OBERSEMINAR PROGRAM SS 25: "AFFINE DELIGNE–LUSZTIG THEORY"

JOÃO LOURENÇO, EVA VIEHMANN, AND IOANNIS ZACHOS

1. Introduction

Starting with a seminal paper from 1976, Deligne–Lusztig realized representations of finite groups of Lie type in the cohomology of suitable subschemes of the associated flag variety and certain coverings. In the early 2000s, Lusztig generalized this to an analogue for algebraic groups that are the so-called jet schemes of connected reductive groups over finite fields. In recent years, and this is the topic of our seminar, Chan and Ivanov generalized this to an emerging affine Deligne-Lusztig theory for algebraic groups that arise as Moy-Prasad quotients of parahoric subgroup schemes of connected reductive groups G over a nonarchimedan local field F (associated with unramified maximal tori). Although this theory is still far from being as complete as in the finite field case, many important foundational results have been shown. Besides the definition of the affine Deligne-Lusztig representations, important goals in the seminar will be to show the analog of the Mackey formula, to compute the scalar product of affine Deligne-Lusztig characters, and to discuss how they (in the case of groups of type A) realize the Jacquet-Langlands correspondence.

The seminar consists of several parts. After an overview of the classical theory of Deligne-Lusztig, talks 2–4 define the affine DL representations and provide the necessary background on Moy–Prasad and parahoric subgroups. In talks 5–7, we get an overview on several important aspects of the theory of supercuspidal representations that we need later on. In talks 8–12 we return to our main topic and prove the above mentioned key results - talks 8 and 10 being concerned with the generic Mackey formula and the scalar product theorem, the last two talks focusing on GL_n and the relations to cuspidality and the Jacquet-Langlands correspondence.

2. Talk summaries

Talk 1: Review of classical Deligne-Lusztig theory (14.4). Recall the construction of the Deligne-Lusztig variety and the virtual representations R_T^θ. Discuss some of the main results of Deligne-Lusztig theory [Yos04, Theorem 1.1, Propositions 4.1, 4.2].

Talk 2: Moy–Prasad quotients and commutator relations (28.4). Introduce the Moy-Prasad filtration [CI21b, §2.4], define the affine smooth group $\mathbb{F}_q$-scheme $\mathbb{G}_r$ and certain subgroups of it [CI21b, §2.5-2.7]. Then describe its Bruhat-decomposition [CI21b, §2.8] and finish the talk mentioning some results on commutation relations [CI21b, §2.9], skipping the proof of [CI21b, Lemma 2.9]. Define the notion of regularity for characters θ of $T(\breve{F})$.

Talk 3: The scheme of triples (5.5). Sketch the proof of [CI21b, Theorem 3.2], which establishes an alternating sum formula for the cohomology of the scheme of triples Σ. You might want to skip some of the details of the proofs of [CI21b, §3.3-3.5].

Talk 4: Parahoric induction (12.5). Define the schemes $S_{T,U}$ and the virtual representations $R_{T,U,r}^\theta$ from [CI21b, §4] and prove [CI21b, Theorem 1.1]. Then, introduce the parahoric Lusztig induction [Cha24, §3].

Talk 5: Depth 0 supercuspidals (19.5). Recall the notion of maximal elliptic F-tori $T \subset G$ (see [Kal19, Subsection 3.4.1] for G unramified), define (extra regular) depth 0 characters θ as in [Kal19, Subsection 3.4.2] and the supercuspidal irreducible representations $\pi_{(S,\theta)}$. State its character formula for strongly regular elements, see [Kal19, Proposition 3.4.14] (sketch the proof, if time permits). Apply it to classify regular depth 0 supercuspidal representations of $G(F)$, check [Kal19, Lemma 3.4.18].

Talk 6: Yu construction of supercuspidals (26.5). Recall the notion of Yu data Ψ as in [Fin21, 2.1] and explain the construction of a smooth representation π_{Ψ}^{Yu} following [Fin21, 2.5]. Prove that it is irreducible and supercuspidal as in [Fin21, Theorem 3.1].

Talk 7: Howe factorization (2.6). Define refactorizations of Ψ as in [Kal19, Section 3.5] (skip proofs of lemmas involving z-extensions) and state that the associated supercuspidals are isomorphic. Define (extra) regular supercuspidals $\pi_{(S,\theta)}^{\mathrm{Yu}}$ and prove [Kal19, Lemma 3.6.9]. Define Howe factorizations of (S, θ) as in [Kal19, Definition 3.7.1] and prove their existence, see [Kal19, Proposition 3.7.4].

Talk 8: Generic Mackey formula (16.6). Goals: Prove the generic Mackey formula [Cha24, Theorem 4.5] which relates the Lusztig induction functors and their adjoints giving a formula for the composition ${}^*R^{\mathbb{G}_r}_{\mathbb{L}_r,\mathbb{Q}_r} \circ R^{\mathbb{G}_r}_{\mathbb{M}_r,\mathbb{P}_r}$.

Talk 9: Infinite-level affine Deligne–Lusztig varieties and the smooth models X_h (23.6). Define the semi-infinite Deligne–Lusztig sets and affine Deligne–Lusztig varieties at higher level [CI21a, Part 1] and discuss [CI21a, Theorem 4.9] and [CI21a, Corollary 4.10]. Then briefly mention their isomorphy for G of type A, see [CI21a, Theorem 6.9, Corollary 6.19]. Under the same assumption, study the geometry of $X^{\mathbb{G}_r}_{\mathbb{T}_r,\mathbb{B}_r}$ following [CI21a, §7], starting maybe with [CI21a, Proposition 7.12].

Talk 10: The scalar product formula (30.6). Discuss the proof of [Cha24, Main Theorem]. To do this first discuss the calculation of the cohomology of the fibers of $X^{\mathbb{G}_r}_{\mathbb{T}_r,\mathbb{B}_r} \to X^{\mathbb{G}_{r-1}}_{\mathbb{T}_{r-1},\mathbb{B}_{r-1}}$ ([Cha24, Theorem 5.1]). Then use this and [Cha24, Theorem 4.5] to prove the Main Theorem.

Talk 11: Cuspidality (7.7). Prove that for G of type A and regular θ, the irreducible representation $R^{\mathbb{G}_r}_{\mathbb{T}_r,\mathbb{B}_r}(\theta)$ does not contain the trivial representation when restricted to $\mathbb{U}^{\sigma}_{r:r+}$, following [CI21a, Theorem 9.1].

Talk 12: Automorphic induction and the Jacquet–Langlands correspondence (14.7). Discuss the main theorems ([CI21a, Theorems 11.3 and 12.5]) about the homology of the affine Deligne–Lusztig variety at infinite level $\dot{X}^{\infty}_{w}(b)$.

References

[Cha24] Charlotte Chan. The scalar product formula for parahoric deligne–lusztig induction, 2024. 1, 2

[CI21a] Charlotte Chan and Alexander Ivanov. Affine Deligne-Lusztig varieties at infinite level. *Math. Ann.*, 380(3-4):1801–1890, 2021. 2

[CI21b] Charlotte Chan and Alexander Ivanov. Cohomological representations of parahoric subgroups. *Represent. Theory*, 25:1–26, 2021. 1

[Fin21] Jessica Fintzen. On the construction of tame supercuspidal representations. *Compos. Math.*, 157(12):2733–2746, 2021. 2

[Kal19] Tasho Kaletha. Regular supercuspidal representations. *J. Amer. Math. Soc.*, 32(4):1071–1170, 2019. 2

[Yos04] Teruyoshi Yoshida. An introduction to Deligne-Lusztig theory, 2004. Available at https://math.umd.edu/~jda/748L/introDL.pdf. 1

Universität Münster, Einsteinstrasse 62, Münster, Germany
Email address: j.lourenco@uni-muenster.de

Universität Münster, Einsteinstrasse 62, Münster, Germany
Email address: viehmann@uni-muenster.de

Universität Münster, Einsteinstrasse 62, Münster, Germany
Email address: io.zachos@uni-muenster.de