OBERSEMINAR WS 2024/25

EUGEN HELLMANN, JOÃO LOURENÇO, LUCAS MANN

1. Summary

The classical Corlette–Simpson correspondence provides a description of representations of the fundamental group of a compact Kähler manifold X (over $\mathbb{C}$) in terms of so-called Higgs bundles (E, θ) on X, which are pairs consisting of a holomorphic vector bundle E on X and a Higgs field θ, i.e., a map $\theta\colon E \to E \otimes \Omega^1$ satisfying $\theta \wedge \theta = 0$. More precisely, there is a canonical equivalence of categories:

$$\left\{\begin{matrix}\text{representations } \pi_1(X) \to \mathrm{GL}(V) \\ \text{on fin. dim. } \mathbb{C}\text{-vector spaces } V\end{matrix}\right\} \cong \left\{\begin{matrix}\text{semistable Higgs bundles on } X \\ \text{with vanishing Chern classes}\end{matrix}\right\}$$

The p-adic Simpson correspondence (also known as p-adic non-abelian Hodge theory) seeks an analog of this result when X is replaced by a rigid-analytic variety over a p-adic field. This field has a rich history, which we do not want to discuss here, and has gained momentum in recent years due to new available methods in p-adic geometry. In particular, Heuer [7] proved the following analog of the above correspondence:

Theorem 1.1. *Let X be a proper smooth rigid-analytic variety over a complete algebraically closed extension K of $\mathbb{Q}_p$. Then there is an equivalence of categories*

$$\{v\text{-vector bundles on } X\} \cong \{\text{Higgs bundles on } X\}.$$

Here Higgs bundles are defined very similarly as in the complex case, while v-vector bundles (equivalently pro-étale vector bundles) form a suitable enlargement of the category of representations of the fundamental group. It is still an open question which conditions on a Higgs bundle guarantee that the associated v-vector bundle comes from an actual representation.

There are several variants of the above correspondence. For example, one can replace vector bundles by G-torsors for an algebraic (or even rigid) group G. Also, one can drop the assumption that X is proper, in which case one speaks of the *local p-adic Simpson correspondence*. One can also relax the condition that K is algebraically closed and instead consider finite extensions of $\mathbb{Q}_p$. In fact, already the case $X = \mathrm{Spa}(K)$ is interesting: Here the two sides of the above correspondence take the following form:

(a) Let C be the completed algebraic closure of K. By descent, a v-vector bundle on $\mathrm{Spa}(K)$ is the same as a representation of $\mathrm{Gal}(C/K)$ on a finite dimensional C-vector space.
(b) A Higgs bundle on $\mathrm{Spa}(K)$ is the same as a finite dimensional K-vector space together with an operator. Such a datum is also called a *Sen module*.

Thus the local p-adic Simpson correspondence on $\mathrm{Spa}(K)$ seeks a relation between Galois representations on C-vector spaces and Sen modules. Classically, this relation was studied by Sen [14] in what is now called *Sen theory*; among others it achieves the construction of a canonical *Sen operator* associated to a suitably nice Galois representation. The recent paper [1] discusses this classical result from a modern geometric perspective by identifying both of the above categories with sheaves on certain stacks. Namely, building on the prismatic site and its cohomology developed by Bhatt–Scholze [6], certain formal algebraic stacks over $\mathrm{Spf}(\mathbb{Z}_p)$ were constructed by Bhatt–Lurie [4]: their formalism provides a notion of Sen operators and Sen modules via coherent sheaves on the Hodge–Tate divisor of the primatization of $\mathcal{O}_K$. The main result of Anschütz–Heuer–Le Bras in [1] is then the following:

Theorem 1.2. *Let K be a finite extension of $\mathbb{Q}_p$. The isogeny category of perfect complexes on $\mathcal{O}_K^{\mathrm{HT}}$ embeds fully faithfully into the category of v-vector bundles on $\mathrm{Spa}(K)$ and the essential image consists precisely of "nearly Hodge–Tate" v-vector bundles.*

2. Talks

Talk 1: Recap on perfectoid spaces.
Recall the notion of perfectoid spaces, e.g. following [10, 1.2, 1.3] and the references therein (the tilting equivalence can be omitted). Show that affinoid perfectoid spaces are sheafy [10, Cor. 1.3.5] and compute the cohomology of the structure sheaf on an affinoid perfectoid [10, 1.3.2.1]. Mention the (far stronger) almost acyclicity [10, 1.3.2.2].

Talk 2: The v-topology.
Introduce the pro-étale site [10, 1.4.1] and in particular the structure sheaf thereon, as well as the v-topology [10, 1.4.2.1], [13, Lecture 17]. Briefly define the notion of a diamond [10, 1.4.2.2], [13, §8.3] (see also [11, §1.4] for a quick overview of the relation between diamonds and rigid varieties). Sketch the proof that v-vector bundles on a perfectoid space are free locally in the analytic/étale topology [13, Lemma 17.1.8]. Show that pro-étale cohomology for certain coverings can be computed in terms of continuous group cohomology [10, 1.4.4.1] in particular Proposition 1.4.38 and Lemma 1.4.39.

Talk 3: $\mathbb{G}_a$ and $\mathbb{G}_m$ in the arithmetic case.
Explain as an example that a v-vector bundle on $X = \mathrm{Spa}(K)$ is the same as a semi-linear Gal_K representation on a $\widehat{\overline{K}}$ vector space. And that this is the same as a semi-linear Γ-representation on a $\widehat{K_\infty}$ vector space, where K_∞ is the cyclotomic extension (this is an application of the fact that v-vector bundles and vector bundles on perfectoid spaces agree). Then compute the direct images of $R\nu_*\mathcal{O}$, where $\nu : X_{\mathrm{proet}} \to X$ (see [10, 1.4.4.2] and [15, 3.3, Theorem 1]) and $R\nu_*(\mathcal{O}^*)$ (see [1, Theorem 5.2, ff.]) in the arithmetic setting.

Talk 4: $\mathbb{G}_a$ in the geometric setting.
Compute $R\nu_*\widehat{\mathcal{O}}_X$, where $\nu : X_{\mathrm{proet}} \to X$, if X is a rigid analytic space over an algebraically closed field. See e.g. [10, 1.4.4, Theorem 1.4.36] and the references therein. Discuss the Hodge-Tate exact sequence

$$0 \to H^1(X, \mathcal{O}_X) \to H^1(X, \mathcal{O}_X) \to H^0(X, \Omega_X(-1)) \to 0,$$

for proper X as in [12, §3] and [3, §13].

Talk 5: $\mathbb{G}_m$ in the geometric setting.
In the same setting as in Talk 4 compute the cohomology $R\nu_*\widehat{\mathcal{O}}_X^\times$ and discuss the sequence

$$0 \to \mathrm{Pic}(X) \to \mathrm{Pic}(X_v) \to H^0(X, \Omega_X(-1)) \to 0,$$

(which can be regarded as the p-adic Simpson correspondence for generalized representations in the rank one situation) following [8, Theorem 1.3].

Talk 6: The local correspondence, arithmetic case.
For K a finite extension of $\mathbb{Q}_p$, the local Simpson correspondence on $X = \operatorname{Spa} K$ encodes classical Sen theory. Explain the main ideas behind Sen theory from [14] and in particular sketch the decompletion process [14, §1.3, Theorem 1] and the construction of the Sen operator in [14, Theorem 4] (see also [**?**, §15.1] in particular Theorem 15.1.2, Proposition 15.1.4 and Theorem 15.1.7 for a different exposition). Follow [1, §1.1] to discuss the subtleties in the correspondence and explain [1, Corollary 5.6] for the rank 1 case.

Talk 7: The local correspondence, geometric case.
Discuss the p-adic Simpson correspondence for small objects when X admits a toric chart $X \to \mathbb{T}^n$ see [7, §4] and [9, Theorem 6.5] (restricted to $G = \mathrm{GL}_n$).

Talk 8: Prismatic site.
Cover the basics of δ-rings following [6, §2.1-2.2] with emphasis on [6, Definition 2.1, Remark 2.2, and Lemma 2.18]. Introduce the notion and key examples of distinguished elements d of a δ-ring A as in [6, Definition 2.19, Example 2.20] and characterize their Teichmüller series if A is a perfect δ-ring, see [6, Lemma 2.33]. Define prisms (A, I) as in [6, Definition 3.2], prove their rigidity as in [6, Lemma 3.5], and classify perfect prisms via perfectoids, see [6, Theorem 3.10]. Finally, define the prismatic site equipped with a map toward the étale site, see [6, Definition 4.1, Construction 4.4], and state the Hodge–Tate comparison as in [6, Construction 4.9 and Theorem 4.11].

Talk 9: Hodge–Tate stack.
Define the prismatization of $\mathcal{O}_K$ following [4, Definition 3.1.4], show that it is representable by the stack quotient of a formal affine scheme by an affine group scheme as in [4, Proposition 3.2.3], and prove that it receives a map from $\mathrm{Spf}(A)$ for every $\mathcal{O}_K$-prism (A, I). Define the Hodge–Tate divisor $\mathcal{O}_K^{\mathrm{HT}}$ of the prismatization as in [4, Definition 3.4.1] or [2, Definition 5.1.6], deduce that perfectoid rings R map canonically to $\mathcal{O}_K^{\mathrm{HT}}$, see [4, Example 3.4.3], and identify it with classifying stack of the group G_π in [1, Definition 2.2] following [4, Theorem 3.4.13] and [5, Proposition 9.5]. Since the main source here [4, §3.1-3.4] assumes $K = \mathbb{Q}_p$, the speaker should refer to [1, §2.1] and [5, Example 9.6] for complements on the ramified case.

Talk 10: Prismatic Sen theory.
Define the category $\mathcal{D}(\mathcal{O}_K^{\mathrm{HT}})$ as in [4, Definition 3.5.1], construct the Sen operator Θ following [4, Construction 3.5.4] and compute it for Breuil–Kisin twists $\mathcal{O}_K^{\mathrm{HT}}\{n\}$. Prove [4, Theorem 3.5.8] and [1, Theorem 2.5] describing Hodge–Tate complexes via the Sen operator. Describe the exponential u^{Θ} for $u \in U_K^1$ as in [4, Proposition 3.7.1] and [1, Lemma 2.14] and reprove Sen's theorem on continuous semilinear representations of Gal_K following [4, Theorem 3.9.5], compare with [1, §2.3]

Talk 11: Galois cohomology of B_{en}.
Follow [1, §3] to analyze the Hodge–Tate stack of $\mathrm{Spf}(\mathcal{O}_K)$. In particular, define the ring B_{en} and compute its Galois cohomology [1, Theorem 3.12]. This will be crucial for the next talk.

Talk 12: Description of v-vector bundles via Hodge–Tate stacks.
Discuss the comparison between v-vector bundles on $\mathrm{Spa}(K)$ and vector bundles on the Hodge–Tate stack [1, §4]. In particular show the fully faithful embedding of the latter into the former [1, Theorem 4.2] and describe the essential image [1, Lemma 4.6], then explain the application to [1, Corollary 5.1]. If time permits,

explain how the *whole* category of v-vector bundles can be obtained by taking modules on the Hodge–Tate stack for increasing extensions L of K [1, Theorem 4.9].

References

[1] J. Anschütz, B. Heuer, A.-C. Le Bras, *v-vector bundles on p-adic fields and Sen theory via the Hodge-Tate stack*, https://arxiv.org/abs/2211.08470.

[2] B. Bhatt, *Prismatic F-gauges*, https://www.math.ias.edu/~bhatt/teaching/mat549f22/lectures.pdf.

[3] B. Bhatt, M. Morrow, P. Scholze, *Integral p-adic Hodge theory*, Publications mathématiques de l'IHÉS volume 128, (2018) 219–397.

[4] B. Bhatt, J. Lurie, *Absolute prismatic cohomology*, https://arxiv.org/abs/2201.06120.

[5] B. Bhatt, J. Lurie, *The prismatization of p-adic formal schemes*, https://arxiv.org/pdf/2201.06124.

[6] B. Bhatt, P. Scholze, *Prisms and prismatic cohomology*, Annals of Mathematics volume 196 (2022), 1135–1275.

[7] B. Heuer, *A p-adic Simpson correspondence for smooth proper rigid varieties.* https://arxiv.org/abs/2307.01303.

[8] B. Heuer, *Line bundles on rigid spaces in the v-topology.* Forum of Math., Sigma (2022) Vol 10:e82 1–46.

[9] B. Heuer, *Moduli spaces in p-adic non-abelian Hodge theory*, https://arxiv.org/abs/2207.13819.

[10] B. Heuer, *Perfectoid spaces*, available at https://bheuer.github.io/PerfectoidNotes.pdf.

[11] M. Mann, *A p-Adic 6-Functor Formalism in Rigid-Analytic Geometry*, https://arxiv.org/abs/2206.02022.

[12] P. Scholze, *Perfectoid spaces: a survey*, Current Developments in Mathematics 2012, 193–227. Int. Press, Somerville (2013).

[13] P. Scholze, J. Weinstein, *Berkeley lectures on p-adic geometry.*

[14] S. Sen, *Continuous cohomology and p-adic Galois representations*, Inventiones mathematicae, volume 62 (1980), 89–116.

[15] J. Tate, *p-divisible groups.*