regard a subsequence of $s_\bullet$ as a smaller word $t_\bullet \leq s_\bullet$ by inserting the identity when needed. Then $q_\bullet$ is still $t_\bullet$-permissible and we get closed immersions $\mathrm{Fl}_{\mathcal{I},t_\bullet,q_\bullet}$. Our goal is to calculate the degree of $\omega_{s_\bullet,q_\bullet}(\partial_{s_\bullet,q_\bullet})$ when restricted to $\mathrm{Fl}_{\mathcal{I},\leq s_i}$ for every $1 \leq i \leq n$: namely, that it equals $-q_i$. By the adjunction formula for canonical divisors along regular immersions, we see that this quantity remains stable under restriction from $s_\bullet$ to any $t_\bullet$ with $t_i = s_i$. By induction we can hence assume that $s_\bullet$ has been truncated to one letter, in which case the result is clear. Indeed, identifying $\mathrm{Fl}_{\mathcal{I},\leq s_i,q_i}$ with $\mathbb{P}^1_k$, this reduces to the usual calculation of the cotangent sheaf, but we need to remember that our definition of degrees in the perfection $\mathbb{P}^{1,\mathrm{pf}}_k$ was twisted by q_i. □

4.4. **Global $+$-regularity.** In equicharacteristic, it is known that $\mathrm{Fl}_{\mathcal{G},\leq w}$ are perfections of globally φ-regular k-varieties by [Cas22, Theorem 1.4] for split G and [FHLR22, Theorem 4.1] for general G. In particular, these deperfections are also globally $+$-regular by [BMP+23, Lemma 6.14]. We want to prove a version of this in mixed characteristic, except it is not always true.

Theorem 4.8. *If $q_\bullet = 1$ is $s_\bullet$-permissible, then $(\mathrm{Fl}_{\mathcal{I},\leq s_\bullet,1}, \partial_{s_\bullet,1} + E)$ is asymptotically globally $+$-regular for any origin avoiding $\mathbb{Q}$-divisor E with degree 1 everywhere.*

Proof. Let $s_\bullet = (s_1, \ldots, s_n)$ be a word of simple reflections, not necessarily reduced, and set $X = \mathrm{Fl}_{\mathcal{I},\leq s_\bullet,1}$, $\partial X = \partial_{s_\bullet,1}$. Consider the effective Cartier divisors $D_i := \mathrm{Fl}_{\mathcal{I},\leq s_{\bullet\neq i},1} \subset \mathrm{Fl}_{\mathcal{I},\leq s_\bullet,1}$ for any $1 \leq i \leq n$ obtained by deleting the letter i. We know that the Cartier divisor $K_X + \partial X$ has degree -1 in every $L^+\mathcal{I}$-stable curve, so up to changing the canonical divisor we identify its negative with the semi-ample E. Since we have $(E + \partial X - S)|_S = E|_S + \partial S$ thanks to origin avoidance of E, our goal is to show that $(X, E + \partial X)$ is asymptotically purely Fano along $S := D_n$, so that we may apply Theorem 3.11. Thus, we have to slightly perturb the coefficients of $E + \partial X = E + \sum_{i\leq n} D_i$ to get a $\mathbb{Q}$-divisor

$$\Delta = r_0 E + \sum_{i\leq n} r_i D_i \tag{4.12}$$

with r_i smaller but arbitrarily close to 1 and $r_n = 1$ so that $K_X + \Delta$ is $\mathbb{Q}$-Cartier (trivial as X is smooth) and anti-ample. For convenience, we set $\epsilon_i := 1 - r_i$. Then, we deduce that

$$-K_X - \Delta = \epsilon_0 E + \sum_i \epsilon_i D_i \tag{4.13}$$

and it is enough that the associated degrees sequence is decreasing. In a lemma below, we construct a sequence ϵ_i stable under homothety such that the right side has decreasing degrees. Thus, we get the required asymptotical pure Fano property. □

We used the following lemma during the previous proof.

Lemma 4.9. *There exists a sequence of rational numbers* $1 > \epsilon_1 > \cdots > \epsilon_n = 0$ *such that the effective* $\mathbb{Q}$*-divisor* $cA + E$ *is ample on* X *for every* $c \in \mathbb{Q}_{>0}$*, where* $A := \sum_i \epsilon_i D_i$ *and* $\deg(E) = (1, \dots, 1)$*.*

Proof. We are going to construct instead a strictly decreasing sequence of positive integers a_i such that $B := \sum_i a_i D_i$ has strictly decreasing non-negative degrees on the $\mathrm{Fl}_{\mathcal{I},\leq s_k}$. Then, we set $\epsilon_i = N^{-1} a_i$ where $N > a_i$ for all i.

By induction on the length of our sequence $s_\bullet$, we can assume that we already have $a_2, \dots, a_n$ satisfying our degree hypothesis for $s_{\bullet>1}$. Choose now a sufficiently large positive integer a_1. Since D_1 has trivial degree on $\mathrm{Fl}_{\mathcal{I},\leq s_k}$ for $k > 1$, we see that the divisor A has strictly decreasing non-negative degrees on the $\mathrm{Fl}_{\mathcal{I},\leq s_k}$ for $k > 1$. On the other hand, D_1 has degree 1 on $\mathrm{Fl}_{\mathcal{I},\leq s_1}$, so the corresponding degree of A grows linearly with a_1. Thus, we may assume that it supersedes those for $k > 1$. □

It is conjectured that global $+$-regularity is an equivalent notion to strong φ-regularity, but it is not currently known. In our situation, we can still deduce strong φ-regularity from our results.

Corollary 4.10. *If* $q_\bullet = 1$ *is* $s_\bullet$*-permissible, then* $\mathrm{Fl}_{\mathcal{I},\leq s_\bullet,1}$ *is globally* φ*-regular and compatibly* φ*-split with* $\mathrm{Fl}_{\mathcal{I},\leq t_\bullet,1}$ *for every* $t_\bullet \leq s_\bullet$*.*

Proof. Take $\Delta_{s_\bullet,1} = (1 - 1/p)\mathrm{Fl}_{\mathcal{I},\leq s_{\bullet\setminus i},1}$. Then global $+$-regularity implies that the map of coherent sheaves

$$\mathcal{O}_{\mathcal{I},\leq s_\bullet,1} \to \varphi_* \mathcal{O}_{\mathcal{I},\leq s_{\bullet\setminus i},1/p}((p-1)\mathrm{Fl}_{\mathcal{I},\leq s_{\bullet\setminus i},1/p}) \tag{4.14}$$

admits a splitting. Twisting it by the ideal sheaf of $\mathrm{Fl}_{\mathcal{I},\leq s_{\bullet\setminus i},1}$ and applying the projection formula, we deduce that $\mathrm{Fl}_{\mathcal{I},\leq s_\bullet,1}$ is compatibly φ-split with each irreducible component of its boundary, compare with [BK05, Theorem 1.4.10]. This proves by induction that it is compatibly φ-split with any intersection and unions of those.

Next, we handle global φ-regularity. We select an ample effective $\mathbb{Q}$-divisor

$$\Delta_{s_\bullet,1} := \sum_i \frac{n_i}{q} \mathrm{Fl}_{\mathcal{I},\leq s_{\bullet\setminus i},1} \tag{4.15}$$

for some sequence of non-negative integers $0 \leq n_i < q$ and some sufficiently large $q = p^e \gg 0$. This is possible by the argument in the previous lemma for example. Then, we can guarantee the existence of a splitting

$$\mathcal{O}_{\mathcal{I},\leq s_\bullet,1} \to \varphi^e_* \mathcal{O}_{\mathcal{I},\leq s_\bullet,1/q}(\sum_i n_i \mathrm{Fl}_{\mathcal{I},\leq s_{\bullet\setminus i},1/q}) \tag{4.16}$$

by global $+$-regularity. Now we can apply [Smi00, Theorem 3.10] to deduce that $\mathrm{Fl}_{\mathcal{I},\leq s_\bullet,1}$ is globally φ-regular. □

Now, we assume that $s_\bullet$ is reduced. Let $\mathrm{Fl}_{\mathcal{G},\leq w}$ be the image of the map $\mathrm{Fl}_{\mathcal{I},\leq s_\bullet} \to \mathrm{Fl}_{\mathcal{G}}$. Then, we can define $\mathrm{Fl}_{\mathcal{G},\leq w,q_\bullet}$ as the normal k-variety modelling the original Schubert variety whose structure sheaf equals the pushforward of $\mathcal{O}_{\mathcal{I},\leq s_\bullet,q_\bullet}$ along the map $\mathrm{Fl}_{\mathcal{I},\leq s_\bullet} \to \mathrm{Fl}_{\mathcal{G},\leq w}$ of perfect varieties (regarded as a topological map). In the $q_\bullet = 1$ case, we can say a lot about the geometry of these schemes:

Proposition 4.11. *If $q_\bullet = 1$ is $s_\bullet$-permissible, then $\mathrm{Fl}_{\mathcal{G},\leq w,1}$ is globally φ-regular and the maps $\mathrm{Fl}_{\mathcal{G},\leq v,1} \to \mathrm{Fl}_{\mathcal{G},\leq w,1}$ are compatibly φ-split closed immersions for $v \leq w$. Moreover, $\mathrm{Fl}_{\mathcal{I},\leq s_\bullet,1} \to \mathrm{Fl}_{\mathcal{G},\leq w,1}$ is a rational resolution.*

Proof. Global φ-regularity is preserved along proper birational maps, so the first claim follows. For the second claim, it is crucial to show that the induced map $\mathcal{O}_{\mathcal{G},\leq w,1} \to \mathcal{O}_{\mathcal{G},\leq v,1}$ is surjective. This is equivalent to showing that the ideal sheaf of $\mathrm{Fl}_{\mathcal{I},\leq s_{\bullet\setminus i},1}$ has vanishing higher direct images along the Demazure resolution. This is true by construction at the perfect level and we can therefore descend it to our deperfection by using a φ-splitting of $\mathrm{Fl}_{\mathcal{I},\leq s_\bullet,1}$ compatible with its boundary. As for rationality of the deperfected Demazure resolution, we can prove it in the same manner: we know that the higher direct images of the structure sheaf vanish at the perfect level and then it descends via a φ-splitting, compare with the discussion around [BS17, Lemma 6.9]. For the canonical sheaf, we know that pushforward respects dualizing complexes by Grothendieck–Serre duality and the preceding higher vanishing of the structure sheaf. Then, it suffices to observe that $\mathrm{Fl}_{\mathcal{G},\leq w,1}$ is globally $+$-regular, hence Cohen–Macaulay. □

We can also compute the Picard group of $\mathrm{Fl}_{\mathcal{G},\leq w,1}$ as follows:

Corollary 4.12. *If $q_\bullet = 1$ is $s_\bullet$-permissible, then the natural map $\mathrm{Pic}(\mathrm{Fl}_{\mathcal{G},\leq w,1}) \to \prod_{s\leq w} \mathrm{Pic}(\mathrm{Fl}_{\mathcal{G},\leq s,1})$ is an isomorphism.*

Proof. Due to rationality and compatible φ-splitness of the $\mathrm{Fl}_{\mathcal{G},\leq v,1}$, we can now write the deperfected Demazure resolution $\mathrm{Fl}_{\mathcal{I},\leq s_\bullet,1} \to \mathrm{Fl}_{\mathcal{G},\leq w,1}$ as a composition of rational maps with non-trivial fibers equal to $\mathbb{P}^1_k$, just like in [FHLR22, Lemma 4.5]. Then, one shows also inductively that the line bundle $\mathcal{O}(\nu)$ on $\mathrm{Fl}_{\mathcal{I},\leq s_\bullet,1}$ for some integral weight ν of the torus $\widehat{S}$ pushes forward along the resolution to a line bundle on $\mathrm{Fl}_{\mathcal{G},\leq w,1}$, see [FHLR22, Lemma 4.20]. □

A consequence of this result is that we can now construct explicit φ-splittings of $\mathrm{Fl}_{\mathcal{I},\leq s_\bullet,1}$ via the Mehta–Ramanathan criterion. We can even characterize their descents to the 1-deperfected Schubert varieties $\mathrm{Fl}_{\mathcal{I},\leq w,1}$ at Iwahori level.

Proposition 4.13. *Each global section $\theta \in H^0(\mathrm{Fl}_{\mathcal{I},\leq w,1}, \mathcal{O}(\rho))$ not vanishing at the origin determines a unique φ-splitting of $\mathrm{Fl}_{\mathcal{I},\leq w,1}$ compatibly splitting $\partial_{w,1}$ and $\mathrm{div}(\theta)$.*

Proof. Let us first note that such a global section θ indeed exists. We must show that every ample line bundle on $\mathrm{Fl}_{\mathcal{I},\leq w,1}$ is globally generated. But restriction to the origin yields a surjection upon taking global sections by the existence of some compatible φ-splitting, and furthermore the base locus is $\mathcal{I}(O)$-equivariant and open, so it equals the whole space.

Let us now fix a 1-permissible reduced word $s_\bullet$ for w. Then, we consider the pullback $f^*\theta$ along $f\colon \mathrm{Fl}_{\mathcal{I},\leq s_\bullet,1} \to \mathrm{Fl}_{\mathcal{I},\leq w,1}$. We may now apply the Mehta–Ramanathan criterion, see [BK05, Proposition 1.3.11], to produce a φ-splitting given by the $(p-1)$-th power of a global section σ of the anticanonical sheaf $\omega^{-1}_{s_\bullet,1}$ such that $\mathrm{div}(\sigma) = \partial_{s_\bullet,1} + \mathrm{div}(f^*\theta)$. We can descend this φ-splitting to $\mathrm{Fl}_{\mathcal{I},\leq w,1}$ and we immediately get that it compatibly splits $\partial_{w,1}$ and $\mathrm{div}(\theta)$ (which is therefore reduced).

In order to prove the claim, we still have to characterize the descended φ-splitting on $\mathrm{Fl}_{\mathcal{I},\leq w,1}$ in unequivocal fashion, i.e., by proving that is given by a specific $(p-1)$-th global section τ of the anticanonical sheaf $\omega^{-1}_{w,1}$. For this, we perform a calculation away from codimension 2. Let

$v \leq w$ be such that $\ell(v) = \ell(w) - 1$. The Bruhat decomposition implies that the open locus of $\mathrm{Fl}_{\mathcal{I},\leq w}$ consisting of the w-orbit and the v-orbit lifts isomorphically to $\mathrm{Fl}_{\mathcal{I},\leq s_\bullet}$, and by using birationality of f and normality of the base, we can descend this to the 1-deperfections. Now, one checks easily that the claimed τ exists and $\mathrm{div}(\tau) = \partial_{w,1} + \mathrm{div}(\theta)$. $\square$

Another corollary of global +-regularity is the Demazure character formula. Recall that the central extension $\widehat{LG}$ acts on any line bundle $\mathcal{O}(\nu)$. In particular, if ν is sufficiently p-divisible, the line bundle $\mathcal{O}(\nu)$ descends to $\mathrm{Fl}_{\mathcal{I},\leq s_\bullet,q_\bullet}$, so the cohomology groups $H^i(\mathrm{Fl}_{\mathcal{I},\leq s_\bullet,q_\bullet}, \mathcal{O}(\nu))$ have an associated character counting dimensions of affine weight spaces, i.e., an element of the group ring $\mathbb{Z}[X^*(\widehat{S})]$ of affine weights. We use exponential notation for this group ring to avoid confusion with sums of coefficients and sums of weights (which correspond to multiplication in the ring).

Corollary 4.14. *Let $\nu \in X^*(\widehat{S})^+$ be a dominant weight. If $q_\bullet = 1$ is $s_\bullet$-permissible, then we have an equality*

$$\mathrm{char} H^0(\mathrm{Fl}_{\mathcal{I},\leq s_\bullet,1}, \mathcal{O}(\nu)) = \Lambda_{s_1} \circ \cdots \circ \Lambda_{s_n}(e^\nu) \tag{4.17}$$

where $\Lambda_s(e^\nu) = (1 - e^{-a_s})^{-1}(e^\nu - e^{\nu - \langle \alpha_s^\vee, \nu+\rho\rangle a_s})$ is the Demazure operator.

Proof. The usual inductive proof identifies the right side with the character of the Euler characteristic $\chi(\mathrm{Fl}_{\mathcal{I},\leq s_\bullet,1}, \mathcal{O}(\nu))$, compare with [Lit98, Theorem 7]. Global +-regularity of $\mathrm{Fl}_{\mathcal{I},\leq s_\bullet,1}$ yields vanishing of higher cohomology, so the equality follows. $\square$

4.5. **Local models.** In this section, we briefly need to refer to the theory of v-sheaves as in [Sch17, SW20], but the reader may treat this as a black box. Consider the Beilinson–Drinfeld Grassmannian $\mathrm{Gr}_\mathcal{G}$ in the sense of [SW20] defined over the v-sheaf $\mathrm{Spd}\, O$. Its generic fiber is isomorphic to the B_{dR}^+-affine Grassmannian and its special fiber equals the v-sheaf attached to the affine flag variety $\mathrm{Fl}_\mathcal{G}$. Let μ be a geometric conjugacy class of coweights with reflex field E. Then, the affine Grassmannian $\mathrm{Gr}_{G,E}$ base changed to E contains a closed subsheaf $\mathrm{Gr}_{G,\leq\mu}$ arising as the L^+G-orbit closure of $\mu(\xi)$. Following [AGLR22], we define the v-sheaf local model $M_{\mathcal{G},\mu}^v$ as the v-sheaf closure of $\mathrm{Gr}_{G,E}$ inside $\mathrm{Gr}_{\mathcal{G},O_E}$.

If F has characteristic p or μ is minuscule, there exists by [AGLR22, Theorem 1.1] and [GL24, Corollary 1.4] a unique flat normal projective O_E-scheme $M_{\mathcal{G},\mu}$ with reduced special fiber whose associated v-sheaf equals $M_{\mathcal{G},\mu}^v$. We call it the scheme-theoretic local model and, with the single exception of wild odd unitary groups (so only when $p = 2$), it was shown in the corresponding statements of [AGLR22, GL24] (relying on [FHLR22, Theorem 1.2]) that this scheme has φ-split special fiber. We can now use our global +-regularity result to remove this assumption from the computation of the special fiber in [AGLR22, FHLR22, GL24] (see Remark 4.18 for Cohen–Macaulayness).

Before we state and prove it, we treat the deperfection of the μ-admissible locus $A_{\mathcal{G},\mu}$. Recall that the μ-admissible set Adm_μ of Kottwitz–Rapoport [KR00] consists of all elements w of the Iwahori–Weyl group bounded by the translation t_λ for some representative of μ. Then, the μ-admissible locus $A_{\mathcal{G},\mu}$ is the union of all Schubert varieties $\mathrm{Fl}_{\mathcal{G},\leq w}$ as w runs over all double cosets with lifts in Adm_μ (after fixing a Iwahori O-model $\mathcal{I}$ mapping to $\mathcal{G}$). Below, we show strong structure results on the so-called canonical deperfection of $A_{\mathcal{G},\mu}$ in the sense of [AGLR22, Definition 3.14], generalizing [AGLR22, Theorem 3.16].

Proposition 4.15. *Assume F has characteristic p or μ is minuscule. Then, $A_{\mathcal{G},\mu}$ has a unique φ-split deperfection $A_{\mathcal{G},\mu,1}$ admitting the deperfections $\mathrm{Fl}_{\mathcal{G},\leq t_\lambda,1}$ as compatibly φ-split closed subschemes for every representative λ of μ.*

Proof. Note that t_λ has a reduced word for which the constant sequence 1 is permissible by the proof of [AGLR22, Lemma 3.15], so the statement is reasonable. Consider more generally a finite subset W of $W_\mathcal{G}\backslash\tilde{W}/W_\mathcal{G}$. Then, we claim that the Schubert scheme $\mathrm{Fl}_{\mathcal{G},\leq W}$ in the sense of

[AGLR22, Definition 3.6] obtained as the reunion of all Schubert varieties $\mathrm{Fl}_{\mathcal{G},\leq w}$ with $w \in W$, is the finite colimit (which is neither filtered nor sifted) of these exact subvarieties along the various natural inclusions. We prove this observation by double induction on the dimension and the number of irreducible components (regarded as lexicographically ordered pairs). If W has a maximal element, this is obvious as the colimit has a final term. Otherwise, we write it as a union $W_1 \cup W_2$ of incomparable sets, and verify that $\mathrm{Fl}_{\mathcal{G},\leq W}$ equals the coproduct $\mathrm{Fl}_{\mathcal{G},\leq W_1} \sqcup_{\mathrm{Fl}_{\mathcal{G},\leq W_{12}}} \mathrm{Fl}_{\mathcal{G},\leq W_2}$, where W_{12} contains the maximal elements below W_1 and W_2. Indeed, the obvious map is a universal homeomorphism and one has a Mayer–Vietoris short exact sequence that computes structure sheaves.

Now, assume that every element w of W has a 1-permissible reduced word $s_\bullet$ defined with respect to a fixed Iwahori $\mathcal{I}$ mapping to $\mathcal{G}$. We define the deperfection $\mathrm{Fl}_{\mathcal{G},\leq W}$ as the analogous colimit of the $\mathrm{Fl}_{\mathcal{G},\leq w,1}$. This exists at least as a functor, and we claim that it is representable by a scheme, that the transition maps $\mathrm{Fl}_{\mathcal{G},\leq V,1} \to \mathrm{Fl}_{\mathcal{G},\leq W,1}$ are closed immersions, and there is a compatible φ-splitting of them all. We perform again the same double induction argument. Assume first that W has a maximal element: then existence is clear, it contains 1-deperfected Schubert varieties as closed subschemes, and these are all compatibly φ-split. As for $V \leq W$ without a maximal element, one can show as in the previous paragraph that the colimit is realized inside $\mathrm{Fl}_{\mathcal{G},\leq W,1}$: for the pushout step, one has to use the fact that compatibly split subschemes have reduced intersection. If W has no maximal element, write it as a union $W_1 \cup W_2$ of incomparable sets. Now, existence of the colimit follows by the existence of pushouts along closed immersions, see [Sta23, Tag 0E25]. If we had a φ-splitting of $\mathrm{Fl}_{\mathcal{G},\leq W,1}$ compatible with all the other maps $\mathrm{Fl}_{\mathcal{G},\leq V,1}$, then the claim would follow.

In order to do that, we assume first that $\mathcal{G} = \mathcal{I}$ is a Iwahori. Then, the line bundle $\mathcal{O}(\rho)$ is defined on $\mathrm{Fl}_{\mathcal{I},\leq W}$ and it carries an origin avoiding global section, again by the same double induction procedure. This shows that we get a compatible φ-splitting of $\mathrm{Fl}_{\mathcal{I},\leq W,1}$ compatible with all the smaller $V \leq W$. If $\mathcal{G}$ is not a Iwahori, then we consider the birational universal homeomorphism $g\colon \mathrm{Fl}_{\mathcal{I},\leq U,1} \to \mathrm{Fl}_{\mathcal{G},\leq W,1}$, where U consists of the obvious lifts defined by the reduced words. It suffices to compute the pushforward of the structure sheaf under g_*: this commutes with finite colimits at the derived level and then one gets the desired result at the abelian level by higher vanishing of direct images. □

The compatible φ-splitting of $A_{\mathcal{G},\mu,1}$ allows us to compute its Picard group in the same manner as in [FHLR22, Proposition 4.23, Corollary 5.8], which is given by the obvious $\mathbb{Z}$-lattice in the Picard group of the corresponding connected component of $\mathrm{Fl}_{\mathcal{G}}$. In particular, we may speak unambigously of the central charge $c_{\mathcal{L}}$ for some line bundle $\mathcal{L}$ on $A_{\mathcal{G},\mu,1}$ by first extending to the connected component and then translating to the neutral component. Next, we compute the dimensions of global sections of an ample $\mathcal{L}$ on the 1-deperfection $A_{\mathcal{G},\mu,1}$. The result below answers the coherence conjecture of Pappas–Rapoport [PR08] in our p-adic setting (and gives a new proof for some wild groups in equicharacteristic).

Theorem 4.16. *Assume F has characteristic p or μ is minuscule. For any ample line bundle $\mathcal{L}$ on $A_{\mathcal{G},\mu,1}$, there is an equality*

$$\dim_k H^0(A_{\mathcal{G},\mu,1}, \mathcal{L}) = \dim_E H^0(\mathrm{Gr}_{G,\leq\mu,1}, \mathcal{O}(c_{\mathcal{L}})) \tag{4.18}$$

of dimensions of global sections of line bundles, where $c_{\mathcal{L}}$ is its central charge.

Here, we have to regard the central charge $c_{\mathcal{L}}$ as a tuple of integers obtained by splitting the adjoint quotient of G into simple factors.

Proof. The first order of business is to calculate the Euler characteristic of any line bundle $\mathcal{L}$ on $A_{\mathcal{G},\mu,1}$. Below, we give a closed formula for the Euler characteristic on more general unions

$\mathrm{Fl}_{\mathcal{G},\leq W,1}$ in terms of its irreducible Schubert subvarieties.

$$\chi(\mathrm{Fl}_{\mathcal{G},\leq W,1},\mathcal{L}) = \sum_{w_0} \mu_W(w_0)\chi(\mathrm{Fl}_{\mathcal{G},\leq w_0},\mathcal{L}) \tag{4.19}$$

where $\mu_W(w_0)$ is the difference between the number of even and odd chains $w_0 < w_1 < \cdots < w_n \leq W$. We note that this formula is rather inefficient, but it would otherwise be impossible to write down an optimal formula, as one would have to somehow capture the optimal intersection and union patterns. We regard it as an analogue of the Möbius inversion formula for posets, but as we could not find an appropriate reference, we give an explicit proof.

Again, one has to perform the same double induction procedure. If W has a maximal element w, then there is a bijection between w-avoiding chains and w-ending positive length chains. Since their lengths have different parity, the coefficient $\mu_W(w_0)$ vanishes for $w_0 \neq w$, and only the term $\chi(\mathrm{Fl}_{\mathcal{G},\leq w,1})$ indexed by w survives. Now write W as the union of two sets W_1 and W_2 and let W_{12} be the set of maximal elements below W_1 and W_2. Then, there is an equality

$$\chi(\mathrm{Fl}_{\mathcal{G},\leq W,1},\mathcal{L}) + \chi(\mathrm{Fl}_{\mathcal{G},\leq W_{12},1},\mathcal{L}) = \chi(\mathrm{Fl}_{\mathcal{G},\leq W_1,1},\mathcal{L}) + \chi(\mathrm{Fl}_{\mathcal{G},\leq W_2,1},\mathcal{L}) \tag{4.20}$$

by Mayer–Vietoris, and we need to match the chain terms appearring in the Möbius inversion formula. If a chain terminates in W_{12}, then it is summed twice on both sides. Otherwise, the term is counted only once on both sides, so we get indeed an equality.

Together with the Demazure character formula and higher vanishing for ample $\mathcal{L}$, we get a purely combinatorial formula for the left side. We remark that, for triality factors, the dependence on μ rather than its image in the inertia coinvariants appears trickier because the splitting field can have either degree 3 or 6, but one checks that the A_3-average and the S_3-average of an arbitrary coweight coincide. This means that we can reduce the proof of the equality to tame G in equicharacteristic (or even equicharacteristic 0!), so the result follows from [Zhu14, Theorem 3], see also [GL24, Theorem 2.1] for another proof. □

We can deduce from the above an identification of the special fiber of scheme-theoretic local models.

Corollary 4.17. *The special fiber of $M_{\mathcal{G},\mu}$ equals $A_{\mathcal{G},\mu,1}$.*

Proof. If $p > 2$ or Φ_G is reduced, this is [AGLR22, Theorem 1.1] and [FHLR22, Theorem 1.2]. The previous theorem states that the Hilbert polynomials of $A_{\mathcal{G},\mu,1}$ and $\mathrm{Gr}_{G,\leq\mu,1}$ coincide. By [GL24, Corollary 1.4], the special fiber of $M_{\mathcal{G},\mu}$ is reduced with weak normalization equal to $A_{\mathcal{G},\mu,1}$ (essentially by definition). By flatness, the special fiber also shares the same Hilbert polynomial as $\mathrm{Gr}_{G,\leq\mu,1}$. Since the weak normalization injects on structure sheaves, the quotient module has trivial Hilbert polynomial, so it necessarily vanishes, and this yields the claim. □

Remark 4.18. Our methods are not enough to show Cohen–Macaulayness for wild odd unitary G. Indeed, the argument in [FHLR22] relies on the existence of a Frobenius φ on the entire local model. We should mention that Yang [Yan24] proved Cohen–Macaulayness for wild odd unitary groups when $p = 2$ at special parahoric level working explicitly with lattice chains. In this case, the special fiber is irreducible, so we can also deduce it from our results by deforming Cohen–Macaulayness. It is plausible that recent developments in [BMP+24] might lead to an abstract uniform proof of Cohen–Macaulayness for arbitrary parahorics.

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