MOD p SHEAVES ON WITT FLAGS

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Abstract. We characterize Cohen–Macaulay and φ-rational perfect schemes in terms of their perverse étale $\mathbb{F}_p$-sheaves. Using inversion of adjunction, we prove that sufficiently small Schubert varieties in the Witt affine flag variety are perfections of globally $+$-regular varieties, and hence they are φ-rational. Our methods apply uniformly to all affine Schubert varieties in equicharacteristic, as well as classical Schubert varieties, thereby answering a question of Bhatt. As a corollary, we deduce that scheme-theoretic local models always have φ-split special fiber.

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1. Introduction

Hecke categories, i.e., categories of sheaves on local Hecke stacks $\mathrm{Hk}_{\mathcal{G}}$, play a major role in geometric representation theory and in geometric approaches to the Langlands program. Here $\mathcal{G}$ is a parahoric model of a connected reductive group G over a local field F with residue field k of characteristic p. If one considers $\mathbb{F}_\ell$ or $\mathbb{Q}_\ell$-étale sheaves for some prime $\ell \neq p$, then these categories are well-studied for F of equal or mixed characteristic. For example, at hyperspecial level one has the geometric Satake equivalence, e.g. [MV07, Zhu17], and at other parahoric levels there is at least a collection of central sheaves, e.g. [Gai01, ALWY23]. Recently, it has even become possible to talk about $\mathbb{Q}$- and $\mathbb{Z}$-linear motivic sheaves on Hecke stacks, e.g. [RS21, CvdHS22, CvdHS24, vdH24].

The situation changes drastically when one considers étale sheaves for $\ell = p$. There is still a perverse t-structure due to Gabber [Gab04], but its behavior can be quite strange, as half of the six functors do not preserve constructibility. When F has characteristic p, the first author constructed a geometric Satake equivalence in [Cas22], where the dual object is a monoid instead of a group, and also a central functor in [Cas21]. The purpose of this paper, as the title suggests, is to launch an investigation of these properties when F has characteristic 0. In this case, the local Hecke stack $\mathrm{Hk}_{\mathcal{G}}$ and the corresponding affine flag variety $\mathrm{Fl}_{\mathcal{G}}$ exist only canonically as functors on perfect k-algebras.

Let us note that in [CX25] the mod p Hecke category for F of characteristic 0 was already studied, but with no concern for the perverse t-structure. Thus, our first order of business is to investigate IC sheaves in the Hecke category. Toward this direction, we prove the following general result.

Theorem 1.1. *Let k be a perfect field of characteristic p and let X be a connected perfectly finitely presented k-scheme. Then X is Cohen–Macaulay (resp., φ-rational) if and only if the shifted constant sheaf $\mathbb{F}_p[\dim X]$ is perverse (resp., perverse and simple).*

In [Cas22] it was shown that the above commutative-algebraic properties of a finite-type k-scheme imply the corresponding properties of perverse $\mathbb{F}_p$-sheaves, but the fact that the converse holds after passing to the perfection lies much deeper. As it turns out, Theorem 1.1 was known to experts in the φ-singularities community and appeared in [BBL+23], after we already found an argument independently. We have included our argument for the benefit of readers unfamiliar with the literature on φ-singularities, and because it differs significantly from the one in [BBL+23] in that we perform most of the key arguments on the coherent as opposed to topological side. Our notions of Cohen–Macaulayness and φ-rationality for a perfect scheme X, which are in fact properties of the local rings of X, require the following notions from commutative algebra.

Recall that a noetherian local ring $(R, \mathfrak{m})$ is Cohen–Macaulay if the local cohomology groups $H^i_{\mathfrak{m}}(R)$ vanish for $i < \dim R$. If R has characteristic p, the absolute Frobenius φ gives $H^i_{\mathfrak{m}}(R)$ a module structure over the non-commutative polynomial ring $R[\varphi]$. When R is also excellent and φ-finite, one says that R is φ-rational if it is Cohen–Macaulay and $H^{\dim R}_{\mathfrak{m}}(R)$ is a simple $R[\varphi]$-module. In the perfect case, we define Cohen–Macaulayness and φ-rationality in exactly the same way. Since the local cohomology of the perfection R^{perf} is the perfection of $H^i_{\mathfrak{m}}(R)$ with respect to φ, these notions capture phenomena from the noetherian case up to elements annihilated by some iterate of φ. We prove in Lemma 2.21 that R^{perf} is φ-rational if and only if R is φ-nilpotent, where the latter property is a topic of active research in commutative algebra.

The proof of the converse direction for both properties in Theorem 1.1 involves a noetherian induction and passage to the strict henselization. To prove Cohen–Macaulayness we apply an argument already present in [Bha20], which allows us to prove that the lower local cohomomlogy groups are supported on a geometric point and are holonomic in the sense of Bhatt–Lurie's Riemann–Hilbert correspondence [BL19]. In particular, their vanishing can be checked after passing to φ-invariants, where it is guaranteed by the perversity of $\mathbb{F}_p[\dim X]$. To prove φ-rationality we also need to invoke a strong finiteness result for simple φ-submodules of the top local cohomology due to Lyubeznik [Lyu97]. Both arguments utilize Matlis duality in an essential way.

We now return to the affine flag variety $\text{Fl}_{\mathcal{G}}$. The Bruhat decomposition yields Schubert subvarieties $\text{Fl}_{\mathcal{G}, \leq w}$ indexed by double cosets w of the Iwahori–Weyl group. At this point, it is natural to formulate the following expectation.

Conjecture 1.2. *The perfect Schubert schemes $\text{Fl}_{\mathcal{G}, \leq w}$ are φ-rational.*

Let us present some evidence for this conjecture. If F has characteristic p, this result follows from [Cas22] for split G and [FHLR22] for general G. If one had an analogue of the Grauert–Riemenschneider theorem in perfect geometry (which is known for finite-type φ-split varieties), then Cohen–Macaulayness would be a consequence of the triviality of the higher direct images of the structure sheaf along Demazure resolutions, which was proved in [CX25]. We do not know of such a result, but for a certain class of sufficiently small

w, we are able to show using inversion of adjunction that there is a certain deperfection $\mathrm{Fl}_{\mathcal{G},\leq w,1}$ which is globally $+$-regular in the sense of [BMP+23]. This property, which we explain next, is stronger than φ-rationality.

The property of φ-rationality is of a local nature, and in particular, it does not descend along proper covers. In order to get proper descent, one has to define a global variant of φ-rationality, but it is unclear how to proceed in the perfect setting. For classical schemes, this is well understood via the property of strong φ-regularity of Hochster–Huneke [HH89] and its global variant [Smi00]. In this paper, we prefer to use the closely related property of global $+$-regularity, as it carries the advantage of making every $\mathbb{Q}$-divisor integral up to passing to a cyclic cover. However, new ideas are still required because the proof strategy in [Cas22, FHLR22] relies on the criterion of Mehta–Ramanathan [MR85]. This pressuposes the existence of certain theta divisors that Faltings [Fal03] constructs in equicharacteristic on the natural deperfection of the whole $\mathrm{Fl}_{\mathcal{G}}$, but such a deperfection does not exist in mixed characteristic. Let us first state our result, and then we will explain the various notations and hypotheses.

Theorem 1.3. *Assume $s_\bullet$ is a reduced word for w, and $q_\bullet = 1$ is $s_\bullet$-permissible. Then $(\mathrm{Fl}_{\mathcal{G},\leq w,1}, \Delta)$ is globally $+$-regular for any $\mathbb{Q}$-divisor $\Delta \leq \partial_{w,1}$ with $\lfloor \Delta \rfloor = 0$.*

When F has characteristic p each $\mathrm{Fl}_{\mathcal{G},\leq w}$ is canonically isomorphic to the perfection of a projective k-scheme $\mathrm{Fl}_{\mathcal{G},\leq w,1}$ (the seminormalization of the affine Schubert variety in [PR08], see also [HLR24, FHLR22] for the necessity of this functor). In this case, the hypothesis that $q_\bullet = 1$ is $s_\bullet$-permissible is automatically satisfied for every reduced word for w. The divisor $\partial_{w,1}$ is the sum of the Iwahori–Schubert subvarieties in codimension one for a fixed choice of Iwahori $\mathcal{I}$ mapping to $\mathcal{G}$. Moreover, whenever the hypotheses of Theorem 1.3 are satisfied (even when F has characteristic 0), we deduce in Proposition 4.11 that $\mathrm{Fl}_{\mathcal{G},\leq w,1}$ is globally φ-regular, and compatibility φ-split with all of its Schubert subvarieties. Thus, when F has characteristic p we obtain a new proof of the global φ-regularity of affine Schubert varieties, first shown in [Cas22, FHLR22], but which avoids the the Mehta–Ramanathan criterion. Moreover, applying this criterion to wildly ramified groups in [FHLR22] required extra casework, whereas our new proof is uniform across all groups.

Remark 1.4. Bhatt [Bha12] proved that Schubert varieties in the classical finite flag variety of GL_n in positive characteristic are derived splinters (an alternative name for globally $+$-regular), using inversion of adjunction. Bhatt asked in [Bha12, Remarks 7.8 and 7.10] if his methods could be generalized to general groups, and our proof of Theorem 1.3 answers this question positively. Indeed, all classical Schubert varieties arise as particular affine Schubert varieties for F of characteristic p. In fact, our proof also applies in the context of Kac–Moody groups, again without any conditions on w. We note here that global φ-regularity of classical Schubert varieties was known much earlier by [LRPT06], which used the Mehta–Ramanathan criterion

Let us now explain the permissibility hypothesis in Theorem 1.3, which cannot be avoided when F has characteristic 0. The first step to prove global $+$-regularity is to replace $\mathrm{Fl}_{\mathcal{G},\leq w}$ by its proper modification $\mathrm{Fl}_{\mathcal{I},\leq s_\bullet}$ for some reduced word $s_\bullet$ for w. Here $\mathcal{I} \subset \mathcal{G}$ is an Iwahori subgroup, and $\mathrm{Fl}_{\mathcal{I},\leq s_\bullet}$ is the Demazure resolution. We then construct a class of deperfections $\mathrm{Fl}_{\mathcal{I},\leq s_\bullet,q_\bullet}$, where $q_\bullet$ is a certain $s_\bullet$-permissible sequence of powers of p. This notion of permissibility is defined by induction on the length of the sequence. The factor $\mathrm{Fl}_{\mathcal{I},\leq s_i,q_i}$ in the twisted product $\mathrm{Fl}_{\mathcal{I},\leq s_\bullet,q_\bullet}$ is the φ_{q_i}-twist of the canonical Iwahori-equivariant smooth deperfection $\mathrm{Fl}_{\mathcal{I},\leq s_i,1} \simeq \mathbb{P}^1_k$ of $\mathrm{Fl}_{\mathcal{I},\leq s_i}$. In order for the twisted product to exist when F has characteristic 0, we are forced to twist every new factor to the right by a nonnegative power of p, which we have no control over. Thus, $q_\bullet$ is is a non-decreasing (and not so rarely increasing) sequence, which ultimately hinders proving global $+$-regularity.

If the constant sequence $q_\bullet = 1$ is $s_\bullet$-permissible, then we can carry out inversion of adjunction. Indeed, a calculation for the boundary pair $(\mathrm{Fl}_{\mathcal{I},\leq s_\bullet,q_\bullet}, \partial_{s_\bullet,q_\bullet})$ reveals that its anti-canonical divisor is semi-ample (and in this case also big) precisely when $q_\bullet$ is non-increasing, so constant sequences are the optimal scenario. This assumption holds for all w if F has characteristic p and for all w in the μ-admissible set of Kottwitz–Rapoport [KR00] associated with some minuscule conjugacy class of geometric coweights μ. The idea for applying the inversion of adjunction criterion for global $+$-regularity of pairs of [BMP+23] is then to slightly perturb the coefficients of the boundary $\partial_{s_\bullet,1}$ in such a way that the anti-canonical divisor of the pair becomes ample.

Finally, we give an application to local models. Recall that [AGLR22, GL24] prove the existence and uniqueness of normal flat O_E-schemes $M_{\mathcal{G},\mu}$ with reduced special fiber representing a certain closed v-subsheaf of the Beilinson–Drinfeld Grassmannian $\mathrm{Gr}_{\mathcal{G}}$, provided either μ is minuscule or F has characteristic p. In [FHLR22] it was proved for all groups except wild odd unitary ones that the special fiber is moreover φ-split. Now, we can generalize this to all groups and prove it uniformly. This finishes the problem of determining the special fiber of $M_{\mathcal{G},\mu}$ in full generality and thus Cohen–Macaulayness is the only property remaining in the above mentioned series of papers for which the infamous hypothesis "$p > 2$ or Φ_G is reduced" is still needed.

Corollary 1.5. *Assume F has characteristic p or μ is minuscule. Then, the special fiber of $M_{\mathcal{G},\mu}$ equals the canonical deperfection $A_{\mathcal{G},\mu,1}$ in the sense of* [AGLR22] *of the μ-admissible locus. Moreover, $A_{\mathcal{G},\mu,1}$ is φ-split compatibly with every $\mathcal{G}(O)$-stable closed subscheme.*

The idea goes as follows: once we know that Schubert varieties in the μ-admissible locus have globally $+$-regular deperfections $\mathrm{Fl}_{\mathcal{G},\leq w,1}$, we can construct a φ-split canonical deperfection $A_{\mathcal{G},\mu,1}$ of the admissible locus $A_{\mathcal{G},\mu}$. Then, it suffices to prove the coherence conjecture of [PR08], i.e., we need to compute global sections of certain line bundles for the previous deperfection and for the generic fiber of $M_{\mathcal{G},\mu}$. The φ-splitness yields higher vanishing of cohomology for ample line bundles, so we get an inclusion-exclusion type formula in terms of Schubert subvarieties for the global sections dimension. But the latter can be computed by the Demazure character formula, so it does not change if we replace G by another group with the same combinatorics. Therefore, we can reduce to tame G and equicharacteristic F, already handled by Zhu [Zhu14]. We also note that the corollary above was used in [Lou23] when F has equicharacteristic and $\pi_1(G)$ is p-torsion free to finish the proof for all G of normality of Schubert varieties embedded in the usual scheme-theoretic affine flag varieties (i.e., before taking seminormalizations).

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2. Perverse $\mathbb{F}_p$-sheaves and φ-singularities

Fix a prime number p. For a scheme X over $\mathbb{F}_p$, let φ be the absolute Frobenius morphism. We will often be concerned with noetherian schemes which are φ-finite, meaning that $\varphi_*\mathcal{O}_X$ is a finite $\mathcal{O}_X$-module. By Kunz's theorem [Kun76, Theorem 2.5], a φ-finite noetherian ring is excellent. Additionally, a noetherian φ-finite scheme admits a coherent dualizing complex [Gab04, Remark 13.6]. The proof in loc. cit. only applies when X is affine, which is the only case we will use. Recall also that the perfection of a scheme X is $X^{\text{perf}} = \lim(\cdots \xrightarrow{\varphi} X \xrightarrow{\varphi} X)$. A deperfection of a perfect scheme X is a scheme X_0 equipped with an isomorphism $X_0^{\text{perf}} \cong X$.

2.1. **Cartier modules.** Let R be a ring over $\mathbb{F}_p$ and let φ_*R be the R-module associated to $\varphi_*\mathcal{O}_{\text{Spec}(R)}$. Recall from [BB11] that a Cartier module over R consists of an R-module M with a map $\varphi_*M \to M$. Homomorphisms between Cartier modules must respect this map. A Cartier module M is said to be nilpotent if $\varphi_*^e M \to M$ is zero for some $e \geq 0$. Furthermore, a Cartier module is said to be coherent if its underlying R-module is finite. We have the following decisive structure theorem for Cartier modules.

Theorem 2.1 (Blickle–Böckle). *Let R be a noetherian φ-finite ring and let M be a coherent Cartier module.*

(1) *There exists a finite composition series $0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$ by coherent Cartier submodules such that each M_i/M_{i+1} is either nilpotent, or non-nilpotent and simple.*
(2) *If M is a simple coherent Cartier module then M has a unique associated prime $\mathfrak{p} \in \text{Spec}(R)$. Furthermore, $M \subset M_{\mathfrak{p}}$, and the latter is a finite-dimensional vector space over $R/\mathfrak{p}$.*

Proof. Part (1) is [BB11, Proposition 4.23], and part (2) is proved in [BB11, Propositions 4.14, 4.15]. ☐

Important examples of coherent Cartier modules include the cohomology sheaves $\mathcal{H}^i(\omega_R^\bullet)$ of dualizing complexes on φ-finite noetherian rings. Here the map $\varphi_*\mathcal{H}^i(\omega_R^\bullet) \to \mathcal{H}^i(\omega_R^\bullet)$ is obtained from exactness of φ_* and the adjoint of the canonical isomorphism $\omega_R^\bullet \to \varphi^!\omega_R^\bullet$ from Grothendieck duality.

2.2. φ**-modules.** Let R be an $\mathbb{F}_p$-algebra, and let $R[\varphi]$ be the non-commutative polynomial ring over R in one variable, also denoted φ, subject to the relation $\varphi a = a^p\varphi$ for all $a \in R$. A left $R[\varphi]$-module is the same as an R-module M with an R-linear map $M \to \varphi_*M$; note that the map goes in the direction opposite to that of Cartier modules.

We recall a decisive structure result for $R[\varphi]$-modules closely related to Theorem 2.1. As in the case of Cartier modules, we say that an $R[\varphi]$-module M is nilpotent if $M \to \varphi^e_* M$ is zero for some $e \geq 0$. Similarly, an $R[\varphi]$-module M is co-finite if it is Artinian as an R-module. Important examples of co-finite $R[\varphi]$ modules include the local cohomology groups $H^i_{\mathfrak{m}}(R)$ of noetherian local $\mathbb{F}_p$-algebras $(R, \mathfrak{m})$ [BS13b, Theorem 7.1.3].

Theorem 2.2 (Lyubeznik). *Let $(R, \mathfrak{m})$ be a noetherian local $\mathbb{F}_p$-algebra and let M be a co-finite $R[\varphi]$-module.*

1. *M admits a finite composition series $0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$ by co-finite $R[\varphi]$-submodules such that each M_i/M_{i+1} is either nilpotent, or non-nilpotent and simple.*
2. *The collection of non-nilpotent simple subquotients of M is independent of the composition series.*

Proof. See [Lyu97, Theorem 4.7]. □

It is worth mentioning the following special case of Lyubeznik's theorem, which has been proved via different means by Hartshorne–Speiser, Lyubeznik, Gabber, and Bhatt–Blickle–Lyubeznik–Singh–Zhang.

Corollary 2.3. *Let $(R, \mathfrak{m})$ be a noetherian local $\mathbb{F}_p$-algebra and let M be a co-finite $R[\varphi]$-module. Then some power of φ annihilates*

$$\{a \in H^i_{\mathfrak{m}}(R) \,:\, \varphi^e(a) = 0 \text{ for some } e > 0\}.$$

Proof. This follows from Theorem 2.2; see also [HS77, Proposition 1.11], [Lyu97, Proposition 4.4], [Gab04, Lemma 13.1] or [BBL⁺23, Corollary 4.24]. □

We conclude by explaining a precise relation between Cartier modules and $R[\varphi]$-modules. Suppose that $(R, \mathfrak{m})$ is a complete, local, noetherian and φ-finite $\mathbb{F}_p$-algebra. Following [Sta23, Tag 0A82], we normalize the dualizing complex $\omega^\bullet_R$ so that $R\Gamma_{\mathfrak{m}}(\omega^\bullet_R) = E[0]$ lies in degree 0, in which case E is an injective hull of $R/\mathfrak{m}$. Recall that Matlis duality $M \mapsto \mathrm{Hom}_R(M, E)$ gives an anti-equivalence between coherent and Artinian R-modules. Then Matlis duality also induces an anti-equivalence between coherent Cartier modules and co-finite $R[\varphi]$-modules [BB11, Proposition 5.2]. The integer $d := \dim R$ is the largest integer such that $\mathcal{H}^{-d}(\omega^\bullet_R) \neq 0$ [Sta23, 0AWN]; the cohomology sheaf $\omega_R := \mathcal{H}^{-d}(\omega^\bullet_R)$ is called the dualizing sheaf. For each i there is a canonical isomorphism $\mathrm{Hom}_R(\mathcal{H}^{-i}(\omega^\bullet_R), E) \cong H^i_{\mathfrak{m}}(R)$ even if R is not complete, e.g. see [BS13a, 10.2.19], [Sta23, Tag 0AAK].

2.3. **Perverse $\mathbb{F}_p$-sheaves.** For a scheme X over $\mathbb{F}_p$ let $D(X, \mathbb{F}_p)$ be the derived category of étale $\mathbb{F}_p$-sheaves on X, and let $D^b_c(X, \mathbb{F}_p)$ be the bounded constructible subcategory. In this subsection we fix a perfect field k of characteristic p. Every scheme of finite type over k is automatically φ-finite.

Definition 2.4. Let X be a k-scheme of finite type. For each point $x \in X$, fix a strict henselization $\mathcal{O}^{\mathrm{sh}}_x$ of the local ring at x, and let $i_x \colon \overline{x} \to \mathrm{Spec}(\mathcal{O}^{\mathrm{sh}}_x)$ be the inclusion of the closed point. We define the full subcategory ${}^pD^{\leq 0}(X, \mathbb{F}_p)$ (resp. ${}^pD^{\geq 0}(X, \mathbb{F}_p)$) of $D(X, \mathbb{F}_p)$ consisting of $\mathcal{F}^\bullet \in D(X, \mathbb{F}_p)$ such that $\mathcal{H}^n(i_x^* \mathcal{F}^\bullet) = 0$ for all $x \in X$ and

$n > -\dim \overline{\{x\}}$ (resp. $\mathcal{F}^\bullet$ has bounded below cohomology sheaves and $\mathcal{H}^n(Ri_x^!\mathcal{F}^\bullet) = 0$ for all $x \in X$ and $n < -\dim \overline{\{x\}}$).

The following special case of a theorem of Gabber implies that the subcategories above give a t-strucutre on $D(X, \mathbb{F}_p)$, cf. [EK04b, Theorem 11.5.4]. We call objects in the heart perverse $\mathbb{F}_p$-sheaves.

Theorem 2.5 (Gabber)**.** *Let X be a k-scheme of finite type.*

(1) *The pair $({}^pD^{\leq 0}(X, \mathbb{F}_p), {}^pD^{\geq 0}(X, \mathbb{F}_p))$ gives rise to a t-structure on $D(X, \mathbb{F}_p)$.*
(2) *The t-structure above restricts to a t-structure on $D^b_c(X, \mathbb{F}_p)$.*
(3) *Every perverse subquotient of a constructible perverse $\mathbb{F}_p$-sheaf is constructible, i.e. lies in $D^b_c(X, \mathbb{F}_p)$.*
(4) *Every constructible perverse $\mathbb{F}_p$-sheaf has finite length.*

Proof. See [Gab04, Theorem 10.4, Corollary 12.4]. □

The t-structure on $D^b_c(X, \mathbb{F}_p)$ has also been studied in [Cas22, BBL+23]. By the topological invariance of the small étale site [Sta23, Tag 04DY], we have a canonical equivalence $D^b_c(X, \mathbb{F}_p) \cong D^b_c(X^{\mathrm{perf}}, \mathbb{F}_p)$, so we also get a t-structure for perfections of k-schemes of finite type.

We now recall the notion of intermediate extension for perverse $\mathbb{F}_p$-sheaves. Let $\mathcal{F}^\bullet$ be a constructible perverse $\mathbb{F}_p$-sheaf on U, and let $j\colon U \to X$ be an open immersion into a k-scheme of finite type. By taking perverse truncations of Rj_* and $Rj_!$, we may define

$$j_{!*}\mathcal{F}^\bullet := \mathrm{Im}\,({}^pj_!\mathcal{F}^\bullet \to {}^pj_*\mathcal{F}^\bullet).$$

Note that while ${}^pj_*\mathcal{F}^\bullet$ may not be constructible, both ${}^pj_!\mathcal{F}^\bullet$ and $j_{!*}\mathcal{F}^\bullet$ are constructible. The intermediate extension $j_{!*}\mathcal{F}^\bullet$ is characterized as the unique perverse extension of $\mathcal{F}^\bullet$ with no quotients or subobjects supported on $X \setminus U$. If $i\colon X \setminus U \to X$ is the inclusion (with the reduced scheme structure), the latter conditions are equivalent to $i^*\mathcal{F}^\bullet \in {}^pD^{\leq -1}(X \setminus U, \mathbb{F}_p)$ and $Ri^!(\mathcal{F}^\bullet) \in {}^pD^{\geq 1}(X \setminus U, \mathbb{F}_p)$, respectively [Cas22, lemma 2.7].

2.4. **The Riemann–Hilbert correspondence.** We now recall the Riemann–Hilbert correspondence of Bhatt–Lurie [BL19]. Let R be an $\mathbb{F}_p$-algebra and let (R, φ) be the ring R regarded as an $R[\varphi]$-module via the Frobenius. For an R-algebra S, extension of scalars provides a functor from $R[\varphi]$-modules to $S[\varphi]$-modules, which is used implicitly in the following definition taken from [BL19, Construction 2.3.1].

Definition 2.6. Let $D(R[\varphi])$ be the derived category of $R[\varphi]$-modules. Define the functor

$$\mathrm{Sol}(-) := \underline{\mathrm{RHom}}_{D(R[\varphi])}((R, \varphi), -)\colon D(R[\varphi]) \to D(\mathrm{Spec}(R), \mathbb{F}_p).$$

Informally, Sol can be thought of as the derived functor of φ-invariants. The functor Sol is not an equivalence of categories because $D(R[\varphi])$ is too large. To solve this issue in the constructible case, Bhatt–Lurie define a notion of holonomicity [BL19, Definition 4.1.1.]. A holonomic $R[\varphi]$-module is an $R[\varphi]$-module isomorphic to one of the form

$$M^{\mathrm{perf}} := \mathrm{colim}(M \to \varphi_* M \to \varphi_*^2 M \to \cdots)$$

for an $R[\varphi]$-module M which is finitely presented as an R-module. Note that a holonomic $R[\varphi]$-module is in particular perfect, meaning that $M \to \varphi_* M$ is an isomorphism. Restriction of scalars along $R \to R^{\mathrm{perf}}$ identifies the categories of perfect $R[\varphi]$-modules and perfect $R^{\mathrm{perf}}[\varphi]$-modules [BL19, Proposition 3.4.3], which by the following theorem is closely related to the topological invariance of the small étale site.

Theorem 2.7 (Bhatt–Lurie)**.** *Let $D_{\mathrm{hol}}(R[\varphi]) \subset D(R[\varphi])$ be the full subcategory of complexes with holonomic cohomology sheaves. Then Sol restricts to an equivalence of categories $D_{\mathrm{hol}}(R[\varphi]) \cong D_c^b(\mathrm{Spec}(R), \mathbb{F}_p)$ which is t-exact for the standard t-structures on the source and target*

Proof. See [BL19, Theorem 7.4.1, Corollary 12.1.7]. □

By t-exactness, if M is a holonomic $R[\varphi]$-module then $\mathrm{Sol}(M)$ is the étale sheaf on $\mathrm{Spec}(R)$ whose value on an étale R-algebra S is

$$\mathrm{Sol}(M)(S) = \{x \in M \otimes_R S \,:\, \varphi(x) = x\}.$$

Remark 2.8. In [BBL+23] the authors use a different definition of the perverse t-structure on $D_c^b(\mathrm{Spec}(R), \mathbb{F}_p)$, in terms of the Riemann–Hilbert correspondence and a perverse t-structure on coherent sheaves, but the two agree by [BBL+23, Theorem 4.43]. Correspondingly, our proofs of Theorem 2.17 and Theorem 2.25 below are quite different from their analogues [BBL+23, Remark 4.39, Corollary 5.15]. Our Theorem 2.25 also differs from [BBL+23, Corollary 5.15] in that we allow a non-complete base and hence have to eliminate the possibility of branching behavior (with the help of [DMP23]).

2.5. **Cohen–Macaulayness.** There are numerous equivalent definitions of Cohen–Macaulayness for a noetherian local ring, for example involving regular sequences, local cohomology, or a dualizing complex. While there is no standard definition of Cohen–Macaulayness in the non-noetherian setting, the one in [Bha20, Definition 2.1] will be useful here (see also [Bha20, Remark 2.4]).

Definition 2.9. Let X be a topologically noetherian scheme. We say that X is Cohen–Macaulay if for every local ring $(R, \mathfrak{m})$ on X, the (Zariski) local cohomology groups $H^i_{\mathfrak{m}}(R) := R^i\Gamma_{\{\mathfrak{m}\}}(\mathcal{O}_{\mathrm{Spec}(R)})$ vanish for $i < \dim R$.

If R is a noetherian local $\mathbb{F}_p$-algebra then $H^i_{\mathfrak{m}}(R)$ has a canonical $R[\varphi]$-module structure as the cohomology of a Koszul complex of $R[\varphi]$-modules [Sta23, Tag 0956]. In this case, $H^i_{\mathfrak{m}}(R)$ is finitely generated as an $R[\varphi]$-module, and since it is Artinian as an R-module it is even a module over the completion $\hat{R}$. Furthermore, if M is an R-module (R is still noetherian) we have $H^i_{\mathfrak{m}}(M) = \mathrm{colim}_n \mathrm{Ext}^i_R(R/\mathfrak{m}^n, M)$ [Sta23, Tag 0955].

Lemma 2.10. *Let $(R, \mathfrak{m})$ be the perfection of a noetherian local $\mathbb{F}_p$-algebra $(R_0, \mathfrak{m}_0)$ of dimension d with normalized dualizing complex $\omega^\bullet_{R_0}$.*

(1) *$H^i_{\mathfrak{m}}(R) = 0$ if and only if $H^i_{\mathfrak{m}_0}(R_0)$ is nilpotent.*
(2) *$H^d_{\mathfrak{m}}(R) \neq 0$.*
(3) *If $H^i_{\mathfrak{m}}(R) = 0$ for all $i < d$ then R is equidimensional.*
(4) *If R_0 is φ-finite then $H^i_{\mathfrak{m}_0}(R_0)$ is nilpotent if and only if $\mathcal{H}^{-i}(\omega^\bullet_{R_0})$ is nilpotent.*
(5) *If R_0 is φ-finite and $H^i_{\mathfrak{m}}(R) = 0$ for all $i < d$, then $\mathrm{Spec}(R)$ is Cohen–Macaulay in the sense of Definition 2.9.*

Remark 2.11. A noetherian local $\mathbb{F}_p$-algebra $(R_0, \mathfrak{m}_0)$ such that $H^i_{\mathfrak{m}_0}(R_0)$ is nilpotent for $i < \dim R_0$ is called weakly φ-nilpotent [Mad19], see also [PQ19, Quy19].

Proof. Since R is an R_0-module and $\mathrm{Spec}(R) \cong \mathrm{Spec}(R_0)$ then $H^i_{\mathfrak{m}}(R) = H^i_{\mathfrak{m}_0}(R)$. Next, $H^i_{\mathfrak{m}_0}(R) = \mathrm{colim}_n \mathrm{Ext}^i_{R_0}(R_0/\mathfrak{m}_0^n, \mathrm{colim}_e \varphi^e_* R_0)$. By taking a resolution of $R_0/\mathfrak{m}_0^n$ by finite free R_0-modules and using that filtered colimits are exact [Sta23, Tag 00DB], the inner colimit over e commutes with $\mathrm{Ext}^i_{R_0}(R_0/\mathfrak{m}_0^n, -)$. Then by exchanging the colimits and using exactness of φ_* to commute the latter with $H^i_{\mathfrak{m}_0}(-)$, we get

$$H^i_{\mathfrak{m}}(R) = \operatorname*{colim}_e \varphi^e_* H^i_{\mathfrak{m}_0}(R_0) = H^i_{\mathfrak{m}_0}(R_0)^{\mathrm{perf}}. \tag{2.1}$$

Thus, $H^i_{\mathfrak{m}}(R) = 0$ if and only if every element of $H^i_{\mathfrak{m}_0}(R_0)$ is annihilated by some power of φ. Now (1) follows from Corollary 2.3.

For (2), we consider two cases. If $d = 0$ then $H^0_{\mathfrak{m}}(R) = R$ is nonzero. On the other hand, if $d > 0$, then by (1) we need only show that $H^d_{\mathfrak{m}_0}(R_0)$ is not nilpotent. But if $H^d_{\mathfrak{m}_0}(R_0)$ were nilpotent, then since it is finitely generated as an $R_0[\varphi]$-module, it would also be finitely generated as an R_0-module, which is impossible [BS13b, Corollary 7.3.3].

Part (3) follows from the proof of [PQ19, Proposition 2.8(3)]; we reproduce the argument here for completeness. If R is not equidimensional, let $\mathfrak{p} \subset R_0$ be a minimal prime such that $n := \dim R_0/\mathfrak{p} < d = \dim R_0$, and let I be the intersection of the other minimal primes. Then we have an exact sequence of $R_0[\varphi]$-modules

$$0 \to R_0 \to R_0/\mathfrak{p} \oplus R_0/I \to R_0/(\mathfrak{p} + I) \to 0$$

where $\dim R_0/(\mathfrak{p} + I) < n$. Applying $R\Gamma_{\{\mathfrak{m}_0\}}$ gives a surjection $H^n_{\mathfrak{m}_0}(R_0) \to H^n_{\mathfrak{m}_0}(R_0/\mathfrak{p})$ by [Sta23, Tag 0DXC]. This implies that $H^n_{\mathfrak{m}_0}(R_0/\mathfrak{p})$ is nilpotent, which contradicts part (2).

Next, we note that if R_0 is complete then (4) follows immediately from Matlis duality. Remarkably this statement is true even if R_0 is not complete, as was observed in [ST17, Lemma 2.3]. The argument is similar to [Sch09, Proposition 4.3], using that the double Matlis duality functor is isomorphic to $(-) \otimes_{R_0} \hat{R}_0$ on finite R_0-modules, and faithful flatness of $R_0 \to \hat{R}_0$; we refer to loc. cit for more details.

For each prime $\mathfrak{p} \subset R_0$, the localization $(\omega^\bullet_{R_0})_{\mathfrak{p}}$ is a dualizing complex for $(R_0)_{\mathfrak{p}}$, and since R_0 is equidimensional, $(\omega_{R_0})_{\mathfrak{p}}$ is a dualizing sheaf for $(R_0)_{\mathfrak{p}}$ [Smi93, Proposition 2.3.2]. Thus, (5) follows from (4). □

Since $H^i_{\mathfrak{m}}(R) = H^i_{\mathfrak{m}_0}(R)$, then $H^i_{\mathfrak{m}}(R)$ has a canonical $R[\varphi]$-module structure as the cohomology of a Koszul complex, constructed from finitely many generators of $\mathfrak{m}$ up to radical, independent of the chosen deperfection. The canonicity of the $R[\varphi]$-module structure can also be deduced by noting that the action of φ comes from applying $H^i_{\mathfrak{m}}(-)$ to the Frobenius map $R \to \varphi_* R$. Furthermore, $H^i_{\mathfrak{m}}(R)$ is a perfect $R[\varphi]$-module in the sense of [BL19, Definition 3.2.1].

We now introduce some notation for working with perfections. If R is an $\mathbb{F}_p$-algebra and $r \in R$, we denote by $r^{1/p^e} \in R^{\mathrm{perf}}$ the p^eth root. If M is an $R[\varphi]$-module and $m \in M$, we denote by $\varphi^{-e}(m) \in M^{\mathrm{perf}}$ the image of m under the map $\varphi^e_* M \to M^{\mathrm{perf}}$. We use this notation in the proof of the following structure result for perfect φ-modules.

Proposition 2.12. *Let $(R, \mathfrak{m})$ be the perfection of a noetherian local $\mathbb{F}_p$-algebra $(R_0, \mathfrak{m}_0)$.*

(1) *The functor* $M_0 \mapsto M_0^{\mathrm{perf}}$ *from* $R_0[\varphi]$*-modules to* $R[\varphi]$*-modules is exact.*
(2) *If* M_0 *is a co-finite* $R_0[\varphi]$*-module then* M_0^{perf} *has finite length as an* $R[\varphi]$*-module.*
(3) *If* M_0 *is a co-finite, non-nilpotent and simple* $R_0[\varphi]$*-module then* M_0^{perf} *is a simple, nonzero* $R[\varphi]$*-module.*

Proof. Part (1) follows from the exactness of φ_*. For (2), we take the perfection of a composition series for M_0 as in Theorem 2.2. This kills the nilpotent subqoutients, so (2) will then follow from (3). For (3), let $M := M_0^{\mathrm{perf}}$, and let $m \in M$ be nonzero. We need to show that $R[\varphi] \cdot m = M$. Write $m = \varphi^{-e}(m')$ for some $m' \in M_0$ and $e \geq 0$. It suffices to show that for all $n \in M_0$ and $f \geq 0$, we have $\varphi^{-e-f}(n) \in R[\varphi] \cdot m$. By simplicity $R_0[\varphi] \cdot \varphi^f(m') = M_0$, so there exist $r_i \in R_0$ such that $\sum_i r_i \cdot \varphi^{i+f}(m') = n$. Then we conclude since $\sum_i r_i^{1/p^{e+f}} \cdot \varphi^i(m) = \varphi^{-e-f}(n)$. □

The following result is the key input in the proof of Theorem 2.17 below.

Proposition 2.13. *Let* $(R, \mathfrak{m})$ *be the perfection of a* φ*-finite noetherian local* $\mathbb{F}_p$*-algebra* $(R_0, \mathfrak{m}_0)$*. Suppose that* R *is equidimensional, the punctured spectrum of* $(R, \mathfrak{m})$ *is Cohen–Macaulay, and* $R/\mathfrak{m}$ *is algebraically closed. Then* $\mathrm{Spec}(R)$ *is Cohen–Macaulay if and only if for all* $i < \dim R$*, there does not exist a nonzero element* $x \in H^i_{\mathfrak{m}}(R)$ *such that* $\varphi(x) = x$*.*

Proof. The necessity of the condition on φ-fixed elements is clear. For sufficiency, by Lemma 2.10 we may suppose for contradiction that that $\mathcal{H}^{-i}(\omega^{\bullet}_{R_0})$ has a non-nilpotent simple Cartier subquotient M for some $i \neq \dim R$. By Theorem 2.1 and our assumption on the punctured spectrum, the unique associated prime of M must be $\mathfrak{m}_0$, so M has finite length as an R_0-module. Since M was arbitrary, $\mathcal{H}^{-i}(\omega^{\bullet}_{R_0})$ has finite length as an R_0-module up to nilpotents. Let E be an injective hull of $R_0/\mathfrak{m}_0$. Then $\mathrm{Hom}_{R_0}(\mathcal{H}^{-i}(\omega^{\bullet}_{R_0}), E) \cong H^i_{\mathfrak{m}_0}(R_0)$, even as $R_0[\varphi]$-modules [BB11, Lemma 5.1], so $H^i_{\mathfrak{m}_0}(R_0)$ has finite length as an R_0-module up to nilpotents.

Now by Proposition 2.12, $H^i_{\mathfrak{m}}(R)$ has a finite composition series with holonomic subqoutients in the sense of [BL19, Definition 4.1.1]. Since the abelian category of holonomic $R_0[\varphi]$-modules is closed under extensions [BL19, Corollary 4.3.3, Remark 3.2.2], then $H^i_{\mathfrak{m}}(R)$ is a holonomic R_0-module which is moreover set-theoretically supported on $\mathrm{Spec}(R/\mathfrak{m})$ (holonomicity can also be deduced from [Bha20, Lemma 2.16]). By construction the functor Sol is compatible with pullback, so that $\mathrm{Sol}(H^i_{\mathfrak{m}}(R))$ is an étale sheaf supported on $\mathrm{Spec}(R/\mathfrak{m})$. Since $R/\mathfrak{m}$ is algebraically closed, an étale sheaf vanishes if and only if its global sections vanish. Now we conclude using that Sol is an equivalence when restricted to holonomic modules by Theorem 2.7. □

Lemma 2.14. *Let* $(R, \mathfrak{m})$ *be a noetherian local ring which is equidimensional and universally catenary. Let* $R \to S$ *be an étale ring map and let* $\mathfrak{q}$ *be a prime of* S *lying above* $\mathfrak{m}$*. Then the localization* $S_{\mathfrak{q}}$ *is equidimensional and* $\dim S_{\mathfrak{q}} = \dim R$*.*

Proof. Let $\mathfrak{q}_0$ be a minimal prime of S contained in $\mathfrak{q}$. Then $\mathfrak{p}_0 := R \cap \mathfrak{q}_0$ is minimal by flatness, and $R/\mathfrak{p}_0$ is universally catenary by [Sta23, Tag 00NK]. By the dimension formula [Sta23, Tag 02IJ] applied to the ring extension $R/\mathfrak{p}_0 \subset S/\mathfrak{q}_0$, we have $\mathrm{ht}(\mathfrak{m}/\mathfrak{p}_0) = \mathrm{ht}(\mathfrak{q}/\mathfrak{q}_0)$. □

Lemma 2.15. *Let* $(R, \mathfrak{m})$ *be a local* $\mathbb{F}_p$*-algebra. Then if* R *satisfies any of the following three properties, so does the strict henselization* $(R^{\mathrm{sh}}, \mathfrak{m}^{\mathrm{sh}})$*.*

(1) *R is noetherian.*
(2) *R is φ-finite.*
(3) *R is excellent and equidimensional.*

Proof. Property (1) is part of [Gro67, Proposition 18.8.8]. Property (2) is in the proof of [BCRG+19, Theorem 4.1], the point being that R^{sh} is an ind-étale R algebra, so $\varphi_* R \otimes_R R^{\mathrm{sh}} = \varphi_* R^{\mathrm{sh}}$. For property (3), excellence is preserved by the last remark in [FK88, Ch. 1 §1], and it remains to prove equidimensionality.

Let $\mathfrak{q}_0$ be a minimal prime of R^{sh}. Then $\mathfrak{p}_0 := R \cap \mathfrak{q}_0$ is minimal by flatness. As in [Sta23, Tag 06LK], write $R^{\mathrm{sh}} = \mathrm{colim}_i R_i$ as a direct limit of local rings R_i which are localizations of étale R-algebras faithfully flat over R. Then $R_i \to R^{\mathrm{sh}}$ is faithfully flat for all i by [Sta23, Tag 00U7, Tag 05UT]. We have the minimal primes $\mathfrak{p}_i := R_i \cap \mathfrak{q}_0$ in the R_i. By prime avoidance, for all large enough i, $\mathfrak{q}_0$ is the only minimal prime of R^{sh} lying over $\mathfrak{p}_i$. Fix such an i and let $C = (\mathfrak{Q}_0 \subset \cdots \subset \mathfrak{Q}_n)$ be a maximal chain of primes in $R^{\mathrm{sh}}/\mathfrak{p}_i R^{\mathrm{sh}}$. By faithful flatness of $R_i/\mathfrak{p}_i \to R^{\mathrm{sh}}/\mathfrak{p}_i$ and the going down property, we have $n \geq \dim R_i/\mathfrak{p}_i$. But by Lemma 2.14, $\dim R_i/\mathfrak{p}_i = \dim R$. Since $\dim R = \dim R^{\mathrm{sh}}$ [Sta23, Tag 06LK], this implies that the preimage $C \cap R^{\mathrm{sh}}$ also a maximal chain of primes in R^{sh}. By our choice of i, the minimal prime of $C \cap R^{\mathrm{sh}}$ is $\mathfrak{q}_0$. Thus, for an arbitrary minimal prime $\mathfrak{q}_0$ we have exhibited a chain C of primes starting at $\mathfrak{q}_0$ and of length $\dim R^{\mathrm{sh}}$, so R^{sh} is equidimensional. □

Lemma 2.16. *Let $(R, \mathfrak{m})$ be the perfection of a noetherian local $\mathbb{F}_p$-algebra, and let $(R^{\mathrm{sh}}, \mathfrak{m}^{\mathrm{sh}})$ be the strict henselization.*

(1) $H^i_{\mathfrak{m}^{\mathrm{sh}}}(R^{\mathrm{sh}}) = H^i_{\mathfrak{m}}(R) \otimes_R R^{\mathrm{sh}}$.
(2) $H^i_{\mathfrak{m}}(R) = 0$ *if and only if* $H^i_{\mathfrak{m}^{\mathrm{sh}}}(R^{\mathrm{sh}}) = 0$.
(3) *If the punctured spectrum of $(R, \mathfrak{m})$ is Cohen–Macaulay, so is the punctured spectrum of $(R^{\mathrm{sh}}, \mathfrak{m}^{\mathrm{sh}})$.*

Proof. It follows from topological invariance of the small étale site [Sta23, Tag 04DY] that perfection commutes with strict henselization, so that R^{sh} is topologically noetherian. Since $\mathfrak{m}^{\mathrm{sh}} = \mathfrak{m} R^{\mathrm{sh}}$ then $H^i_{\mathfrak{m}^{\mathrm{sh}}}(R^{\mathrm{sh}}) = H^i_{\mathfrak{m}}(R^{\mathrm{sh}})$. Via the description of $H^i_{\mathfrak{m}}(R)$ as the cohomology of a Koszul complex of R-modules, (1) and (2) follow from faithful flatness of $R \to R^{\mathrm{sh}}$ [Sta23, Tag 07QM]. For (3), let $\mathfrak{q}$ be a non-maximal prime of R^{sh}, and let $\mathfrak{p}$ be its preimage in R. We claim that $\mathfrak{q} R^{\mathrm{sh}}_{\mathfrak{q}} = \mathfrak{p} R^{\mathrm{sh}}_{\mathfrak{q}}$ and $\dim R^{\mathrm{sh}}_{\mathfrak{q}} = \dim R_{\mathfrak{p}}$. Granting these claims, $H^i_{\mathfrak{q}}(R^{\mathrm{sh}}_{\mathfrak{q}}) = H^i_{\mathfrak{p}}(R_{\mathfrak{p}}) \otimes R^{\mathrm{sh}}_{\mathfrak{q}}$ and (3) follows.

To prove the claims, use [Sta23, Tag 06LK] to write $R^{\mathrm{sh}} = \mathrm{colim}_i R_i$ as a direct limit of local rings R_i which are localizations of étale R-algebras. If $\mathfrak{p}_i$ is the preimage of $\mathfrak{q}$ in R_i, then $R^{\mathrm{sh}}_{\mathfrak{q}} = \mathrm{colim}_i (R_i)_{\mathfrak{p}_i}$. We have $\mathfrak{p}_i (R_i)_{\mathfrak{p}_i} = \mathfrak{p}(R_i)_{\mathfrak{p}_i}$ by [Sta23, Tag 00U4], so the first claim follows. For the claim about dimensions, note that $\dim R^{\mathrm{sh}}_{\mathfrak{q}} \geq \dim R_{\mathfrak{p}}$ by faithful flatness and the going down property. For the other direction, if $\mathfrak{q}_0 \subset \mathfrak{q}_1 \subset \cdots \subset \mathfrak{q}_n = \mathfrak{q}$ is a chain of primes in R^{sh}, then for some i this restricts to a chain of primes in R_i of the same length. Now we conclude since $\dim (R_i)_{\mathfrak{p}_i} = \dim R_{\mathfrak{p}}$ by [Sta23, Tag 07QP]. □

We now come to the main result of this subsection. In the proof, we use the Artin–Schreier sequence

$$0 \to \mathbb{F}_p \to \mathcal{O}_{\mathrm{Spec}(R)} \xrightarrow{\varphi - 1} \mathcal{O}_{\mathrm{Spec}(R)} \to 0 \tag{2.2}$$

to translate between quasi-coherent cohomology and étale cohomology. This sequence is exact in the étale topology on $\mathrm{Spec}(R)$ for any $\mathbb{F}_p$-algebra R. If $(R,\mathfrak{m})$ is the perfection of a local $\mathbb{F}_p$-algebra R_0 then $\mathcal{O}_{\mathrm{Spec}(R)}$ may be viewed as a quasi-coherent sheaf on $\mathrm{Spec}(R_0)$, and then (2.2) is also exact in the étale topology on $\mathrm{Spec}(R_0)$. In particular, if R_0 is noetherian then $R^i\Gamma_{\{\mathfrak{m}\}}(\mathcal{O}_{\mathrm{Spec}(R)}) = H^i_{\mathfrak{m}_0}(R)$ is the same in both the Zariski and étale topology by [Sta23, Tag 04DY] and descent for quasi-coherent sheaves.

Theorem 2.17. *Let k be a perfect field of characteristic p and let X be a scheme isomorphic to the perfection of a connected finite-type k-scheme. Then the following are equivalent.*

(1) *X is Cohen–Macaulay in the sense of Definition 2.9.*
(2) *The shifted constant sheaf $\mathbb{F}_p[\dim X] \in D^b_c(X,\mathbb{F}_p)$ is perverse.*

Furthermore, X is equidimensional in both cases.

Proof. We first show that (1) implies (2). By Lemma 2.10, X is equidimensional. It is immediate that $\mathbb{F}_p[\dim X] \in {}^pD^{\leq 0}(X,\mathbb{F}_p)$. To prove that $\mathbb{F}_p[\dim X] \in {}^pD^{\geq 0}(X,\mathbb{F}_p)$, fix a point $x \in X$. Let $(R,\mathfrak{m})$ be the strict henselization of the corresponding perfect local ring. Let $d := \dim R$, which also agrees with the dimension before strict henselization [Sta23, Tag 06LK]. By Lemma 2.16, (1) is equivalent to the statement that $H^i_{\mathfrak{m}}(R) = 0$ for all $i < d$ and points $x \in X$. On the other hand, as X is equidimensional, (2) is equivalent to the statement that for all points x, we have $\mathcal{H}^i(Ri_x^!\mathbb{F}_p) = 0$ for $i < d$, where $i_x \colon \overline{x} \to \mathrm{Spec}(R)$ is the inclusion of the closed point. Since $\overline{x}$ is a geometric point, $\mathcal{H}^i(Ri_x^!\mathbb{F}_p) = R^i\Gamma_{\{\mathfrak{m}\}}(\mathbb{F}_p)$. Now the fact that (1) implies (2) follows from applying $R\Gamma_{\{\mathfrak{m}\}}$ to the Artin–Schreier sequence (2.2).

Next, we show that (2) implies X is equidimensional. A straightforward argument using the Artin–Schreier sequence as above shows that if Y is Cohen–Macaulay, irreducible, and of finite-type over k, then $\mathbb{F}_p[\dim Y] \in D^b_c(Y,\mathbb{F}_p)$ is perverse. Now let X_0 be a deperfection of X by a finite-type k-scheme, which we may assume is reduced. Since X_0 is of finite type over the perfect field k, we may let Y be a smooth dense open subscheme of X_0, so the perversity of $\mathbb{F}_p[\dim Y]$ implies that X_0 is equidimensional.

To show that (2) implies (1), we proceed by descending induction on $\dim \overline{\{x\}}$. By Lemma 2.16, it suffices to show the strict henselization $(R,\mathfrak{m})$ of the local ring at x satisfies $H^i_{\mathfrak{m}}(R) = 0$ for $i < d := \dim R = \dim X - \dim \overline{\{x\}}$, and we may assume the punctured spectrum of $(R,\mathfrak{m})$ is Cohen–Macaulay. By Lemma 2.15 the hypotheses of Proposition 2.13 are satisfied, so we are reduced to checking that $H^i_{\mathfrak{m}}(R)$ has trivial φ-invariants for $i < d$. But this condition follows from induction and the long exact sequence obtained from applying $R\Gamma_{\{\mathfrak{m}\}}$ to the Artin–Schreier sequence (2.2), together with the perversity of $\mathbb{F}_p[\dim X]$. □

2.6. **φ-rationality.** Next we discuss a perfect notion of φ-rationality. Recall that by a theorem of Smith [Smi97], an excellent local $\mathbb{F}_p$-algebra $(R_0,\mathfrak{m}_0)$ is φ-rational if and only if it is Cohen–Macaulay and $H^{\dim R_0}_{\mathfrak{m}_0}(R_0)$ is a simple $R_0[\varphi]$-module (this differs from the original definition in terms of tight closure, cf. [FW89, HH94]). One of the present authors showed in [Cas22, Theorem 1.7] that if X is an irreducible scheme of finite type over an algebraically closed field, all of whose local rings are φ-rational, then $\mathbb{F}_p[\dim X]$ is simple as a perverse sheaf. In the opposite direction, we will show that if

$\mathbb{F}_p[\dim X]$ is simple, then X^{perf} is Cohen–Macaulay and $H^{\dim R_0}_{\mathfrak{m}_0}(R_0)^{\mathrm{perf}}$ is simple. The latter properties are encapsulated by the following definition.

Definition 2.18. Let X be the perfection of a noetherian φ-finite $\mathbb{F}_p$-scheme. We say that X is φ-rational if it is Cohen–Macaulay in the sense of Definition 2.9, and, for every local ring $(R, \mathfrak{m})$ on X, the top local cohomology group $H^{\dim R}_{\mathfrak{m}}(R)$ is a simple $R[\varphi]$-module.

It will be useful to characterize φ-rationality of a perfect scheme in terms of a property of one (equivalently, every) deperfection. The latter property turns out to be φ-nilpotence, first introduced by Blickle–Bondu [BB05] under the name close to F-rational, and further studied e.g. in [ST17, PQ19, DMP23, KMPS23].

Definition 2.19. Let $(R, \mathfrak{m})$ be a φ-finite noetherian local $\mathbb{F}_p$-algebra of dimension d.

(1) The tight closure of the zero submodule in $H^d_{\mathfrak{m}}(R)$, denoted $0^*_{H^d_{\mathfrak{m}}(R)}$, is the $R[\varphi]$-submodule consisting of all elements $x \in H^d_{\mathfrak{m}}(R)$ such that there exists some $c \in R$ not contained in any minimal prime with the property that $c\varphi^e(x) = 0$ for all $e \gg 0$.

(2) The ring R is said to be φ-nilpotent if each of the the $R[\varphi]$-modules $H^0_{\mathfrak{m}}(R), \ldots, H^{d-1}_{\mathfrak{m}}(R), 0^*_{H^d_{\mathfrak{m}}(R)}$ is nilpotent.

By [PQ19, Proposition 2.8 (2)], R is φ-nilpotent if and only if its reduction is φ-nilpotent, so we will usually assume R is reduced. To relate this notion to coherent objects, note that Matlis duality gives a canonical pairing $f\colon H^d_{\mathfrak{m}}(R) \otimes_{\hat{R}} \omega_{\hat{R}} \to E$. The parameter test module $\tau(\omega_R) \subset \omega_R$ is the Cartier submodule consisting of all $\eta \in \omega_R$ such that $f(x \otimes \eta) = 0$ for all $x \in 0^*_{H^d_{\mathfrak{m}}(R)}$. The parameter test module is well-behaved under localization [HT04, Proposition 3.1] (cf. [Bli13, Proposition 3.2 (e)]), completion [HT04, Proposition 3.2], and more generally under flat base change when the residue field extension is separable [ST17, Lemma 1.5]. By construction, $0^*_{H^d_{\mathfrak{m}}(R)}$ is the Matlis dual of $\omega_R/\tau(\omega_R)$, even if R is not complete. When combined with Lemma 2.10, the following gives a characterization of φ-nilpotence in terms of $\omega^\bullet_R$.

Lemma 2.20. *Let $(R, \mathfrak{m})$ be a reduced, φ-finite noetherian local $\mathbb{F}_p$-algebra of dimension d. Then $0^*_{H^d_{\mathfrak{m}}(R)}$ is nilpotent if and only if $\omega_R/\tau(\omega_R)$ is nilpotent.*

Proof. It is observed [ST17, Lemma 2.3] that this follows from an argument similar to [HT04, Lemma 2.1]. □

Lemma 2.21. *Let $(R, \mathfrak{m})$ be the perfection of a φ-finite noetherian local $\mathbb{F}_p$-algebra $(R_0, \mathfrak{m}_0)$ of dimension d. Then the following are equivalent.*

(1) $\mathrm{Spec}(R)$ *is φ-rational in the sense of Definition 2.19.*

(2) $\mathrm{Spec}(R^{\mathrm{sh}})$ *is φ-rational in the sense of Definition 2.19.*

(3) R_0 *is φ-nilpotent.*

(4) R_0^{sh} *is φ-nilpotent.*

Furthermore, if any of the above conditions is satisfied then R is geometrically unibranch.

Proof. By [PQ19, Proposition 2.8] we may assume that R_0 is reduced. The equivalence of (3) and (4) then follows from Lemma 2.16 and Lemma 2.20, together with the compatibility of $\tau(\omega_{R_0})$ and its structure map as a Cartier module under faithfully flat base change to R_0^{sh} as in [ST17, Proposition 2.4 (4)], cf. [KMPS23, Theorem 4.4]. Once we prove the equivalence of (1) and (3), the equivalence with (2) will then follow.

First suppose that R_0 is φ-nilpotent. Then the completion $\hat{R}_0$ is also φ-nilpotent by [PQ19, Proposition 2.8 (4)]. Thus $\hat{R}_0$ is a domain by [DMP23, Theorem 3.1], and after the equivalence of (1)-(4) is established, loc. cit. will also imply the final claim about geometric unibranchedness. By [Smi93, Theorem 3.1.4], $0^*_{H^d_{\mathfrak{m}_0}(R_0)}$ is the unique maximal proper $R_0[\varphi]$-submodule of $H^d_{\mathfrak{m}_0}(R_0)$. Since this submodule is nilpotent, then Proposition 2.12 implies that $H^d_{\mathfrak{m}}(R)$ is a simple $R[\varphi]$-module. Furthermore, $H^i_{\mathfrak{m}}(R) = 0$ for $i < d$ by Lemma 2.10. For every prime $\mathfrak{p} \subset R_0$ the localization $(R_0)_{\mathfrak{p}}$ is φ-nilpotent by [ST17, Proposition 2.4 (3)] or [PQ19, Corollary 5.17], so the same arguments apply to $(R_0)_{\mathfrak{p}}$ and hence $\mathrm{Spec}(R)$ is φ-rational.

Now suppose that $\mathrm{Spec}(R)$ is φ-rational. Then $H^i_{\mathfrak{m}_0}(R_0)$ is nilpotent for $i < d$ and R_0 is equidimensional by Lemma 2.10. By [Bli04, Corollary 3.9] (and the surrounding discussion if R_0 is not complete), $0^*_{H^d_{\mathfrak{m}_0}(R_0)}$ is the intersection of the maximal proper $R_0[\varphi]$-submodules of $H^d_{\mathfrak{m}_0}(R_0)$. Furthermore, the quotient of $H^d_{\mathfrak{m}_0}(R_0)$ by each of these maximal proper $R_0[\varphi]$-submodules is non-nilpotent by [Bli04, Theorem 3.8]. Since $H^d_{\mathfrak{m}_0}(R_0)$ is simple up to nilpotents then $0^*_{H^d_{\mathfrak{m}_0}(R_0)}$ must be nilpotent, so R_0 is φ-nilpotent. □

Lemma 2.22. *Let $(R, \mathfrak{m})$ be the perfection of a φ-finite noetherian local $\mathbb{F}_p$-algebra $(R_0, \mathfrak{m}_0)$. If the punctured spectrum of $(R, \mathfrak{m})$ is φ-rational, then so is the punctured spectrum of $(R^{\mathrm{sh}}, \mathfrak{m}^{\mathrm{sh}})$.*

Proof. By Lemma 2.21 the punctured spectrum of $(R_0, \mathfrak{m}_0)$ is φ-nilpotent, and it suffices to show the same is true of $(R_0^{\mathrm{sh}}, \mathfrak{m}_0^{\mathrm{sh}})$. We can assume R_0 is reduced. If $\mathfrak{q} \subset R_0^{\mathrm{sh}}$ is a non-maximal prime then it lies over some non-maximal prime $\mathfrak{p} \in R_0$. The map of local rings $((R_0)_{\mathfrak{p}}, \mathfrak{p}(R_0)_{\mathfrak{p}}) \to ((R_0^{\mathrm{sh}})_{\mathfrak{q}}, \mathfrak{q}(R_0^{\mathrm{sh}})_{\mathfrak{q}})$ is faithfully flat and the residue field extension is separable (for separability, use that R_0^{sh} is a filtered colimit of étale R_0-algebras [Sta23, Tag 04GW] and apply [Sta23, Tag 00U4]). Since $(R_0)_{\mathfrak{p}}$ is φ-nilpotent, so is $(R_0^{\mathrm{sh}})_{\mathfrak{q}}$ by [ST17, Proposition 2.4 (4)]. □

The following results generalizes [ST17, Proposition 2.5] to the case where the deperfected punctured spectrum of $(R_0, \mathfrak{m}_0)$ is φ-nilpotent instead of φ-rational.

Proposition 2.23. *Let $(R, \mathfrak{m})$ be the perfection of a φ-finite noetherian local $\mathbb{F}_p$-algebra $(R_0, \mathfrak{m}_0)$ of dimension $d > 0$. Suppose that R is equidimensional, the punctured spectrum of $(R, \mathfrak{m})$ is φ-rational, and $R/\mathfrak{m}$ is algebraically closed. Then* $\mathrm{Spec}(R)$ *is φ-rational if and only if for all i, there does not exist a nonzero element $x \in H^i_{\mathfrak{m}}(R)$ such that $\varphi(x) = x$.*

Proof. First suppose that $\mathrm{Spec}(R)$ is φ-rational. We may assume that R_0 is reduced. By Lemma 2.13 we only need to deal with the conditions on $H^d_{\mathfrak{m}}(R)$. As in the proof of Lemma 2.21, R_0 is a domain and $H^d_{\mathfrak{m}_0}(R_0)$ has a unique simple $R_0[\varphi]$-module quotient M_0, which is also non-nilpotent. We claim that $\mathrm{Ann}_{R_0}(M_0) = (0)$. This follows from $\mathrm{Ann}_{\hat{R}_0}(M_0) = (0)$, which in turn follows from the fact that Matlis duality preserves annihilators [BS13a, 10.2.14] and torsion-freeness of the dualizing sheaf $\omega_{\hat{R}_0}$ [Sta23, Tag 0AWK]. Since R is φ-rational then $H^d_{\mathfrak{m}}(R) = M_0^{\mathrm{perf}}$, and it follows that $\mathrm{Ann}_R(H^d_{\mathfrak{m}}(R)) =$

(0). Now if there exists a nonzero $x \in H^d_{\mathfrak{m}}(R)$ with $\varphi(x) = x$, then by simplicity x generates $H^d_{\mathfrak{m}}(R)$ as an R-module. But every element of $H^d_{\mathfrak{m}}(R)$ is also annihilated by some collection of elements which generate $\mathfrak{m}$ up to radical, so $\mathfrak{m} = (0)$, a contradiction since $d > 0$.

Now suppose the punctured spectrum of $(R_0, \mathfrak{m}_0)$ is φ-nilpotent. Again, we only need to deal with the conditions on $H^d_{\mathfrak{m}}(R)$, and R_0 is equidimensional by Lemma 2.10. For contradiction we may assume that R_0 is reduced and $\omega_{R_0}/\tau(\omega_{R_0})$ is non-nilpotent (Lemma 2.20). Let M be a simple non-nilpotent Cartier subqoutient of $\omega_{R_0}/\tau(\omega_{R_0})$, and let $\mathfrak{p} \subset R_0$ be its unique associated prime (Theorem 2.1). By the compatibility of $\tau(\omega_{R_0})$ with localization [HT04, Proposition 3.1], our assumption on the punctured spectrum implies $\mathfrak{p} = \mathfrak{m}_0$. Now we conclude as in the proof of Proposition 2.13. Briefly, the Matlis dual of $\omega_{R_0}/\tau(\omega_{R_0})$ is $0^*_{H^d_{\mathfrak{m}_0}(R_0)}$, which therefore has finite length as an R_0-module up to nilpotents. Thus, the perfection of $0^*_{H^d_{\mathfrak{m}_0}(R_0)}$ inside $H^d_{\mathfrak{m}}(R)$ is a holonomic R_0-module whose image under Sol is an étale sheaf supported on $\operatorname{Spec}(R/\mathfrak{m})$. Since $R/\mathfrak{m}$ is algebraically closed, the condition on φ-fixed elements implies that $0^*_{H^d_{\mathfrak{m}_0}(R_0)}$ is nilpotent. □

Lemma 2.24 (Emerton–Kisin). *Let X be a smooth irreducible scheme of finite type over a perfect field k. Let $\mathcal{L}$ be an étale local system of $\mathbb{F}_p$-vector spaces on X. Then $\mathcal{L}[\dim X]$ is simple as a perverse sheaf if and only if it is simple as a local system.*

Proof. The property of being perverse is étale-local so that Theorem 2.17 implies $\mathcal{L}[\dim X]$ is perverse. The part about simplicity follows from the claim that every perverse subsheaf of $\mathcal{L}[\dim X]$ is again a shifted local system. Indeed, Gabber's result [Gab04, Corollary 12.4] implies every perverse subsheaf is constructible, and then there are multiple ways to proceed; here we sketch the argument of Emerton–Kisin in [EK04a, Corollary 4.3.3]. Via their Riemann–Hilbert correspondence, $\mathcal{L}[\dim X]$ corresponds to a unit φ-crystal, i.e., an $\mathcal{O}_X[\varphi]$-module M, locally free of finite rank over $\mathcal{O}_X$, where the unit condition means that the adjoint map $\varphi^* M \to M$ is an isomorphism. Their correspondence is a perverse t-exact anti-equivalence, so that perverse subsheaves of $\mathcal{L}[\dim X]$ correspond to unit $\mathcal{O}_X[\varphi]$-module qoutients of M. Then the key input is that any such quotient is is locally free [EK04a, Proposition 1.2.3], so that its Riemann–Hilbert partner is a shifted local system. □

Our main result in this subsection characterizes those schemes for which a simple local system corresponds to a simple perverse sheaf.

Theorem 2.25. *Let k be a perfect field of characteristic p and let X be a connected scheme isomorphic to the perfection of a finite-type k-scheme. Then the following are equivalent.*

(1) *X is φ-rational in the sense of Definition 2.19.*
(2) *The shifted constant sheaf $\mathbb{F}_p[\dim X]$ is a simple perverse sheaf.*

Proof. First suppose that X is φ-rational. Then $\mathbb{F}_p[\dim X]$ is perverse by Theorem 2.17. Furthermore, X is irreducible by Lemma 2.21 and [DMP23, Theorem 3.1]. Let $U \subset X$ be a nonempty open subscheme isomorphic to the perfection of a smooth finite-type k-scheme, which exists by [Sta23, Tag 056V]. Let $i\colon X \setminus U \to X$ be a complementary closed immersion. Then $\mathbb{F}_p[\dim X]_U$ is perverse and simple on U by Lemma 2.24, so it suffices to show the intermediate extension to X is $\mathbb{F}_p[\dim X]$. Clearly $i^*\mathbb{F}_p[\dim X] \in {}^pD^{\leq -1}(X \setminus U, \mathbb{F}_p)$, and it remains to verify that $Ri^!\mathbb{F}_p[\dim X] \in {}^pD^{\geq 1}(X \setminus U, \mathbb{F}_p)$. Let $(R, \mathfrak{m})$ be a strict henselization of the local ring at a point in $X \setminus U$. By Lemma 2.21 the hypothesis of Lemma 2.23 are satisfied, and in particular R is a domain. We must therefore verify that $R^i\Gamma_{\{\mathfrak{m}\}}(\mathbb{F}_p) = 0$ for $i \leq \dim R$, where $\mathbb{F}_p$ is viewed as an étale sheaf on $\mathrm{Spec}(R)$. When $i < \dim R$ this vanishing follows from perversity, and furthermore $H^i_{\mathfrak{m}}(R) = 0$ by Cohen–Macaulayness. The case $i = \dim R$ then follows from Proposition 2.23, by applying $R\Gamma_{\{\mathfrak{m}\}}$ the Artin–Schreier sequence (2.2).

For the other direction, let $j\colon U \to X$ be an irreducible open subscheme isomorphic to the perfection of a smooth finite-type k-scheme. By simplicity we must have $\mathbb{F}_p[\dim X] \cong j_{!*}(\mathbb{F}_p[\dim X]_U)$. On the other hand, $j_{!*}(\mathbb{F}_p[\dim X])$ is supported on the closure of U, so X is irreducible. We now show by descending induction on $\dim \overline{\{x\}}$ that the local ring at $x \in X$ is φ-rational. If $x \in U$ this follows since regular local rings φ-rational even before passing to the perfection [HH89, Theorem 2.1 a)]. If $x \in X \setminus U$ we may assume the punctured spectrum of the local ring at x is φ-rational, so the same is true of the strict henselization $(R, \mathfrak{m})$ by Lemma 2.22. Since X is irreducible, the condition $j_{!*}(\mathbb{F}_p[\dim X]_U) = \mathbb{F}_p[\dim X]$ implies that $R^i\Gamma_{\{\mathfrak{m}\}}(\mathbb{F}_p) = 0$ for $i \leq \dim R$, where $\mathbb{F}_p$ is viewed as an étale sheaf on $\mathrm{Spec}(R)$. Perversity of $\mathbb{F}_p[\dim X]$ implies that $H^i_{\mathfrak{m}}(R) = 0$ for $i < \dim R$ (Theorem 2.17 and Lemma 2.16). Then by the Artin–Schreier sequence (2.2), $H^{\dim R}_{\mathfrak{m}}(R)$ has trivial φ-invariants. The remaining hypotheses of Proposition 2.23, equidimensionality in particular, are satisfied for R by Lemma 2.15. Thus, R is φ-rational, and hence so is the local ring at x Lemma 2.21. □

A priori, the fact that the irreducibility of a scheme is not an étale-local property could prevent the simplicity of $\mathbb{F}_p[\dim X]$ from being an étale-local property. However, the relation with φ-nilpotence shows that this is not the case.

Corollary 2.26. *Let k be a perfect field of characteristic p and let X_0 be a finite-type k-scheme. If shifted constant sheaf $\mathbb{F}_p[\dim X_0]$ is a simple perverse sheaf, then X_0 is geometrically unibranch.*

Proof. By Theorem 2.25 and Lemma 2.21, the local rings of X_0 are φ-nilpotent, so the result follows from [DMP23, Theorem 3.1, Remark 3.2]. □

Remark 2.27. Let X be a normal irreducible scheme of finite type over a perfect field k. Two important theorems in commutative algebra assert that the absolute integral closure X^+ of X in its field of fractions is Cohen–Macaulay (due to Hochster–Huneke [HH92]) and φ-rational (due to Smith [Smi94]) in an appropriate sense. This is true more generally over a φ-finite base. As observed in [BBL+23, §5.6], these theorems can be recovered from results such as Theorem 2.25. Informally, the idea is to show that $\mathbb{F}_p[\dim X]_{X^+}$ is a simple perverse sheaf on X^+. To prove this, one must compute $i^*\mathbb{F}_p[\dim X]_{X^+}$ and $Ri^!\mathbb{F}_p[\dim X]_{X^+}$, where $i\colon Z \to X^+$ is a proper closed subscheme. But the $*$-pullback is constant, and the !-pullback vanishes since $\mathbb{F}_p[\dim X]_{X^+}$ is the $*$-extension of its restriction to any open subset [Bha20, Proposition 3.10]. We refer to [BBL+23, §5.6] for more details.

3. Global +-regularity and inversion of adjunction

In this section, we review some of the material from [BMP+23] on globally +-regular varieties, explain the proof of their criterion for inversion of adjunction, and adapt it to a certain asymptotic analogue. This is going to be applied later to certain Demazure varieties.

Remark 3.1. In positive characteristic, ideas such as these have been known for at least a decade in advance. For instance, Das [Das15] proved inversion of adjunction for strong φ-regularity in characteristic p, but we would rather avoid his treatment, because it circumvents the Kawamata–Viehweg +-vanishing of [Bha12] and because it forces us to work with $\mathbb{Z}_{(p)}$-divisors everywhere instead of $\mathbb{Q}$-divisors.

3.1. **Global +-regularity.** Let k be a perfect field and X be a finite type connected normal k-scheme. It is helpful to consider the notion of boundary and subboundary $\mathbb{Q}$-divisors, as they constitute a very flexible tool in studying singularities.

Definition 3.2. Let $\Delta = \sum_i r_i D_i$ be a $\mathbb{Q}$-divisor on X, i.e., a finite rational linear combination of prime divisors on X. We say that Δ is a boundary (resp. a subboundary) if $0 \leq r_i \leq 1$ for all i (resp. if $0 \leq r_i < 1$). We refer to (X, Δ) as a boundary (resp. subboundary) pair.

Let us recall the notion of global +-regularity following [BMP+23, Definition 6.1].

Definition 3.3. We say that the pair (X, Δ) is globally +-regular if for every finite cover $f\colon Y \to X$ with Y connected normal, the natural map $\mathcal{O}_X \to f_*\mathcal{O}_Y(\lfloor f^*\Delta \rfloor)$ splits in the category of $\mathcal{O}_X$-modules.

If $\Delta = 0$, then we simply say that X is globally +-regular. Note that this condition only has to be verified for a cofinal family of finite covers f. By the cyclic covering trick, we may even assume that $f^*\Delta$ is integral, compare with [BMP+23, Remark 6.2]. Let us start with the first basic stability property.

Lemma 3.4. *If the boundary pair (X, Δ) is globally +-regular, the same holds true for (X, Δ') for any boundary $\Delta' \leq \Delta$.*

Proof. Compose the inclusion $f_*\mathcal{O}_Y(\lfloor f^*\Delta' \rfloor) \to f_*\mathcal{O}_Y(\lfloor f^*\Delta \rfloor)$ with the given section of $\mathcal{O}_X \to f_*\mathcal{O}_Y(\lfloor f^*\Delta \rfloor)$. □

In particular, (X, Δ) being globally +-regular implies that $(X, \epsilon\Delta)$ also is for every $0 \leq \epsilon \leq 1$. More importantly, global +-regularity satisfies proper descent:

Proposition 3.5. *Let $f\colon X \to Y$ be a proper birational map of normal connected k-schemes of finite type. If (X, Δ) is globally +-regular, then so is $(Y, f_*\Delta)$.*

Proof. This is [BMP+23, Proposition 6.19]. Note that pushforwards and pullbacks of $\mathbb{Q}$-divisors along alterations of normal connected finite type are defined locally in codimension 1 on principal divisors via the norm map and the inclusion map, respectively, see [Sta23, Tag 02RS], and then extended by normality to the entire space. Let $g\colon Z \to Y$ be a finite cover by a normal integral k-scheme such that $g^*f_*\Delta$ is an integral divisor. Let W be the normalization of $X \times_Y Z$ with base maps $g'\colon W \to X$ and $f'\colon W \to Z$. Then, we know that $\mathcal{O}_X \to g'_*\mathcal{O}_W(g'^*\Delta)$ splits in $\mathcal{O}_X$-modules. The same holds therefore for $\mathcal{O}_Y \to g_*\mathcal{O}_Z(f'_*g'^*\Delta)$ in $\mathcal{O}_Y$-modules, because there is a natural map $g_*\mathcal{O}_Z(f'_*g'^*\Delta) \to f'_*g'_*\mathcal{O}_W(g'^*\Delta)$. Noticing that $g^*f_*\Delta = f'_*g'^*\Delta$, we deduce our desired splitting. □

Next, we translate the notion of globally +-regularity in terms of trace maps by applying Grothendieck–Serre duality. Recall that there is a 6-functor formalism on the category of quasi-coherent sheaves and the canonical sheaves ω_X arise as the $H^{-\dim(X)}$ of the complex $Rp^!k$, where $p\colon X \to \mathrm{Spec}(k)$ denotes the structure morphism. Since we assume X to be normal and connected, it turns out that ω_X is a reflexive sheaf and we usually fix an arbitrary canonical divisor K_X such that $\omega_X \simeq \mathcal{O}_X(K_X)$.

Proposition 3.6. *Assume $K_X + \Delta$ is $\mathbb{Q}$-Cartier. Then, the boundary pair (X, Δ) is globally +-regular if and only if the trace map*

$$H^0(Y, \mathcal{O}_Y(K_Y - \lfloor f^*(K_X + \Delta)\rfloor)) \to H^0(X, \mathcal{O}_X) \tag{3.1}$$

is surjective for all connected normal finite covers $f\colon Y \to X$.

Proof. This is a particular case of [BMP+23, Proposition 6.8]. By duality, the natural map $\mathcal{O}_X \to f_*\mathcal{O}_Y(\lfloor f^*\Delta\rfloor)$ of $\mathcal{O}_X$-modules is a split injection if and only if the map

$$\mathcal{H}\mathrm{om}_{\mathcal{O}_X}(f_*\mathcal{O}_Y(\lfloor f^*\Delta\rfloor), \mathcal{O}_X) \to \mathcal{O}_X \tag{3.2}$$

of $\mathcal{O}_X$-modules is a split surjection. By Grothendieck–Serre duality, the left side identifies with $f_*\mathcal{H}\mathrm{om}_{\mathcal{O}_Y}(\mathcal{O}_Y(\lfloor f^*\Delta\rfloor), f^!\mathcal{O}_X)$ where $f^!$ is the abelian truncation of the shriek pullback $Rf^!$. Note that this is reflexive, so to compute it we are allowed to restrict to the smooth locus of Y. Over there, $\lfloor f^*\Delta\rfloor$ becomes an actual Cartier divisor, so, in particular, we can write the left side as $f_*f^!\mathcal{O}_X(-\lfloor f^*\Delta\rfloor)$ by pulling the divisor across the Hom. On the other hand, by definition of the canonical divisor, we have $f^!\mathcal{O}_X(K_X) = \mathcal{O}_Y(K_Y)$, and since we are over the smooth locus of Y, we get the identity $f^!\mathcal{O}_X = \mathcal{O}_Y(K_Y - K_X)$ and our sheaf identifies with $f_*\mathcal{O}_Y(K_Y - \lfloor f^*(K_X + \Delta)\rfloor)$, just like in the statement of the proposition. Now, since $\mathcal{O}_X$ is free, the surjectivity of the trace map can be tested at the level of global sections. □

Motivated by the previous proposition, one has the k-module of +-stable sections $B^0(X, \Delta; \mathcal{O}_X)$ in [BMP+23, Definition 4.2] given by the intersection across all normal finite covers $f\colon Y \to X$ of the images of the trace maps (3.1) appearing in the statement of Proposition 3.6. In particular, global +-regularity amounts to demanding an equality $B^0(X, \Delta, \mathcal{O}_X) = H^0(X, \mathcal{O}_X)$. We finish this subsection with the following quite non-standard notion

Definition 3.7. We say that a boundary pair (X, Δ) is $\mathbb{Q}$-Fano if the $\mathbb{Q}$-divisor $K_X + \Delta$ is $\mathbb{Q}$-Cartier and anti-ample.

Eventually, we will want to provide an inductive criterion for lifting global +-regularity along closed subschemes and this positivity condition will play a significant role.

3.2. **Pure variant and inversion of adjunction.** Our next topic consists of a variant of global +-regularity defined along a prime divisor $S \subset X$. The following notion is a simplification of [BMP+23, Definition 6.24]. Let (X, Δ) be a pair consisting of a finite type k-scheme and an effective $\mathbb{Q}$-divisor Δ on X. Assume $\Delta = S + B$ where S is a prime divisor and B an effective $\mathbb{Q}$-divisor on X with irreducible components different from S.

Definition 3.8. We say that $(X, S+B)$ is purely globally $+$-regular along S if the map of $\mathcal{O}_X$-modules $\mathcal{O}_X \to f_*\mathcal{O}_Y(-S_Y + \lfloor f^*(S+B) \rfloor)$ splits for every finite cover $f\colon Y \to X$ with Y connected normal, where the $S_Y \subset Y$ form a compatible family of prime divisors lying over $S \subset X$.

Using the Galois action on X^+ over X, we can show that the previous definition is independent of the choice of the prime divisors $S_Y \subset Y$ (equivalently, of an absolute integral closure $S^+ \subset X^+$). There is a close relationship between pure global $+$-regularity and global $+$-regularity after slightly tweaking the divisors.

Lemma 3.9. *If $(X, S+B)$ is purely globally $+$-regular along S, then $(X, (1-\epsilon)S + B)$ is globally $+$-regular for every rational number $0 < \epsilon \leq 1$.*

Proof. This is [BMP+23, Lemma 4.26]. We just have to notice that $f^*(\epsilon S + B) \leq -S_Y + f^*(S+B)$ for sufficiently large normal finite covers $f\colon Y \to X$, so that the pure $+$-splitting of $(X, S+B)$ along S factors over a $+$-splitting for the pair $(X, (1-\epsilon)S+B)$. □

Proposition 3.10. *The pair $(X, S+B)$ is purely globally $+$-regular along S if and only if the trace map*

$$H^0(Y, \mathcal{O}_Y(K_Y + S_Y - \lfloor f^*(K_X + S + B) \rfloor)) \to H^0(X, \mathcal{O}_X) \tag{3.3}$$

is surjective for all normal finite covers $f\colon Y \to X$.

Proof. The proof is the same as the non-pure along S case, requiring us to check that the $\mathcal{O}_X$-module dual of $f_*\mathcal{O}_Y(-S_Y + \lfloor f^*(S+B) \rfloor)$ equals $f_*\mathcal{O}_Y(K_Y + S_Y - \lfloor f^*(K_X + S + B) \rfloor)$. □

Again, there exists a module $B^0_S(X, S+B; \mathcal{O}_X)$ of pure $+$-stable sections along S, see [BMP+23, Definition 4.21], and pure global $+$-regularity along S translates into an equality $B^0_S(X, S+B; \mathcal{O}_X) = H^0(X, \mathcal{O}_X)$. The pure along S variant of global $+$-regularity was setup in this way, precisely because we want to study how to lift global $+$-regularity from a prime divisor S to the whole k-variety X – this is known as inversion of adjunction.

Theorem 3.11 ([BMP+23])**.** *Let X be a connected normal proper k-scheme, $S \subset X$ a normal prime divisor, and B a subboundary with components different from S. If $(X, S+B)$ is $\mathbb{Q}$-Fano, then $(X, S+B)$ is purely globally $+$-regular along S if and only if $(S, B|_S)$ is globally $+$-regular.*

Proof. This is a particular case of [BMP+23, Theorem 7.2], see also [BMP+23, Corollary 7.5], and we give a sketch of the argument. During the proof, we use the shorthand $\Delta = S + B$. By Serre duality, we can identify the trace maps with the natural maps

$$H^d(X, \mathcal{O}_X(K_X)) \to H^d(Y, \mathcal{O}_Y(\lfloor f^*(K_X + \Delta) \rfloor)) \tag{3.4}$$

induced by pullback along f^* and multiplication by the divisor $\lfloor f^*\Delta \rfloor$, and similarly

$$H^d(X, \mathcal{O}_X(K_X)) \to H^d(Y, \mathcal{O}_Y(-S_Y + \lfloor f^*(K_X + \Delta) \rfloor)) \tag{3.5}$$

in the pure along S case. Note that $\mathcal{O}_X(K_X + S)$ pulls back to $\mathcal{O}_S(K_S)$ because this holds away from codimension 2, and then we apply Hartogs' theorem by normality of S and X. In particular, the associated long exact sequence yields a connecting homomorphism

$$H^{d-1}(S, \mathcal{O}_S(K_S)) \to H^d(X, \mathcal{O}_X(K_X)) \tag{3.6}$$

which is surjective by normality and connectedness: indeed, it arises by dualising on k-modules the non-zero ring map $H^0(X, \mathcal{O}_X) \to H^0(S, \mathcal{O}_S)$ between finite field extensions of k. Similarly, we can connect the right sides of the pullback maps via the following map

$$H^{d-1}(S^+, \mathcal{O}_S^+(\nu_S^*(K_S + B|_S))) \to H^d(X^+, \mathcal{O}_X^+(-S^+ + \nu_X^*(K_X + \Delta))) \tag{3.7}$$

where we let the $+$-notation denote the colimit with respect to a family of connected normal finite covers $f \colon Y \to X$, and ν is the structure map of the absolute integral closures. The kernel of the connecting homomorphism at the $+$-level is given by the image of $H^{d-1}(X^+, \mathcal{O}_X^+(\nu_X^*(K_X + \Delta)))$. The latter vanishes by anti-ampleness of the $\mathbb{Q}$-Cartier divisor $K_X + \Delta$ and the Kodaira $+$-vanishing theorem of [Bha12]. A diagram chase reveals that injectivity for S gives rise to injectivity for X, and vice-versa. □

3.3. **An assymptotic variant.** Theorem 3.11 provides a criterion for inversion of adjunction of global $+$-regularity but has the somewhat unpleasant feature that it lifts global $+$-regularity to at most pure global $+$-regularity. At the same time, the latter comes pretty close to global $+$-regularity itself by Lemma 3.9. This leads us to formulate a variant that treats boundary pairs asymptotically and increases the clarity of our exposition when applying the criterion to Demazure varieties.

Definition 3.12. Given a boundary decomposition $\Delta = S + B$ with S prime and $B \geq 0$ with no common components with S, we similarly say that the boundary pair (X, Δ) is asymptotically purely $\mathbb{Q}$-Fano along S if there exist arbitrarily close subboundaries $B' < B$ such that $K_X + S + B'$ is an anti-ample Cartier divisor.

The definition above is again quite non-standard, but it fits well within our paper. The next step is to define the asymptotic analogue of global $+$-regularity.

Definition 3.13. We say that the boundary pair (X, Δ) is asymptotically globally $+$-regular if for all subboundaries $\Delta' < \Delta$, the pair (X, Δ') is globally $+$-regular in the usual sense.

Note that we do not here the condition applies to all smaller subboundaries, because global $+$-regularity is stable under parallelipepides, unlike ampleness. We can safely ignore a corresponding asymptotic notion of pure global $+$-regularity along a prime divisor, as the criterion for inversion of adjunction now takes the following form.

Corollary 3.14. *Let X be a connected normal proper k-scheme, $S \subset X$ a normal prime divisor, and B a boundary with components different from S. If $(X, S + B)$ is asymptotically purely $\mathbb{Q}$-Fano along S and $(S, B|_S)$ is asymptotically globally $+$-regular, then $(X, S + B)$ is asymptotically globally $+$-regular.*

Proof. Let $B' < B$ be a subboundary such that the corresponding pair (X, Δ') with $\Delta' = S + B'$ is Fano. Now, since we know that $(S, B'|_S)$ is globally $+$-regular, we may apply Theorem 3.11 to get that (X, Δ') is purely globally $+$-regular along S. But then $(X, (1 - \epsilon)S + B')$ is actually globally $+$-regular for any $\epsilon > 0$. Letting ϵ go to 0 and B' to B, we get arbitrarily close to the original boundary Δ, so it induces a asymptotically globally $+$-regular pair. □

Remark 3.15. There is a corresponding version of the corollary which is an equivalence between the behavior of the pairs (X,Δ) and $(S, B|_S)$, but it requires defining asymptotic pure global $+$-regularity along a divisor. The only thing happening here is that the asymptotic pure version for (X,Δ) implies the asymptotic non-pure one for the same pair without tampering with the divisor, precisely because of the asymptoticity.

4. Affine flag varieties

4.1. **Affine Schubert varieties.** Throughout this section k denotes an algebraically closed field of characteristic $p > 0$. Let F be a complete discretely valued field with ring of integers $\mathcal{O}$ and residue field k. Fix a connected reductive group G over F and a parahoric O-model $\mathcal{G}$ in the sense of Bruhat–Tits [BT84]. We introduce the affine Schubert schemes following [AGLR22, §3.2], with some simplifications since we assume k is algebraically closed.

Let $\mathrm{Alg}_k^{\mathrm{perf}}$ denote the category of perfect k-algebras. For $R \in \mathrm{Alg}_k^{\mathrm{perf}}$ let $W(R)$ be the ring of p-typical Witt vectors over R. The ring of $\mathcal{O}$-Witt vectors over R is defined as

$$W_{\mathcal{O}}(R) = \begin{cases} W(R) \otimes_{W(k)} \mathcal{O}, & \mathrm{char}(F) = 0 \\ R \,\widehat{\otimes}_k\, \mathcal{O}, & \mathrm{char}(F) = p. \end{cases}$$

Note that if $\mathrm{char}(F) = p$ and $t \in \mathcal{O}$ is a uniformizer, then $\mathcal{O} \cong k[\![t]\!]$ and $W_{\mathcal{O}}(R) \cong R[\![t]\!]$.

We define the following two functors $\mathrm{Alg}_k^{\mathrm{perf}} \to \mathrm{Grp}$,

$$LG(R) = G(W_{\mathcal{O}}(R) \otimes_{\mathcal{O}} F), \quad L^+G(R) = \mathcal{G}(W_{\mathcal{O}}(R)).$$

The affine flag variety for $\mathcal{G}$ is the étale quotient

$$\mathrm{Fl}_{\mathcal{G}} = LG/L^+\mathcal{G}.$$

The functor $\mathrm{Fl}_{\mathcal{G}}$ is represented by an increasing union of perfections of projective k-schemes. Indeed, if $\mathrm{char}(F) = p$ then $\mathrm{Fl}_{\mathcal{G}}$ is the perfection of the affine flag variety in the sense of [PR08] (which admits a natural moduli problem for all k-algebras), and if $\mathrm{char}(F) = 0$ we obtain the affine flag variety in the sense of [Zhu17] whose representability was proved in [BS17, Corollary 9.6].

The Schubert varieties for the parahoric group scheme $\mathcal{G}$ arise as the $L^+\mathcal{G}$-orbit closures inside $\mathrm{Fl}_{\mathcal{G}}$. As we explain now, these are enumerated via double cosets of the Iwahori–Weyl group and this combinatorics captures their closure relations. Let $\mathbf{f}$ be the unique facet in the Bruhat–Tits building $\mathscr{B}(G,F)$ whose connected stabilizer is $\mathcal{G}(O)$. Let $S \subset G$ be a maximal F-split torus whose apartment contains $\mathbf{f}$. The centralizer $T = Z_G(S)$ is a maximal F-torus and we let $\mathcal{T}$ be its connected Néron O-model. The Iwahori–Weyl group associated to S is $\tilde{W} := N(F)/\mathcal{T}(\mathcal{O})$. The choice of an alcove $\mathbf{a}$ in the apartment of S gives rise to a split exact sequence

$$1 \to W_{\mathrm{af}} \to \tilde{W} \to \pi_1(G)_I \to 1 \tag{4.1}$$

where $\pi_1(G)$ is the algebraic fundamental group and I is the inertia group of F. The affine Weyl group W_{af} is the Coxeter group generated by the reflections in the walls of $\mathbf{a}$. By declaring elements of $\pi_1(G)_I$ to have length zero, $\tilde{W}$ is a quasi-Coxeter group. Let

$W_{\mathcal{G}} \subset \tilde{W}$ be the subgroup generated by reflections in the walls of $\mathbf{f}$. Then we have the Bruhat decomposition

$$L^+\mathcal{G}(k)\backslash LG(k)/L^+\mathcal{G}(k) = W_{\mathcal{G}}\backslash\tilde{W}/W_{\mathcal{G}} \tag{4.2}$$

describing the k-valued points of the Hecke stack $\mathrm{Hk}_{\mathcal{G}} := [L^+\mathcal{G}\backslash\mathrm{Fl}_{\mathcal{G}}]$. Since these capture the entirety of the $L^+\mathcal{G}$-orbits, we can now give the formal definition of Schubert varieties.

Definition 4.1. Let $w \in W_{\mathcal{G}}\backslash\tilde{W}/W_{\mathcal{G}}$. The affine Schubert variety $\mathrm{Fl}_{\mathcal{G},\leq w} \subset \mathrm{Fl}_{\mathcal{G}}$ is the closure of the $\mathcal{G}$-orbit of any choice of lift of w to $\mathrm{Fl}_{\mathcal{G}}(k)$.

The affine Schubert variety $\mathrm{Fl}_{\mathcal{G},\leq w}$ is isomorphic to the perfection of a projective k-scheme. The notation $\mathrm{Fl}_{\mathcal{G},\leq w}$ reflects the fact that $\mathrm{Fl}_{\mathcal{G},\leq w}$ is set-theoretically a disjoint union of the finitely many $L^+\mathcal{G}$-orbits for the $v \in W_{\mathcal{G}}\backslash\tilde{W}/W_{\mathcal{G}}$ bounded by w in the Bruhat order $\leq$. There is a refinement of this collection of closed subschemes obtained as $L^+\mathcal{I}$-orbit closures $\mathrm{Fl}_{(\mathcal{I},\mathcal{G}),\leq w}$ inside $\mathrm{Fl}_{\mathcal{G}}$, called Iwahori–Schubert varieties and indexed by any $w \in \tilde{W}/W_{\mathcal{G}}$.

We will also need convolution Schubert varieties, in order to have access to Demazure resolutions. Let $w_\bullet = (w_1, \dots, w_n)$ be a sequence of elements in $\tilde{W}$. We define the convoluted Schubert variety

$$\mathrm{Fl}_{\mathcal{G},\leq w_\bullet} := (LG)_{\leq w_1} \times^{L^+\mathcal{G}} \dots \times^{L^+\mathcal{G}} \mathrm{Fl}_{\mathcal{G},\leq w_n}, \tag{4.3}$$

where $(LG)_{\leq w} \subset LG$ is the pullback of the Schubert variety $\mathrm{Fl}_{\mathcal{G},\leq w} \subset \mathrm{Fl}_{\mathcal{G}}$ along the natural projection $LG \to \mathrm{Fl}_{\mathcal{G}}$ and the notation $\times^{L^+\mathcal{G}}$ stands for the étale quotient by the diagonal $L^+\mathcal{G}$-action on the adjacent factors. If $\mathcal{G} = \mathcal{I}$ is a Iwahori and all the $w_i =: s_i$ have length 1, then $(LG)_{\leq s_i}$ identifies with the jet group $L^+\mathcal{G}_{s_i}$ of the unique parahoric O-model of G such that $\mathcal{G}_{s_i}(O)$ stabilizes the codimension 1 subfacet $\mathbf{f}_i \subset \bar{\mathbf{a}}$ fixed under s_i. Thus, we can write

$$\mathrm{Fl}_{\mathcal{I},\leq s_\bullet} = L^+\mathcal{G}_{s_1} \times^{L^+\mathcal{I}} \dots \times^{L^+\mathcal{I}} L^+\mathcal{G}_{s_n}/L^+\mathcal{I}. \tag{4.4}$$

and call this a Demazure variety. This is a perfectly smooth variety of dimension n and any convolution Schubert variety admits a proper birational cover given by a Demazure variety under the natural multiplication map. It is costumary to demand that the s_i are simple reflections in W_{aff}, but this forces one to explicitly deal with translations.

Let us compute the Picard group at Iwahori level.

Proposition 4.2. *Suppose $\mathcal{G} = \mathcal{I}$ is an Iwahori model. Then there is an isomorphism $\deg\colon \mathrm{Pic}(\mathrm{Fl}_{\mathcal{I},\leq w_\bullet}) \xrightarrow{\sim} \prod_{i,s\leq w_i} \mathbb{Z}[1/p]^n$ given by the degree of the restriction to $\mathrm{Fl}_{\mathcal{I},\leq s}$, where s runs through length 1 words s bounded by w_i for each i.*

Proof. See [FHLR22, Lemma 4.8] when F has characteristic p and [AGLR22, Theorem 3.8] when F has characteristic 0. While the degree isomorphism $\mathrm{Pic}(\mathbb{P}_k^{1,\mathrm{pf}}) \simeq \mathbb{Z}[1/p]$ implicitly uses the choice of a deperfection, there is no ambiguity when it comes to $\mathrm{Fl}_{\mathcal{I},\leq s}$, as we can take the natural smooth deperfection $\mathrm{Fl}_{\mathcal{I},\leq s,1}$ that comes with a $\mathcal{I}_k$-action and smooth stabilizers, see [AGLR22, Definition 3.14] and the next section on Demazure deperfections. □

In order to apply our criterion on inversion of adjunction for asymptotic global +-regularity, we shall need to have strong control over positivity of line bundles on Demazure varieties.

Lemma 4.3. *Suppose $\mathcal{G} = \mathcal{I}$ is an Iwahori model. Then, a line bundle $\mathcal{L}$ on $\mathrm{Pic}(\mathrm{Fl}_{\mathcal{I},\leq w_\bullet})$ is ample (resp. semi-ample) if and only if $\deg(\mathcal{L})$ is a sequence of positive (resp. non-negative) rationals and the subsequence indexed by any s is strictly decreasing (resp. decreasing).*

Proof. This essentially follows from [HZ20, Theorem 3.1], but we give a self-contained proof. For the forward direction, notice that ampleness (resp. semi-ampleness) is preserved under pull-back along a closed immersion (resp. an arbitrary map). By restricting to $\mathrm{Fl}_{\mathcal{I},\leq s}$, it follows that $\deg(\mathcal{L})$ consists of positive (resp. non-negative) rationals if $\mathcal{L}$ is ample (resp. semi-ample). In order to obtain the monotonicity condition, we restrict $\mathcal{L}$ to the convolution $\mathrm{Fl}_{\mathcal{I},\leq(s,s)}$, and identify it with the usual product $\mathrm{Fl}^2_{\mathcal{I},\leq s}$ via the first projection and multiplication. Observe that this isomorphism maps $\mathrm{Fl}_{\mathcal{I},\leq(s,1)} \subset \mathrm{Fl}_{\mathcal{I},\leq(s,s)}$ (resp. $\mathrm{Fl}_{\mathcal{I},\leq(1,s)} \subset \mathrm{Fl}_{\mathcal{I},\leq(s,s)}$) to the diagonal (resp. second factor) of the untwisted product $\mathrm{Fl}^2_{\mathcal{I},\leq s}$, so the claim is clear.

For the converse, we can reduce to the case where $w_\bullet = w$ by considering the natural embedding $\mathrm{Fl}_{\mathcal{I},\leq w_\bullet} \subset \mathrm{Fl}^n_{\mathcal{I}}$ whose i-th coordinate is the projection $\mathrm{Fl}_{\mathcal{I},\leq w_\bullet} \to \mathrm{Fl}_{\mathcal{I},\leq w_{\bullet \leq i}}$ post-composed with the multiplication $\mathrm{Fl}_{\mathcal{I},\leq w_{\bullet\leq i}} \to \mathrm{Fl}_{\mathcal{I}}$. Indeed, we can extend $\mathcal{L}$ to the right side preserving the positivity condition on degrees. We may also pass to the adjoint quotient of G and then to each of its simple F-factors, and hence assume that G is an almost simple F-group. Now, we consider the closed embedding $\mathrm{Fl}_{\mathcal{I}} \to \prod_s \mathrm{Fl}_{\mathcal{G}^s}$, where s runs through all simple reflections and $\mathcal{G}^s$ is the unique maximal parahoric O-model such that $\mathcal{I}(O) \subset \mathcal{G}^s(O)$ but $s \notin \mathcal{G}^s(O)$. Since $\mathrm{Pic}(\mathrm{Fl}_{\mathcal{G}^s}) = \mathbb{Z}[1/p]$, positivity equals ampleness for these partial flag varieties and the result is clear by pullback. □

4.2. **Central extension.** In this section, we discuss a special central extension of the loop group building on [FHLR22, §4.1.3]. This relates to equivariance of line bundles on $\mathrm{Fl}_{\mathcal{I}}$ with respect to the loop group LG. We assume from now on that G is simply connected and almost simple. One can show by reduction to maximal parahorics $\mathcal{G}^s$ as in the previous lemma and then to GL_n and its determinant line bundle, see [BS17, Theorem 8.8], that every line bundle in $\mathrm{Pic}(\mathrm{Fl}_{\mathcal{I}})$ is $LG(k)$-equivariant. However, it is not true that $LG(k)$-equivariance implies LG-equivariance. Indeed, we have

$$\mathrm{Pic}([LG\backslash \mathrm{Fl}_{\mathcal{I}}]) \simeq \mathrm{Pic}([*/L^+\mathcal{I}]) \simeq X^*(S) \tag{4.5}$$

and we can describe the map towards $\mathrm{Pic}(\mathrm{Fl}_{\mathcal{I}})$ as follows, see also [FHLR22, Lemma 4.10]. Let $\nu \in X^*(S)$ and denote by $\mathcal{O}(\nu)$ the associated line bundle. Given a simple reflection $s \in W_{\mathrm{af}}$ with associated affine root α_s, we can check that the degree of $\mathcal{L}(\nu)$ over $\mathrm{Fl}_{\mathcal{I},\leq s}$ equals $\langle a_s^\vee, \nu\rangle$, where a is the euclidean root underlying α_s, compare with [FHLR22, Lemma 4.12]. The map $X^*(S) \to \mathrm{Pic}(\mathrm{Fl}_{\mathcal{I}})$ is a split injection, but cannot possibly be surjective, as the image has rank equal to that of W (the *euclidean rank* of G), whereas $\mathrm{Pic}(\mathrm{Fl}_{\mathcal{I}})$ has rank equal to that of W_{af} (the *affine rank* of G). The cokernel of $X^*(S) \to \mathrm{Pic}(\mathrm{Fl}_{\mathcal{I}})$ is a free $\mathbb{Z}[1/p]$-module of rank 1, again by the same proof as in [FHLR22, Lemma 4.12]. Employing the same ordering as in [FHLR22, Lemma 4.13] after decomposing G into simple factors, we can select a certain semi-ample line bundle $\mathcal{L}$ to define a direct summand $\mathbb{Z}[1/p]$ inside $\mathrm{Pic}(\mathrm{Fl}_{\mathcal{I}})$ complementary to $X^*(S)$. This leads to the central charge map

$$c\colon \mathrm{Pic}(\mathrm{Fl}_{\mathcal{I}}) \to \mathbb{Z}[1/p] \tag{4.6}$$

with kernel equal to $X^*(S)$. We define the central extension

$$1 \to \mathbb{G}_{m,k}^{\mathrm{pf}} \to \widehat{LG} \to LG \to 1 \tag{4.7}$$

that classifies isomorphisms between $\mathcal{L}$ and $g^*\mathcal{L}$ for all the i. This is independent of the choice of $\mathcal{L}$ by the proof of [FHLR22, Lemma 4.27]. Observe now that we have

$$\mathrm{Pic}([\widehat{LG}\backslash \mathrm{Fl}_{\mathcal{I}}]) \simeq \mathrm{Pic}([*/\widehat{L^+\mathcal{I}}]) \simeq X^*(\widehat{S}) \tag{4.8}$$

and the forgetful map $X^*(\widehat{S}) \to \mathrm{Pic}(\mathrm{Fl}_{\mathcal{I}})$ is an isomorphism by construction. Therefore, every line bundle on $\mathrm{Fl}_{\mathcal{I}}$ has become $\widehat{LG}$-equivariant. Using this, we can define affine coroots.

Definition 4.4. Let s be a simple reflection and $\mathcal{G}_s$ be the associated parahoric group scheme. The affine coroot $\alpha_s^\vee \in X_*(\widehat{S})$ is defined as the unique lift of $a_s^\vee \in X_*(S)$ such that the associated map $\mathbb{G}_{m,k}^{\mathrm{pf}} \to \widehat{LG}$ lands in the subgroup generated by $L^+\mathcal{U}_{\pm\alpha_s}$.

In the definition, we are using the fact that the pullback of $LU_a \subset LG$ along the central extension splits canonically, which is a consequence of the group U_a being unipotent. The dual weight ω_s corresponds under our isomorphism to the line bundle $\mathcal{L}_s$ with degree 1 on $\mathrm{Fl}_{\mathcal{I},\leq s}$ and 0 on every other $L^+\mathcal{I}$-stable $\mathbb{P}_k^{1,\mathrm{pf}}$. The sum of the dual weights $\rho = \sum_s \omega_s$ corresponds to the critical line bundle with degree 1 on every $L^+\mathcal{I}$-stable $\mathbb{P}_k^{1,\mathrm{pf}}$, compare with the terminology in [FHLR22, Lemma 4.17].

Remark 4.5. It is slightly confusing that there are central weights of $\widehat{LG}$ giving rise to non-trivial line bundles by Iwahori induction, but the difference for this very large ind-group is that the center and the cocenter are not isogenous. Indeed, one checks that $\widehat{LG}$ is equal to its own derived subgroup. If there were a rotation $\mathbb{G}_m$-action on LG (e.g., in equicharacteristic and for tame G), we could produce affine roots and their dual coweights as well. In general, however, there does not seem to be a loop interpretation for the affine roots for p-adic G.

4.3. **Equivariant $q_\bullet$-twisted deperfections.** If F has characteristic p, then the moduli space that the perfect flag variety $\mathrm{Fl}_{\mathcal{G}}$ underlies extends naturally to arbitrary k-algebras and is represented by an ind-scheme. As in the perfect case, one defines Schubert varieties as L^+G-orbit closures, where now L^+G is also functor on arbitrary k-algebras. However, since these can fail to be normal in certain cases by [HLR24], one is led to consider their seminormalizations in [FHLR22]. Similarly, one can form convolution products via the moduli interpretation of $\mathrm{Fl}_{\mathcal{G}}$ and we set $\mathrm{Fl}_{\mathcal{G},\leq w_\bullet}$ for the seminormalization (which is again normal).

If F has characteristic 0, then a finite type deperfection $\mathrm{Fl}_{\mathcal{G},\leq w}^{\mathrm{can}}$ was proposed in [AGLR22, Definition 3.14], but it is not clear how well behaved it is beyond low dimensional cases. For us, it is actually preferable to deperfect Demazure varieties. Recall that the loop group $L^+\mathcal{G}$ associated with a parahoric model admits a deperfection $\mathrm{Res}_{O/k}\mathcal{G}$ given by the Greenberg realization. Let $s_\bullet = (s_1, \dots, s_n)$ be a not necessarily reduced sequence of simple reflections in the Iwahori–Weyl group. Then, we get a stacky deperfection of $\mathrm{Fl}_{\mathcal{I},\leq s_\bullet}$ as follows:

$$\mathrm{Fl}_{\mathcal{I},\leq s_\bullet}^{\mathrm{stk}} := \mathrm{Res}_{O/k}\mathcal{G}_{s_1} \times^{\mathrm{Res}_{O/k}\mathcal{I}} \cdots \times^{\mathrm{Res}_{O/k}\mathcal{I}} \mathrm{Res}_{O/k}\mathcal{G}_{s_n}/\mathrm{Res}_{O/k}\mathcal{I} \tag{4.9}$$

where $\mathcal{G}_{s_i}$ is the unique parahoric model such that $\mathcal{G}_{s_i}(O) = \mathcal{I}(O) \cup \mathcal{I}(O)s_i\mathcal{I}(O)$. This is never a scheme because the maps $\mathrm{Res}_{O/k}\mathcal{I} \to \mathrm{Res}_{O/k}\mathcal{G}_{s_i}$ are never injective due to p-torsion in the Witt rings of imperfect rings. Getting actual schemes requires twisting by the Frobenius φ as follows.

We define a certain k-smooth deperfection $\mathrm{Fl}_{\mathcal{I},\leq s_\bullet,q_\bullet}$ by induction on the length of the sequence $s_\bullet$. Here, $q_\bullet$ is going to be an increasing sequence of powers of p defined also in an inductive manner, which we call $s_\bullet$-permissible. Suppose we have constructed the k-smooth variety $\mathrm{Fl}_{\mathcal{I},\leq t_\bullet,r_\bullet}$, where $s_\bullet = (s_1, t_\bullet)$ and $r_\bullet$ is $t_\bullet$-permissible. Suppose that the congruence subgroup $L^+\mathcal{I} \cap L^{\geq n}\mathcal{G}_{s_1}$ acts trivially on the perfect scheme $\mathrm{Fl}_{\mathcal{I},\leq t_\bullet}$. Then, the smooth k-group $\mathrm{Res}_{O_n/k}(\mathcal{I}, \mathcal{G}_{s_1}) := \mathrm{im}(\mathrm{Res}_{O_n/k}\mathcal{I} \to \mathrm{Res}_{O_n/k}\mathcal{G}_{s_1})$ necessarily acts on $(\mathrm{Fl}_{\mathcal{I},\leq t_\bullet,r_\bullet})^q$ for some sufficiently large power q of p. By rescaling $r_\bullet$ with this same power of q, we may assume that there was an action from the start. We then define

$$\mathrm{Fl}_{\mathcal{I},\leq s_\bullet,q_\bullet} := \mathrm{Res}_{O_n/k}\mathcal{G}_{s_1} \times^{\mathrm{Res}_{O_n/k}(\mathcal{I},\mathcal{G}_{s_1})} \mathrm{Fl}_{\mathcal{I},\leq t_\bullet,r_\bullet} \tag{4.10}$$

with $q_\bullet := (1, r_\bullet)$. This is a smooth k-scheme by induction and the fact that $\mathrm{Res}_{O_n/k}\mathcal{G}_{s_1} \to \mathrm{Fl}_{\mathcal{I},\leq s_1}$ has Zariski local sections.

Definition 4.6. For any $s_\bullet$-permissible sequence, we define the equivariant $q_\bullet$-twisted deperfection of $\mathrm{Fl}_{\mathcal{I},\leq s_\bullet}$ as the smooth k-variety $\mathrm{Fl}_{\mathcal{I},\leq s_\bullet,q_\bullet}$ constructed above by induction.

Technically, we are abusing notation by not including the corresponding sequence of integers $n_\bullet$ used to truncate the deperfected jets, as enlarging $n_\bullet$ could force $q_\bullet$ to become large for the twisted product to be well-defined. However, $\mathrm{Fl}_{\mathcal{I},\leq s_\bullet,q_\bullet}$ is still independent of $n_\bullet$ in the following sense: if $q_\bullet$ works for both $n_\bullet$ and $m_\bullet$, then the resulting smooth deperfections are isomorphic.

Next, we compute the canonical sheaf of the deperfection $\mathrm{Fl}_{\mathcal{I},\leq s_\bullet,q_\bullet}$. Note that this variety carries a natural effective divisor $\partial_{s_\bullet,q_\bullet}$ regarded as its boundary, and given by the sum of all the prime divisors $\mathrm{Fl}_{\mathcal{I},\leq s_{\bullet\setminus i},q_{\bullet\setminus i}}$ for all $1 \leq i \leq n$.

Proposition 4.7. *There is an isomorphism*

$$\omega^{-1}_{s_\bullet,q_\bullet} \simeq \mathcal{O}(\partial_{s_\bullet,q_\bullet}) \otimes_{\mathcal{O}} \mathcal{O}(q_\bullet) \tag{4.11}$$

of line bundles on $\mathrm{Fl}_{\mathcal{I},\leq s_\bullet,q_\bullet}$.

Proof. Regard a subsequence of $s_\bullet$ as a smaller word $t_\bullet \leq s_\bullet$ by inserting the identity when needed. Then $q_\bullet$ is still $t_\bullet$-permissible and we get closed immersions $\mathrm{Fl}_{\mathcal{I},t_\bullet,q_\bullet}$. Our goal is to calculate the degree of $\omega_{s_\bullet,q_\bullet}(\partial_{s_\bullet,q_\bullet})$ when restricted to $\mathrm{Fl}_{\mathcal{I},\leq s_i}$ for every $1 \leq i \leq n$: namely, that it equals $-q_i$. By the adjunction formula for canonical divisors along regular immersions, we see that this quantity remains stable under restriction from $s_\bullet$ to any $t_\bullet$ with $t_i = s_i$. By induction we can hence assume that $s_\bullet$ has been truncated to one letter, in which case the result is clear. Indeed, identifying $\mathrm{Fl}_{\mathcal{I},\leq s_i,q_i}$ with $\mathbb{P}^1_k$, this reduces to the usual calculation of the cotangent sheaf, but we need to remember that our definition of degrees in the perfection $\mathbb{P}^{1,\mathrm{pf}}_k$ was twisted by q_i. □

4.4. **Global $+$-regularity.** In equicharacteristic, it is known that $\mathrm{Fl}_{\mathcal{G},\leq w}$ are perfections of globally φ-regular k-varieties by [Cas22, Theorem 1.4] for split G and [FHLR22, Theorem 4.1] for general G. In particular, these deperfections are also globally $+$-regular by [BMP+23, Lemma 6.14]. We want to prove a version of this in mixed characteristic, except it is not always true.

Theorem 4.8. *If $q_\bullet = 1$ is $s_\bullet$-permissible, then $(\mathrm{Fl}_{\mathcal{I},\leq s_\bullet,1}, \partial_{s_\bullet,1})$ is asymptotically globally $+$-regular.*

Proof. Let $s_\bullet = (s_1, \dots, s_n)$ be a word of simple reflections, not necessarily reduced, and set $X = \mathrm{Fl}_{\mathcal{I},\leq s_\bullet,1}$, $\partial X = \partial_{s_\bullet,1}$. Consider the effective Cartier divisors $D_i := \mathrm{Fl}_{\mathcal{I},\leq s_{\bullet \neq i},1} \subset \mathrm{Fl}_{\mathcal{I},\leq s_\bullet,1}$ for any $1 \leq i \leq n$ obtained by deleting the letter i. We know that the Cartier divisor $K_X + \partial X$ has degree -1 in every $L^+\mathcal{I}$-stable curve, so its negative is semi-ample. Since we have $(\partial X - S)|_S = \partial S$, our goal is to show that $(X, \partial X)$ is asymptotically purely Fano along $S := D_n$, so that we may apply Theorem 3.11. Thus, we have to slightly perturb the coefficients of $\partial X = \sum_{i\leq n} D_i$ to get a $\mathbb{Q}$-divisor

$$\Delta = \sum_{i\leq n} r_i D_i \tag{4.12}$$

with r_i smaller but arbitrarily close to 1 and $r_n = 1$ so that $K_X + \Delta$ is $\mathbb{Q}$-Cartier (trivial as X is smooth) and anti-ample. For convenience, we set $\epsilon_i := 1 - r_i$ and $E := -K_X - \partial X$ as shorthand for the everywhere degree 1 Cartier divisor. Then, we deduce that

$$-K_X - \Delta = E + \sum_i \epsilon_i D_i \tag{4.13}$$

and it is enough that the associated degrees sequence is decreasing. In a lemma below, we construct a sequence ϵ_i stable under homothety such that the right side has decreasing degrees. Thus, we get the required asymptotical pure Fano property. ☐

We used the following lemma during the previous proof.

Lemma 4.9. *There exists a sequence of rational numbers $1 > \epsilon_1 > \dots > \epsilon_n = 0$ such that the effective $\mathbb{Q}$-divisor $cA + E$ is ample on X for every $c \in \mathbb{Q}_{>0}$, where $A := \sum_i \epsilon_i D_i$ and $\deg(E) = (1, \dots, 1)$.*

Proof. We are going to construct instead a strictly decreasing sequence of positive integers a_i such that $B := \sum_i a_i D_i$ has strictly decreasing non-negative degrees on the $\mathrm{Fl}_{\mathcal{I},\leq s_k}$. Then, we set $\epsilon_i = N^{-1} a_i$ where $N > a_i$ for all i.

By induction on the length of our sequence $s_\bullet$, we can assume that we already have $a_2, \dots, a_n$ satisfying our degree hypothesis for $s_{\bullet > 1}$. Choose now a sufficiently large positive integer a_1. Since D_1 has trivial degree on $\mathrm{Fl}_{\mathcal{I},\leq s_k}$ for $k > 1$, we see that the divisor A has strictly decreasing non-negative degrees on the $\mathrm{Fl}_{\mathcal{I},\leq s_k}$ for $k > 1$. On the other hand, D_1 has degree 1 on $\mathrm{Fl}_{\mathcal{I},\leq s_1}$, so the corresponding degree of A grows linearly with a_1. Thus, we may assume that it supersedes those for $k > 1$. ☐

It is conjectured that global $+$-regularity is an equivalent notion to strong φ-regularity, but it is not currently known. In our situation, we can still deduce strong φ-regularity from our results.

Corollary 4.10. *If $q_\bullet = 1$ is $s_\bullet$-permissible, then $\mathrm{Fl}_{\mathcal{I},\leq s_\bullet,1}$ is globally φ-regular and compatibly φ-split with $\mathrm{Fl}_{\mathcal{I},\leq t_\bullet,1}$ for every $t_\bullet \leq s_\bullet$.*

Proof. Take $\Delta_{s_\bullet,1} = (1 - 1/p)\mathrm{Fl}_{\mathcal{I},\leq s_{\bullet\setminus i},1}$. Then global $+$-regularity implies that the map of coherent sheaves

$$\mathcal{O}_{\mathcal{I},\leq s_\bullet,1} \to \varphi_* \mathcal{O}_{\mathcal{I},\leq s_{\bullet\setminus i},1/p}((p-1)\mathrm{Fl}_{\mathcal{I},\leq s_{\bullet\setminus i},1/p}) \tag{4.14}$$

admits a splitting. Twisting it by the ideal sheaf of $\mathrm{Fl}_{\mathcal{I},\leq s_{\bullet\setminus i},1}$ and applying the projection formula, we deduce that $\mathrm{Fl}_{\mathcal{I},\leq s_\bullet,1}$ is compatibly φ-split with each irreducible component of its boundary, compare with [BK05, Theorem 1.4.10]. This proves by induction that it is compatibly φ-split with any intersection and unions of those.

Next, we handle global φ-regularity. We select an ample effective $\mathbb{Q}$-divisor

$$\Delta_{s_\bullet,1} := \sum_i \frac{n_i}{q}\mathrm{Fl}_{\mathcal{I},\leq s_{\bullet\setminus i},1} \tag{4.15}$$

for some sequence of non-negative integers $0 \leq n_i < q$ and some sufficiently large $q = p^e \gg 0$. This is possible by the argument in the previous lemma for example. Then, we can guarantee the existence of a splitting

$$\mathcal{O}_{\mathcal{I},\leq s_\bullet,1} \to \varphi^e_*\mathcal{O}_{\mathcal{I},\leq s_\bullet,1/q}(\sum_i n_i\mathrm{Fl}_{\mathcal{I},\leq s_{\bullet\setminus i},1/q}) \tag{4.16}$$

by global $+$-regularity. Now we can apply [Smi00, Theorem 3.10] to deduce that $\mathrm{Fl}_{\mathcal{I},\leq s_\bullet,1}$ is globally φ-regular. □

Now, we assume that $s_\bullet$ is reduced. Let $\mathrm{Fl}_{\mathcal{G},\leq w}$ be the image of the map $\mathrm{Fl}_{\mathcal{I},\leq s_\bullet} \to \mathrm{Fl}_{\mathcal{G}}$. Then, we can define $\mathrm{Fl}_{\mathcal{G},\leq w,q_\bullet}$ as the normal k-variety modelling the original Schubert variety whose structure sheaf equals the pushforward of $\mathcal{O}_{\mathcal{I},\leq s_\bullet,q_\bullet}$ along the map $\mathrm{Fl}_{\mathcal{I},\leq s_\bullet} \to \mathrm{Fl}_{\mathcal{G},\leq w}$ of perfect varieties (regarded as a topological map). In the $q_\bullet = 1$ case, we can say a lot about the geometry of these schemes:

Proposition 4.11. *If $q_\bullet = 1$ is $s_\bullet$-permissible, then $\mathrm{Fl}_{\mathcal{G},\leq w,1}$ is globally φ-regular and the maps $\mathrm{Fl}_{\mathcal{G},\leq v,1} \to \mathrm{Fl}_{\mathcal{G},\leq w,1}$ are compatibly φ-split closed immersions for $v \leq w$. Moreover, $\mathrm{Fl}_{\mathcal{I},\leq s_\bullet,1} \to \mathrm{Fl}_{\mathcal{G},\leq w,1}$ is a rational resolution.*

Proof. Global φ-regularity is preserved along proper birational maps, so the first claim follows. For the second claim, it is crucial to show that the induced map $\mathcal{O}_{\mathcal{G},\leq w,1} \to \mathcal{O}_{\mathcal{G},\leq v,1}$ is surjective. This is equivalent to showing that the ideal sheaf of $\mathrm{Fl}_{\mathcal{I},\leq s_{\bullet\setminus i},1}$ has vanishing higher direct images along the Demazure resolution. This is true by construction at the perfect level and we can therefore descend it to our deperfection by using a φ-splitting of $\mathrm{Fl}_{\mathcal{I},\leq s_\bullet,1}$ compatible with its boundary. As for rationality of the deperfected Demazure resolution, we can prove it in the same manner: we know that the higher direct images of the structure sheaf vanish at the perfect level and then it descends via a φ-splitting, compare with the discussion around [BS17, Lemma 6.9]. For the canonical sheaf, we know that pushforward respects dualizing complexes by Grothendieck–Serre duality and the preceding higher vanishing of the structure sheaf. Then, it suffices to observe that $\mathrm{Fl}_{\mathcal{G},\leq w,1}$ is globally $+$-regular, hence Cohen–Macaulay. □

We can also compute the Picard group of $\mathrm{Fl}_{\mathcal{G},\leq w,1}$ as follows:

Corollary 4.12. *If $q_\bullet = 1$ is $s_\bullet$-permissible, then the natural map $\mathrm{Pic}(\mathrm{Fl}_{\mathcal{G},\leq w,1}) \to \prod_{s\leq w}\mathrm{Pic}(\mathrm{Fl}_{\mathcal{G},\leq s,1})$ is an isomorphism.*

Proof. Due to rationality and compatible φ-splitness of the $\mathrm{Fl}_{\mathcal{G},\leq v,1}$, we can now write the deperfected Demazure resolution $\mathrm{Fl}_{\mathcal{I},\leq s_\bullet,1} \to \mathrm{Fl}_{\mathcal{G},\leq w,1}$ as a composition of rational maps with non-trivial fibers equal to $\mathbb{P}^1_k$, just like in [FHLR22, Lemma 4.5]. Then, one shows also inductively that the line bundle $\mathcal{O}(\nu)$ on $\mathrm{Fl}_{\mathcal{I},\leq s_\bullet,1}$ for some integral weight

ν of the torus $\widehat{S}$ pushes forward along the resolution to a line bundle on $\mathrm{Fl}_{\mathcal{G},\leq w,1}$, see [FHLR22, Lemma 4.20]. □

Remark 4.13. A consequence of this result is that we could now reprove global φ-regularity of $\mathrm{Fl}_{\mathcal{I},\leq s_\bullet,1}$ via the Mehta–Ramanathan criterion. Indeed, every ample line bundle on $\mathrm{Fl}_{\mathcal{G},\leq w,1}$ is globally generated, as restriction to the origin is surjective by φ-splitness, and the base locus is $\mathcal{I}(O)$-equivariant. Then, we can define a φ-splitting via the $(p-1)$-th power of a global section of $H^0(\mathrm{Fl}_{\mathcal{I},\leq w,1}, \mathcal{O}(\rho))$ not vanishing at the origin.

Another corollary of global $+$-regularity is the Demazure character formula. Recall that the central extension $\widehat{LG}$ acts on any line bundle $\mathcal{O}(\nu)$. In particular, if ν is sufficiently p-divisible, the line bundle $\mathcal{O}(\nu)$ descends to $\mathrm{Fl}_{\mathcal{I},\leq s_\bullet,q_\bullet}$, so the cohomology groups $H^i(\mathrm{Fl}_{\mathcal{I},\leq s_\bullet,q_\bullet}, \mathcal{O}(\nu))$ have an associated character counting dimensions of affine weight spaces, i.e., an element of the group ring $\mathbb{Z}[X^*(\widehat{S})]$ of affine weights. We use exponential notation for this group ring to avoid confusion with sums of coefficients and sums of weights (which correspond to multiplication in the ring).

Corollary 4.14. *Let $\nu \in X^*(\widehat{S})^+$ be a dominant weight. If $q_\bullet = 1$ is $s_\bullet$-permissible, then we have an equality*

$$\mathrm{char} H^0(\mathrm{Fl}_{\mathcal{I},\leq s_\bullet,1}, \mathcal{O}(\nu)) = \Lambda_{s_1} \circ \cdots \circ \Lambda_{s_n}(e^\nu) \tag{4.17}$$

where $\Lambda_s(e^\nu) = (1 - e^{-a_s})^{-1}(e^\nu - e^{\nu - \langle \alpha_s^\vee, \nu+\rho\rangle a_s})$ is the Demazure operator.

Proof. The usual inductive proof identifies the right side with the character of the Euler characteristic $\chi(\mathrm{Fl}_{\mathcal{I},\leq s_\bullet,1}, \mathcal{O}(\nu))$, compare with [Lit98, Theorem 7]. Global $+$-regularity of $\mathrm{Fl}_{\mathcal{I},\leq s_\bullet,1}$ yields vanishing of higher cohomology, so the equality follows. □

4.5. **Local models.** In this section, we briefly need to refer to the theory of v-sheaves as in [Sch17, SW20], but the reader may treat this as a black box. Consider the Beilinson–Drinfeld Grassmannian $\mathrm{Gr}_{\mathcal{G}}$ in the sense of [SW20] defined over the v-sheaf $\mathrm{Spd}\, O$. Its generic fiber is isomorphic to the B_{dR}^+-affine Grassmannian and its special fiber equals the v-sheaf attached to the affine flag variety $\mathrm{Fl}_{\mathcal{G}}$. Let μ be a geometric conjugacy class of coweights with reflex field E. Then, the affine Grassmannian $\mathrm{Gr}_{G,E}$ base changed to E contains a closed subsheaf $\mathrm{Gr}_{G,\leq\mu}$ arising as the L^+G-orbit closure of $\mu(\xi)$. Following [AGLR22], we define the v-sheaf local model $M^v_{\mathcal{G},\mu}$ as the v-sheaf closure of $\mathrm{Gr}_{G,E}$ inside $\mathrm{Gr}_{\mathcal{G},O_E}$.

If F has characteristic p or μ is minuscule, there exists by [AGLR22, Theorem 1.1] and [GL24, Corollary 1.4] a unique flat normal projective O_E-scheme $M_{\mathcal{G},\mu}$ with reduced special fiber whose associated v-sheaf equals $M^v_{\mathcal{G},\mu}$. We call it the scheme-theoretic local model and, with the single exception of wild odd unitary groups (so only when $p = 2$), it was shown in the corresponding statements of [AGLR22, GL24] (relying on [FHLR22, Theorem 1.2]) that this scheme has φ-split special fiber. We can now use our global $+$-regularity result to remove this assumption from the computation of the special fiber in [AGLR22, FHLR22, GL24] (see Remark 4.17 for Cohen–Macaulayness).

Before we state and prove it, we treat the deperfection of the μ-admissible locus $A_{\mathcal{G},\mu}$. Recall that the μ-admissible set Adm_μ of Kottwitz–Rapoport [KR00] consists of all elements w of the Iwahori–Weyl group bounded by the translation t_λ for some representative of μ. Then, the μ-admissible locus $A_{\mathcal{G},\mu}$ is the union of all Schubert varieties $\mathrm{Fl}_{\mathcal{G},\leq w}$ as w runs over all double cosets with lifts in Adm_μ (after fixing a Iwahori O-model $\mathcal{I}$ mapping to $\mathcal{G}$). Below, we show strong structure results on the so-called canonical deperfection of $A_{\mathcal{G},\mu}$ in the sense of [AGLR22, Definition 3.14], generalizing [AGLR22, Theorem 3.16].

Theorem 4.15. *Assume F has characteristic p or μ is minuscule. Then, $A_{\mathcal{G},\mu}$ has a unique φ-split deperfection $A_{\mathcal{G},\mu,1}$ admitting the deperfections $\mathrm{Fl}_{\mathcal{G},\leq t_\lambda,1}$ as compatibly φ-split closed subschemes for every representative λ of μ. Moreover, there is an equality*

$$\dim_k H^0(A_{\mathcal{G},\mu,1},\mathcal{L}) = \dim_E H^0(\mathrm{Gr}_{G,\leq\mu,1},\mathcal{O}(c_{\mathcal{L}})) \tag{4.18}$$

of dimensions of global sections of line bundles, where $\mathcal{L}$ is ample and $c_{\mathcal{L}}$ is its central charge.

Here, we have to regard the central charge $c_{\mathcal{L}}$ as a tuple of integers obtained by splitting the adjoint quotient of G into simple factors, then translating connected components of their flag varieties to the neutral one, and finally taking the central charge for the simply connected cover.

Proof. Note that t_λ has a reduced word for which the constant sequence 1 is permissible by the proof of [AGLR22, Lemma 3.15], so the statement is reasonable. We write $A_{\mathcal{G},\mu}$ as the finite colimit of its Schubert subvarieties $\mathrm{Fl}_{\mathcal{G},\leq w}$ along the various natural inclusion maps. Then, we define $A_{\mathcal{G},\mu,1}$ as the analogous colimit of the $\mathrm{Fl}_{\mathcal{G},\leq w,1}$: existence follows by writing is as successive pushouts along closed immersions and invoking [Sta23, Tag 0E25]. This is the unique deperfection birational to all the $\mathrm{Fl}_{\mathcal{G},\leq t_\lambda,1}$, and one checks that the natural maps are closed immersions.

We still have to construct a φ-splitting and we can assume that $\mathcal{G} = \mathcal{I}$ is a Iwahori. In order to glue the various φ-splittings that we have defined on the $\mathrm{Fl}_{\mathcal{I},\leq w,1}$, we must characterize them in unequivocal fashion. The φ-splitting provided by the Mehta–Ramanathan criterion depends only on the Demazure resolution and some global section θ of $\mathcal{O}((q-1)\rho)$ whose divisor $\mathrm{div}(\theta)$ avoids the origin. It is determined uniquely by a global section σ of the $(q-1)$-th power of the anti-canonical sheaf of $\mathrm{Fl}_{\mathcal{I},\leq w,1}$. A calculation away from codimension 2 strata shows that $\mathrm{div}(\sigma) = \mathrm{div}(\theta) + (q-1)\partial_{w,1}$. In other words, once we choose a global section θ of a $(q-1)$-th power of the ample line bundle $\mathcal{O}(\rho)$ of $A_{\mathcal{I},\mu,1}$ such that $\mathrm{div}(\theta)$ avoids the origin, we get compatible φ-splittings of all its subvarieties $\mathrm{Fl}_{\mathcal{I},\leq w,1}$.

Finally, we need to handle the calculation of the Euler characteristics. A Möbius inversion formula for posets yields

$$\chi(A_{\mathcal{G},\mu,1},\mathcal{L}) = \sum_{w<w_1<\cdots<w_n} (-1)^n \chi(\mathrm{Fl}_{\mathcal{G},\leq w},\mathcal{L}) \tag{4.19}$$

where the sum runs through all strict chains of μ-admissible double cosets. Together with the Demazure character formula and higher vanishing for ample $\mathcal{L}$, we get a purely combinatorial formula for the left side. We remark that, for triality factors, this appears trickier because the splitting field can have either degree 3 or 6, but one checks that the A_3-average and the S_3-average of an arbitrary coweight coincide. This means that we can reduce the proof of the equality to tame G in equicharacteristic (or even equicharacteristic 0!), so the result follows from [Zhu14, Theorem 3], see also [GL24, Theorem 2.1] for another proof. □

We can deduce from the above an identification of the special fiber of scheme-theoretic local models.

Corollary 4.16. *The special fiber of $M_{\mathcal{G},\mu}$ equals $A_{\mathcal{G},\mu,1}$.*

Proof. If $p > 2$ or Φ_G is reduced, this is [AGLR22, Theorem 1.1] and [FHLR22, Theorem 1.2]. The previous theorem states that the Hilbert polynomials of $A_{\mathcal{G},\mu,1}$ and $\mathrm{Gr}_{G,\leq\mu,1}$ coincide. By [GL24, Corollary 1.4], the special fiber of $M_{\mathcal{G},\mu}$ is reduced with weak normalization equal to $A_{\mathcal{G},\mu,1}$ (essentially by definition). By flatness, the special fiber also shares the same Hilbert polynomial as $\mathrm{Gr}_{G,\leq\mu,1}$. Since the weak normalization injects on structure sheaves, the quotient module has trivial Hilbert polynomial, so it necessarily vanishes, and this yields the claim. □

Remark 4.17. Our methods are not enough to show Cohen–Macaulayness for wild odd unitary G. Indeed, the argument in [FHLR22] relies on the existence of a Frobenius φ on the entire local model. We should mention that Yang [Yan24] proved Cohen–Macaulayness for wild odd unitary groups when $p = 2$ at special parahoric level working explicitly with lattice chains. In this case, the special fiber is irreducible, so we can also deduce it from our results by deforming Cohen–Macaulayness. It is plausible that recent developments in [BMP+24] might lead to an abstract uniform proof of Cohen–Macaulayness for arbitrary parahorics.

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