ALGEBRAIC GEOMETRY II

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Abstract. These are lecture notes for the SS 24 course on Algebraic Geometry II at the Universität Münster.

Contents

1. Introduction

In the first installment of your journey through the sometimes monstruous world of algebraic geometry, you learned some of the fundamental notions of the subjects. This includes, but is likely not limited to, *sheaves*, i.e., the rigorous means by which we glue *functions* on *spaces*, and *schemes*, i.e., the rigorous means by which we glue *rings* into *spaces*. You spent a lot of time learning about different *geometric* properties of schemes involving dimensions, immersions, properness, flatness, etc.

The goal for this second part is to look instead at *cohomological* features in algebraic geometry, more concretely, coherent cohomology of schemes. Recall that in algebraic topology, one is interested in the study of homology $H_n(X, \mathbb{Z})$ and cohomology groups $H^n(X, \mathbb{Z})$ of a topological space X with $\mathbb{Z}$-coefficients. These invariants are important because they allow us to distinguish topological spaces up to homotopy. In algebraic geometry, there is a similar interest in understanding the cohomology groups $H^n(X, \mathcal{F})$ of a quasi-coherent sheaf $\mathcal{F}$ on a scheme X. We will explain the abstract framework needed to define these invariants, the various finiteness and continuity properties they enjoy under favorable situations, as well as how to compute them in concrete situations. The ultimate goal of the course is to explain Serre duality for proper smooth varieties over fields, which plays the same role for coherent cohomology that Poincaré duality does for singular cohomology.

The first chapter in our series covers infinitesimal properties. Last semester, you saw what a *smooth* map $f\colon X \to Y$ of schemes is, and studied their fibers in detail. A special class of smooth maps that one has to pay special attention to are those of dimension

0, also known as *étale* maps. They generate a very important topology that lead to a breakthrough in the late 60s/early 70s towards the Weil conjectures. Our approach to smoothness this time will center around the key notion of differential sheaves $\Omega_{X/Y}$, certain quasi-coherent sheaves on X that govern $\mathcal{O}_Y$-linear derivations of $\mathcal{O}_X$ towards arbitrary $\mathcal{O}_X$-modules. Another special focus will lie in characterizing smoothness and étaleness via lifting criteria along infinitesimal thickenings. We can summarize our main findings as follows.

Theorem 1.1. *The following properties hold*

1. *A map $f: X \to Y$ of schemes is smooth if and only if it is flat, locally of finite presentation, and with regular geometric fibers.*
2. *A map $f: X \to Y$ of schemes is étale if and only if it is flat, locally of finite presentation, and with discrete and reduced geometric fibers.*
3. *If f is smooth, then $\Omega_{X/Y}$ is a locally free $\mathcal{O}_X$-module of rank given by the dimension of the fibers of f.*
4. *If f is étale, then $\Omega_{X/Y} = 0$.*
5. *If f is finitely presented and formally smooth (i.e., it has the left lifting property against any ring map with nilpotent kernel), then f is smooth.*
6. *If f is finitely presented and formally étale (i.e., the lifts above are unique), then f is étale.*

It is unfortunate that there is no direct criterion for smoothness in terms only of $\Omega_{X/Y}$. If we wanted to do this, we would have to use an object of the derived category $D_{\mathrm{qc}}(X)$ called the *cotangent complex* $L_{X/Y}$, which is isomorphic to $\Omega_{X/Y}[0]$ exactly when f is smooth. In this course, we will try our best to avoid diving into derived methods, however we will not shy away from mentioning their existence, especially when it would lead to a more neat proof. Alongside smooth and étale maps, we will also look at unramified maps, but it should be made clear to all of you that these are much less important.

The next topic that we will handle will be homological algebra. This might disappoint some of you and appear like a dry subject, but it is not possible to discuss cohomology seriously without the rigorous tools required to defining it. The main idea in the subject goes more or less as follows: in algebraic geometry, one encounters plenty of functors that are either left or right exact, but not actually exact, such as the tensor product $\otimes$, the Hom functor, or the global sections functor Γ. This is quite a sad state of affairs, as it prevents us from computing the values of these functors via short exact sequences. However, we know that if we restrict to injectives I, then left exact functors F suddenly become exact. So, the solution is to *resolve* every object of our abelian category $A \to I^\bullet$ by injectives, then look at the cohomology groups of the complex $F(I^\bullet)$. This turns out to be independent of the choice of an injective resolution and defines the left derived functors of F.

Then, we will specialize this discussion to the category of quasi-coherent sheaves $\mathcal{M}$ on schemes X, whose definitions and basic properties will be reviewed. For projective schemes X, there is a nice construction of quasi-coherent sheaves in terms of graded modules, and we will cover these in some detail. The next order of business is coherent cohomology, so we define the cohomology groups $H^i(X, \mathcal{M})$ in terms of our homological machinery from the previous chapter. There is a lot to know about these groups, and we will compute them explicitly for line bundles on the projective space $\mathbb{P}^n_k$ over a field k. We summarize now some of the main features of coherent cohomology that we will look at:

Theorem 1.2. *The following properties hold:*

(1) *Let A be a ring, X be a proper A-scheme, and $\mathcal{M}$ be a coherent sheaf on X. Then, $H^i(X, \mathcal{M})$ is a finite A-module.*
(2) *Let $\mathcal{L}$ be an ample line bundle. Then $H^i(X, \mathcal{M} \otimes \mathcal{L}^j)$ vanishes for $i > 0$ and $j \gg 0$.*
(3) *Assume $\mathcal{M}$ is A-flat. Then, the function $s \mapsto \dim_{\kappa(s)} H^i(X_s, \mathcal{M}_s)$ on $\operatorname{Spec}(A)$ with X_s and $\mathcal{M}_s$ standing for the fibers over s, is upper semicontinuous.*

The first property is called finiteness of cohomology for proper morphisms and it ensures that we can have a good control over them. As you can probably guess, affines have vanishing higher cohomology, but quasi-affines can have a lot of higher cohomology (try out $\mathbb{A}^2_k \setminus \{0\}$). This is one of the main reasons why algebraic geometers love proper maps so much, because we can try to bound the ranks of their cohomology groups and it is much easier to fabricate situation where most of them vanish to suit our purposes. Serre vanishing is the main example of such a situation described in the second item. We need to use *line bundles* $\mathcal{L}$, i.e., locally free sheaves of rank 1: they hahave multiplicative inverses and are sometimes called invertible sheaves. Fix an ample line bundle, meaning a power of it is the pullback of $\mathcal{O}(1)$ along a closed immersion $X \to \mathbb{P}^n_A$ (in particular, X has to be projective). Then, we can make the higher cohomology of any coherent sheaf disappear by means of tensoring with large powers of $\mathcal{L}$. The final property mentioned above gives some constraints on the behavior of fiber dimensions of cohomology groups as we let points vary: while it is not true that the function is constant, we can show that the pre-image of $[0, c]$ for any non-negative integer is actually open. One can say more about the Euler characteristic $\chi(X, \mathcal{M}) = \sum(-1)^i H^i(X, \mathcal{M})$, which is constant on fibers. The Euler characteristic is a fundamental invariant that we will revisit below in the Riemann–Roch formula.

We plan on discussing also the topic of sites and descent. Long ago, Grothendieck had the decisive idea that one can generalize the notion of topology by sieving our space with charts that are *not* local isomorphisms. An example of this would be flat covers, which lead to the so-called fppf and fpqc covers. It is very natural to ask then if quasi-coherent sheaves are also sheaves for these new topologies. This question will lead us to consider glueing data for quasi-coherent sheaves along an fpqc cover $X \to Y$, and then wonder how to *descend* them to an actual quasi-coherent sheaf on Y. We will also show which properties of morphisms are local for which topologies. Descent plays an important role in proving flat base change for coherent cohomology.

As we saw above, line bundles are extremely important in algebraic geometry, and there is a lot of effort put into understanding their cohomology in the projective case. We will study the Picard group $\operatorname{Pic}(X)$ of line bundles on a scheme and relate it to divisors, i.e., codimension 1 cycles of X. Algebraic cycles are key objects in modern algebraic geometry, and some of the most important open conjectures in the field, such as the Hodge and the Tate conjecture revolve around understanding their contribution to Betti cohomology. Then, we will restrict our attention to projective curves C, where a lot of these concepts become rather simple, and compute the Euler characteristic of line bundles.

Theorem 1.3. *Let C be a projective smooth curve over a field k, and D be a divisor on C. Then, $\chi(C, \mathcal{O}(D)) = \deg(D) + 1 - g(C)$, where the degree $\deg(D)$ is the sum of zeros and poles with multiplicity of D and the genus $g(C) = \dim_k H^1(C, \mathcal{O})$.*

This gives a very explicit formula for computing the Euler characteristic of any line bundle $\mathcal{L}$ on C. Note that if we let $k = \mathbb{C}$, then there is a Riemann surface C^{an} obtained by analytifying C, and $g(C)$ is counting the number of holes in the resulting chain of aglutinated donuts. You could still complain that knowing $\chi(C, \mathcal{L})$ still leaves us hanging if we want to compute either $H^0(C, \mathcal{L})$ or $H^1(C, \mathcal{L})$. We will have learned that in certain situation, such as if the degree is strictly negative or quite large relatively to $g(C)$, then we can ensure the vanishing of H^0 (resp. H^1). We should also mention that there is a higher dimensional analogue of the Riemann–Roch formula, which is called the Grothendieck–Riemann–Roch formula (and builds on work of Hirzebruch on complex manifolds), but it goes beyond the scope of our course.

There is an underlying symmetry going on in the Riemann–Roch formula, which we will explain towards the end of the course. As you should already know by this point, mathematicians are fascinated with duality. Dualizing objects or maps can sometimes lead to a simplification of the original problem, we often encounter natural pairing maps between different but related quantities, etc. Poincaré duality, for instance, leads you to expect some type of palyndromic nature in the homology of closed real manifolds. In algebraic geometry, the corresponding notion is called Serre duality.

Theorem 1.4. *Let k be a field and X be a proper smooth k-variety of dimension n. We let $\omega_{X/k} = \wedge^n \Omega_{X/k}$ be the top exterior power of the differential sheaf. Then, there is a canonical isomorphism $H^i(X, \mathcal{E})^\vee \simeq H^{n-i}(X, \omega_{X/k} \otimes \mathcal{E}^\vee)$ for any vector bundle $\mathcal{E}$ on X.*

Here, a vector bundle $\mathcal{E}$ is a locally free coherent sheaf and $\vee$ denotes either the dual vector space or the dual vector bundle. Applied to curves, Serre duality tells us that $\dim_k H^1(C, \mathcal{O}(D)) = \dim_k H^0(C, \mathcal{O}(K_C - D))$, where K_X is a canonical divisor for C, i.e., such that $\mathcal{O}(K_C) = \omega_{C/k}$, and this often yields more concrete information revolving around the Riemann–Roch formula. Our proof of Serre duality uses cup product to study the top cohomology group $H^n(X, \omega_X)$, and then we need to define a trace isomorphism with k. For this, we will perform some reductions to the case when $X = \mathbb{P}^n_k$, so that there is no simple local definition of this map. Recent developments by Clausen–Scholze made it possible to define the local trace maps, but one should note that because cohomology on opens is usually finite, the duality in this theory takes place in some kind of complete topological sheaves they called *rigid sheaves*. As beautiful as this construction might be, it is too damned involved for us to really scratch its surface. Another more classical generalization of Serre duality is the so called *Grothendieck–Serre duality*: it uses the derived category, it applies to possibly non-smooth finitely presented maps of schemes $f\colon X \to Y$ with the main player being a certain *canonical complex* living in $D_{\mathrm{qc}}(X)$.

If time permits, we might be able to look at formal schemes, which are glued out of complete topological rings, prove the theorem on formal functions, a.k.a., formal GAGA for proper schemes, and then discuss Zariski's main theorem that places some severe constraints on how quasi-finite maps can look like and where certain completions play a decisive role. But this is very far from a promise, and quite unlikely!

1.1. **Book sources.** We recommend you to use the following 3 book sources besides these notes: Liu, Görtz–Wedhorn, and Vakil. We highly recommend consulting the Stacks Project, but only for technical minutia. We do not recommend using Hartshorne at all: a lot of respect is owed to this book for educating generations of algebraic geometers, but it has since become quite dated. Similarly, one should if possible avoid consulting the historically significant EGAs, because it is much more tricky to track down a result among all the humongous volumes.

2. Infinitesimal properties

2.1. **Conormal sheaf of an immersion.** Let $i : Z \to X$ be a closed immersion of schemes with defining ideal sheaf $\mathcal{I}$. We consider the pullback $i^*\mathcal{I} = \mathcal{I}/\mathcal{I}^2$ regarded as an $\mathcal{O}_Z$-module. More generally, we have the following definition.

Definition 2.1. Let $i\colon Z \to X$ be an immersion and $U \subset X$ is an open such that i factors through a closed immersion $Z \to U$. The *conormal sheaf* $\mathcal{C}_{Z/X}$ is the quasi-coherent $\mathcal{O}_Z$-module $\mathcal{I}/\mathcal{I}^2$ described above for $Z \to U$.

Remark 2.2. There is also a notion of *normal sheaf*

$$\mathcal{N}_{Z/X} = \mathcal{H}\mathrm{om}_{\mathcal{O}_Z}(\mathcal{C}_{Z/X}, \mathcal{O}_Z) \tag{2.1}$$

obtained by dualizing $\mathcal{C}_{Z/X}$. This provides an algebraic analogue of the normal bundle of a manifold embedding. The reason we work with the conormal objects is that unlike their duals they behave well under base change.

It is clear from the definition that, for any affine open $\mathrm{Spec}(R) = U \subset X$ such that $\mathcal{I}(U) = I$, we get $\Gamma(Z \cap U, \mathcal{C}_{Z/X}) = I/I^2$.

Lemma 2.3. *Let $i\colon Z \to X$ be an immersion of schemes. Given a map $g\colon X' \to X$, we denote by $i'\colon Z' \to X'$ the resulting immersion, resp. $f\colon Z' \to Z$ obtained by base change. There is a natural surjection of $\mathcal{O}_{Z'}$-modules*

$$f^*\mathcal{C}_{Z/X} \to \mathcal{C}_{Z'/X'} \tag{2.2}$$

which is an isomorphism if g is flat.

Proof. We may assume i is a closed immersion after replacing X by an adequate open. We are also allowed to work locally on the source and target of g, and hence assume that every scheme is affine. Clearly, any power of the ideal $I \subset R$ defining $Z \subset X$ maps to the same power of $I' = \mathrm{im}(I \otimes_R R' \subset R')$, so we have a map $I/I^2 \otimes R \to I'/I'^2$. This is clearly surjective, and in the flat case we get that $I^n \otimes_R R' \to (I')^n$ is an isomorphism for any n, which forces it to also be injective. □

Lemma 2.4. *Let $Z \xrightarrow{i} Y \xrightarrow{j} X$ be immersions of schemes. Then there is a canonical exact sequence*

$$i^*\mathcal{C}_{Y/X} \to \mathcal{C}_{Z/X} \to \mathcal{C}_{Z/Y} \to 0 \tag{2.3}$$

where the maps come from Lemma 2.3 and $i : Z \to Y$ is the first morphism.

Proof. This reduces immediately to the case where i and j are closed immersions, and all the involved schemes are affines. Passing to global sections, let $C \to B \to A$ be the corresponding ring surjections. Set $I = \mathrm{Ker}(B \to A)$, $J = \mathrm{Ker}(C \to A)$ and $K = \mathrm{Ker}(C \to B)$ so that $I = J/K$. This yields an exact sequence

$$K/K^2 \otimes_B A \to J/J^2 \to I/I^2 \to 0 \tag{2.4}$$

of A-modules, which translates into the desired statement. □

2.2. **Sheaves of differentials.** In this subsection, we introduce the notion of differentials on schemes. First, we need to understand this notion at the level of rings.

Definition 2.5. Let $A \to B$ be a ring map and let M be a B-module. An *A-derivation* into M is an A-linear map $D : B \to M$ satisfying the *Leibniz rule*: $D(ab) = aD(b) + bD(a)$.

We have a natural structure of B-module on the set $\mathrm{Der}_A(B, M)$ of A-linear derivations $D\colon B \to M$. Given a B-linear map $\alpha\colon M \to N$, we get an induced B-linear map $\mathrm{Der}_A(B, M) \to \mathrm{Der}_A(B, N)$ induced by post-compositing with α.

Definition 2.6. Let $A \to B$ be a homomorphism of rings. The *module of differentials* $\Omega_{B/A}$ is the B-module representing the functor $M \to \mathrm{Der}_A(B, M)$.

This means that we have natural isomorphisms $\mathrm{Hom}_B(\Omega_{B/A}, M) \simeq \mathrm{Der}_A(B, M)$, and applied to the identity map of $\Omega_{B/A}$ it yields a universal A-derivation $d_{B/A}\colon B \to \Omega_{B/A}$, provided $\Omega_{B/A}$ actually exists. This is what we now verify.

Lemma 2.7. *The module of differentials $\Omega_{B/A}$ always exists.*

Proof. Consider the free B-module $B^{\oplus B}$ on the elements of B itself with the canonical basis elements denoted by e_b. We want to realize $\Omega_{B/A}$ as a quotient of this free B-module, so we have to impose the relations coming from linearity and the Leibniz rule. Consider the submodule generated by either $e_{b+b'} - e_b - e_{b'}$ or $e_{bb'} - be_{b'} - b'e_b$ for every pair $(b, b') \in B^2$, and finally e_a for any $a \in A$. We let $\Omega_{B/A}$ be the corresponding quotient and define $d_{B/A}\colon B \to \Omega_{B/A}$ via $b \mapsto e_b$. This is an A-linear derivation by the relations imposed on $\Omega_{B/A}$. More generally, assume $d\colon B \to M$ is an A-linear derivation. We can define a B-linear map $B^{\oplus B} \to M$ taking e_b to $d(b)$. Applying the relations satisfied by A-linear derivations, we see that this map factors through $\Omega_{B/A}$, and it is also clear that this is the only map that works. □

Example 2.8. Let $B = A[x_1, \dots, x_n]$. Then $\Omega_{B/A}$ is a free B-module on the $d_{B/A}(x_i)$. Indeed, it is straightforward to check that an arbitrary collection of elements $m_i \in M$ for $i = 1, \dots, n$ induces a unique A-linear derivation $d\colon B \to M$ with $d(x_i) = m_i$: it is given by $d(p) = \sum_i (\partial p/\partial x_i) m_i$ for any $p \in A[x_1, \dots, x_n]$ due to the Leibniz rule.

Lemma 2.9. *Let*

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ A' & \longrightarrow & B' \end{array} \tag{2.5}$$

be a commutative diagram of rings. There is a canonical map

$$\Omega_{B/A} \otimes_B B' \to \Omega_{B/'A'} \tag{2.6}$$

of B'-modules. If (2.5) is cocartesian, then (2.6) is an isomorphism.

Proof. The universal A'-derivation $d_{B'/A'}\colon B' \to \Omega_{B/'A'}$ restricts to an A-derivation $B \to \Omega_{B'/A'}$, and hence we get a B-linear map $\Omega_{B/A} \to \Omega_{B/'A'}$, as required. If the diagram is a pushout, it suffices to show that $d_{B/A} \otimes_B B'$ is the universal A'-derivation of B. Given a derivation $d\colon B' \to M'$, we can restrict it to B and get a B-linear map $\Omega_{B/A} \to M'$. Now, its B'-linearization $\Omega_{B/A} \otimes_B B' \to M'$ composed with $d_{B/A} \otimes_B B'$ recovers d by construction. □

The previous lemma already shows that the module of differentials is local on the target and could be used with some care to construct the quasi-coherent sheaf $\Omega_{X/S}$ for a map $X \to S$ of schemes. We will define it instead as the conormal sheaf of the diagonal $X \to X \times_S X$ after the next two lemmas.

Lemma 2.10. *Let $A \to B \to C$ be maps of rings. Then, there is a canonical exact sequence*

$$\Omega_{B/A} \otimes_B C \to \Omega_{C/A} \to \Omega_{C/B} \to 0 \tag{2.7}$$

of C-modules.

Proof. We already saw how to produce the first map in Lemma 2.9 and the second one is quite similar. By the presentation of differential modules Ω, we see that the generators dc of $\Omega_{C/B}$ are in the image of those of $\Omega_{C/A}$. The new relations are obtained by the vanishing of db for $b \in B$ and hence lie in the image of $\Omega_{B/A}$. □

Lemma 2.11. *Let $p\colon B \to C$ be a ring epimorphism with kernel I and $A \to B$ be an arbitrary ring map. There is a canonical exact sequence*

$$I/I^2 \to \Omega_{B/A} \otimes_B C \to \Omega_{C/A} \to 0 \tag{2.8}$$

of C-modules. If p has a section, then (2.8) extends to a split short exact sequence.

Proof. If we consider the derivation $d_{B/A}\colon I \to \Omega_{B/A}$, then I^2 gets sent to $I\Omega_{B/A}$ by the Leibniz rule, so this yields the first arrow. The second arrow comes from Lemma 2.9 and is certainly surjective as p is. Clearly the two maps compose to 0. Using the presentation of $\Omega_{C/A}$, we can check that the kernel of the second arrow is generated by di with $i \in I$ as desired. Suppose now that p admits a section $C \to B$ and write $B = C \oplus I$. The functoriality of differential modules implies that the right arrow splits. On the other hand, the projection $B \to I/I^2$ is easily checked to be a derivation, and thus we get a section also on the left side (implying injectivity on the left as well). □

The following lemma gives a very clean definition of the sheaf of differentials without refering to derivations.

Lemma 2.12. *Let $A \to B$ be a map of rings. There is a canonical isomorphism between $\Omega_{B/A}$ and the conormal module I/I^2 for the multiplication map $B \otimes_A B \to B$.*

Proof. We have a short exact sequence

$$0 \to I/I^2 \to \Omega_{B\otimes_A B/B} \otimes_{B\otimes_A B} B \to \Omega_{B/B} \to 0 \tag{2.9}$$

because the multiplication map $B \otimes_A B \to B$ admits a section by any of the factor inclusions. Clearly, $\Omega_{B/B} = 0$ as all the generators db must vanish. Also, by base change, we have $\Omega_{B\otimes_A B/B} \simeq \Omega_{B/A} \otimes_B (B \otimes_A B)$, so the middle term identifies with $\Omega_{B/A}$ again, and we get the desired isomorphism. □

In particular, we see immediately that if $A \to B$ is an immersion, then $\Omega_{B/A}$ vanishes, and if $A \to B$ is finitely presented, so is the B-module $\Omega_{B/A}$.

Definition 2.13. Let $f\colon X \to S$ be a morphism of schemes. We define its *sheaf of differentials* by $\Omega_{X/S} := \mathcal{C}_{X/X\times_S X}$.

It is now clear that if f restricts to a map of affine opens $U \to V$, then $\Omega_{X/S}(U) = \Omega_{B/A}$, where $B := \Gamma(U, \mathcal{O}_X)$ and $A := \Gamma(V, \mathcal{O}_S)$. Therefore, this is truly a sheafified version of the module of differentials, and thus all exact sequences of modules of differential globalize to sheaves of differentials.

Corollary 2.14. *Let $f\colon X \to S$ be a morphism of schemes admitting a section $s\colon S \to X$. Then, we get a canonical isomorphism $s^*\Omega_{X/S} \simeq \mathcal{C}_{S/X}$.*

Proof. Apply Lemma 2.11 to the immersion $s\colon S \to X$ and the map $f\colon X \to S$. Then the right term $\Omega_{S/S}$ vanishes and we get an surjection $\mathcal{C}_{S/X} \to s^*\Omega_{X/S}$. Since the immersion s is split by f, we conclude that this is an isomorphism. □

Remark 2.15. In particular, if S is the spectrum of a field k, and x a k-valued point of X, then we see that the fiber $x^*\Omega_{X/k}$ is the k-linear dual of the tangent space $T_{X,x}$. Indeed, the latter had been defined as the k-linear dual of $\mathcal{C}_{x/X}$.

We complete our collection of right exact sequences as follows:

Lemma 2.16. *Let $i\colon Z \to X$ and $j\colon Z \to Y$ be immersions and $f\colon X \to Y$ a map with $f \circ i = j$. Then, the sequence*

$$\mathcal{C}_{Z/Y} \to \mathcal{C}_{Z/X} \to i^*\Omega_{X/Y} \to 0 \tag{2.10}$$

is exact.

Proof. Exactness on the right follows from Lemma 2.11 because $\Omega_{Z/Y}$ vanishes as $Z \to Y$ is an immersion. Exactness in the middle will be proved later with the help of the cotangent complex. □

2.3. **Flat morphisms.** In this subsetion, we will discuss the notion of flatness, which is quite particular to algebraic geometry and has no classical differential geometric analogue. In order to do this, we need to first discuss the corresponding notion over rings.

Recall that the tensor product functor $N \mapsto M \otimes_R N$ *commutes with colimits* and it is only *right exact*. The notion of flatness is meant to identify a fix for the lack of exactness.

Definition 2.17. Let A be a ring and M be an A-module. Then, an A-module M is called *flat* if the functor $N \mapsto M \otimes_A N$ is exact. It is moreover *faithfully flat* if the same functor is conservative, i.e., it reflects isomorphism.

We call a ring map $A \to B$ (faithfully) flat if B has the same property as an A-module. Localizations $B = S^{-1}A$ are clearly flat. We review some basic properties of flat modules.

Lemma 2.18. *Let A be a ring and Let $I, J \subset A$ be ideals. Let M be a flat A-module. Then $IM \cap JM = (I \cap J)M$.*

Proof. Tensor the exact sequence $0 \to I \cap J \to R \to R/I \oplus R/J$ with M and use that the kernel of $M \to M/IM \oplus M/JM$ is equal to $IM \cap JM$. □

Lemma 2.19. *(Faithful) flatness is stable under composition and base change.*

Proof. Composition is clear, because it preserves exactness and conservativity of functors. As for base change, the isomorphism $N \otimes_{A'} B' \simeq N \otimes_A B$ shows that we can identify the base changed tensor functor with the original one precomposed with the forgetful functor $\mathrm{Mod}_{A'} \to \mathrm{Mod}_A$, which is also exact and conservative. □

Remark 2.20. It is reassuring to know that to check flatness of an A-module M, it suffices to check injectivity of $I \otimes_A M \to M$ for finitely generated ideals $I \subset A$.

Lemma 2.21. *Let $A \to B$ be a faithfully flat ring map, M be an A-module, and set $N = M \otimes_A B$. Then M is A-flat if and only if N is B-flat.*

Proof. One of the directions was seen already. If N is B-flat, then $\otimes_A M$ becomes exact after composing with the conservative $\otimes_A B$, thus M is A-flat. □

Lemma 2.22. *Let $A \to B \to C$ be ring maps. If $A \to C$ is (faithfully) flat and $B \to C$ is faithfully flat, then $A \to B$ is also (faithfully) flat.*

Proof. We use the same idea as in the previous lemma. The functor $\otimes_A B$ becomes exact (and conservative) after tensoring with the conservative functor $\otimes_B C$, and thus B is (faithfully) flat. □

Lemma 2.23. *Let M be a flat R-module. The following are equivalent:*

1. *M is faithfully flat,*
2. *for all primes $\mathfrak{p} \subset R$, the fiber $M \otimes_R \kappa(\mathfrak{p})$ is non-zero,*
3. *for all maximal ideals $\mathfrak{m} \subset R$, the fiber $M \otimes_R \kappa(\mathfrak{m}) = M/\mathfrak{m}M$ is non-zero.*

Proof. If M were faithfully flat with trivial fiber over $\mathfrak{p}$, then $0 = \kappa(\mathfrak{p})$ by conservativity, so (1) implies (2) which trivially implies (3). Assume conversely that all maximal fibers of M are non-zero. It suffices to show that $\otimes_R M$ reflects zero objects. Clearly by flatness, we may assume $N = R/I$ and unless $I = R$ simply choose a maximal ideal $\mathfrak{m} \supset I$, so that $N \otimes_R M$ is non-zero. □

Lemma 2.24. *Let $R \to S$ be a flat ring map. The following are equivalent:*

1. *$A \to B$ is faithfully flat,*
2. *the induced map on spectra is surjective,*
3. *the induced map on spectra contains all closed points in its image.*

Proof. This follows quickly from Lemma 2.23 as the last two conditions relate to the fibers of $A \to B$ being non-zero. □

Lemma 2.25. *Let $A \to B$ be flat. The image of the induced map on spectra is stable under generalization.*

Proof. The maps of local rings induced by $A \to B$ are necessarily faithfully flat, because the maximal fibers become non-zero, see Lemma 2.24. The same result now tell us that these local maps are surjective on spectra, so the global map is generalizing. □

Now, we can finally define what it means for a map of schemes to be flat.

Definition 2.26. Let $f : X \to S$ be a morphism of schemes. We say that f is *flat* if $\mathcal{O}_{X,x}$ is a flat $\mathcal{O}_{S,f(x)}$-module.

More generally, for a quasi-coherent sheaf $\mathcal{M}$ on X, we say that it is S-flat if $\mathcal{M}_x$ is a flat $\mathcal{O}_{S,f(x)}$-module (so, f is flat iff the structure sheaf is S-flat).

Lemma 2.27. *Let $f : X \to S$ be a morphism of schemes. Then, f is flat if and only if, for any affine opens $U \subset X$ and $f(U) \subset V \subset Y$, the restriction $U \to V$ comes from a flat map of rings. Moreover, this can be checked on a single open affine cover.*

Proof. Let $A \to B$ be a ring map. As localizations are flat, we see that $A_{\mathfrak{p}} \to B_{\mathfrak{q}}$ also is for all primes. Conversely, if this is the case for all primes, then we can apply faithfully flat descent along $B \to \prod B_{\mathfrak{q}}$ to see that so is $A \to B$. □

A similar statement holds for quasi-coherent sheaves, with the exact same proof.

Lemma 2.28. *Flatness is stable under composition and base change.*

Proof. This reduces to the corresponding statement over rings, see Lemma 2.19. □

Lemma 2.29. *Let $f\colon X \to S$ be a flat map locally of finite presentation. Then, f is universally open.*

Proof. By Lemma 2.25, we know that f is generalizing. Since flatness and finite presentation are stable under base change, it suffices to show that the image of f is open. According to Chevalley's theorem, f has constructible image, and a generalizing constructible subset is necessarily open. □

2.4. **Smooth morphisms.** Recall that in differential geometry a smooth manifold has local charts that are isomorphic to opens of $\mathbb{R}^d$. In this subsection, we define a similar notion for maps of schemes $f\colon X \to Y$.

Definition 2.30. We say that a map of rings $A \to B$ is standard smooth if there exists a presentation $B = A[x_1, \dots, x_n]/(f_1, \dots, f_c)$ such that the Jacobian matrix

$$\mathrm{Jac}(f_1, \dots, f_c) := \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_c}{\partial x_1} & \cdots & \frac{\partial f_c}{\partial x_n} \end{pmatrix} \tag{2.11}$$

taken inside B has rank c. Let $f : X \to S$ be a morphism of schemes. We say that f is *smooth at* $x \in X$ if there are affine open neigborhoods $x \in U \subset X$ and $f(x) \in f(U) \subset V \subset S$ such that $U \to V$ is standard smooth. We say that f is *smooth* if it is smooth at every point of X.

A pleasing feature of this definition is that the smooth locus is automatically open. It is also a local property on source and target, by definition.

Remark 2.31. If $S = \mathrm{Spec}(k)$ and k is a field, then we are in the situation of smooth varieties. In particular, we know that a k-scheme X is smooth if and only if $X_{\bar{k}}$ is regular, where $\bar{k}$ denotes an algebraic closure of k.

Lemma 2.32. *Smooth morphisms are stable under composition and base change. Moreover, open immersions are smooth.*

Proof. Open immersions are clearly smooth as one takes the identity as the presentation itself. Stability under base change is also clear, because the presentation remains with of maximal rank Jacobian after extending scalars. For composition, we are reduced to considering two standard smooth maps $A \to B \to C$. We can write $C = B[y_1, \ldots, y_m]/(g_1, \ldots, g_d)$ and $B = A[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$. The only way to proceed is to lift the coefficients of the g_i to elements of $A[x_1, \ldots, x_n]$ and get the presentation $C = A[x_1, \ldots, x_n, y_1, \ldots, y_m]/(f_1, \ldots, f_c, g_1, \ldots, g_c)$. The Jacobian matrix $\mathrm{Jac}(f_1, \ldots, f_c, g_1, \ldots, g_d)$ with C-coefficients is block upper triangular with the diagonal blocks equal to $\mathrm{Jac}(f_1, \ldots, f_c)$ (taken in C) and $\mathrm{Jac}(g_1, \ldots, g_d)$, so it has maximal rank. □

We can understand the cotangent sheaf $\Omega_{X/S}$ of a smooth map $f \colon X \to S$ pretty well.

Lemma 2.33. *Let $f : X \to S$ be a smooth morphism of schemes. Then, the sheaf of differentials $\Omega_{X/S}$ is finite locally free of local rank given by the relative dimension of f.*

Proof. We may assume that f is a standard smooth map of affines. Given a presentation $B = A[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$, we have an exact sequence by Lemma 2.11

$$(f_1, \ldots, f_c)/(f_1, \ldots, f_c)^2 \to \oplus_{1 \le i \le n} B dx_i \to \Omega_{B/A} \to 0. \tag{2.12}$$

The image is a rank c free B-module on a cardinality c subset of the dx_i as it is given by the rank c matrix $\mathrm{Jac}(f_1, \ldots, f_c)$. This results in $\Omega_{B/A}$ also being a free B-module of rank $n - c$. Passing to geometric fibers and using the theory of tangent spaces, we know that the fibers of $\Omega_{B/A}$ have rank equal to the relative dimension d of f, and thus $d = n - c$. □

Thanks to Lemma 2.33, the following definition makes sense.

Definition 2.34. Let $d \geq 0$ be an integer. We say a morphism of schemes $f : X \to S$ is *smooth of relative dimension d* if f is smooth and $\Omega_{X/S}$ is finite locally free of constant rank d.

In other words, f is smooth and the nonempty fibres are equidimensional of dimension d. Unfortunately, local freeness of $\Omega_{X/S}$ does not imply smoothness. In order to fix this, we need to look at a notion of regularity for closed immersions.

Definition 2.35. Let A be a ring and $f_1, \ldots, f_c \in A[x_1, \ldots, x_n]$ be a sequence. Then, we say that the given sequence has regular fibers if, for every prime $\mathfrak{p} \subset A$, the reduction of f_i is not a zero divisor in $\kappa(\mathfrak{p})[x_1, \ldots, x_n]/(f_1, \ldots, f_{i-1})$. In this case, we say that $A \to B = A[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$ has global complete intersection (gci) fibers.

The order in which the f_i are placed is important in general, but will not matter when choosing a standard smooth presentation.

Lemma 2.36. *A standard smooth map $A \to B$ is flat with gci fibers. Moreover, I/I^2 is a rank c free B-module, where $I = (f_1, \dots, f_c)$.*

Proof. It will be convenient for us to define the intermediate quotient rings $B_i := A[x_1, \dots, x_n]/(f_1, \dots, f_i)$ for $0 \le i \le c$, so that we have $B_0 = A[x_1, \dots, x_n]$ a polynomial A-algebra, $B_i = B_{i-1}/f_i B_{i-1}$, and in the end $B_c = B$. We can check the assertion on gci fibers after passing to geometric fibers, and therefore assume that $A = k$ is an algebraically closed field. Indeed, regularity of a sequence is expressed in terms of injectivity of the multiplication maps of the corresponding elements, and this can be checked after faithfully flat base change. Note that all of the intermediate quotient rings $B_i := k[x_1, \dots, x_n]/(f_1, \dots, f_i)$ are regular because $\mathrm{Jac}(f_1, \dots, f_i)$ must have rank equal to i. In particular, they are all integral domains, and it suffices to check that $f_i \neq 0$ in B_{i-1} to conclude. But if any term in the sequence were superfluous $f_i \in (f_1, \dots, f_{i-1})$, then the rank $\mathrm{Jac}(f_1, \dots, f_i)$ could not possibly equal i, so we win.

Now, we prove that $A \to B$ is flat by induction on c and assume that A is a local noetherian ring with maximal ideal $\mathfrak{m}$ (we omit the necessary approximation argument in the non-noetherian case). Furthermore, we assume by induction that $A \to B_{i-1}$ is flat, the initial case $i = 0$ being the polynomial map $A \to A[x_1, \dots, x_n]$. By the first paragraph, we know f_i is not a zero divisor in $B_{i-1} \otimes \kappa(\mathfrak{m})$. Next, we show that f_i is also not a zero divisor in $B_{i-1}/\mathfrak{m}^n B_{i-1}$ for all n by induction on n. Note that we have a short exact sequence

$$0 \to \mathfrak{m}^{n-1} B_{i-1}/\mathfrak{m}^n B_{i-1} \to B_{i-1}/\mathfrak{m}^n B_{i-1} \to B_{i-1}/\mathfrak{m}^{n-1} B_{i-1} \tag{2.13}$$

of A-modules, where the left term identifies with $\mathfrak{m}^{n-1}/\mathfrak{m}^n \otimes_A B_{i-1}$ by flatness, hence f_i-torsion free. Assume by induction on n that the right term is f_i-torsion free as well. Then, it is not hard to show that the middle term of the short exact sequence is also f_i-torsion free, either by a diagram chase or by invoking the long exact sequence of Tor groups. This means that the submodule $B_i[f_i] \subset B_i$ of f_i-torsion is contained in the intersection of all the $\mathfrak{m}^n$. By Krull's intersection theorem applied to every local ring of B_{i-1}, we deduce that f_i is also not a zero divisor in B_i. Finally, we wish to show that $A \to B_i$ is flat, for which we consider the short exact sequence

$$0 \to B_{i-1} \xrightarrow{\cdot f_i} B_{i-1} \to B_i \to 0 \tag{2.14}$$

of A-modules, the left and middle term being A-flat. Either by a diagram chagewe or by using Tor_1, we quickly reduce to B_{i-1}/IB_{i-1} being f_i-torsion free for every ideal $I \subset A$. But if we run the same argument from the beginning replacing A by A/I, then we get the desired conclusion. By the way, the argument in this paragraph is sometimes called, e.g., Vakil, the slicing lemma for flatness.

For the last assertion, there is a surjective map $B^{\oplus c} \to I/I^2$ of B-modules. To check its injectivity, assume we can find polynomials $p_i \in A[x_1, \dots, x_n]$ with $1 \le i \le c$ such that $\sum p_i f_i$ lies in I^2. Notice that while proving flatness in the middle paragraph, we also saw that f_i is not a zero divisor in B_{i-1}. Hence, we see that $p_c f_c$ is a multiple of f_c^2 in B_{c-1}, and thus $p_c \in I$. By symmetry, we conclude that all the p_i lie in I and our map of B-modules is thus an isomorphism. □

The following characterizes a smooth morphism as a flat, finitely presented morphism with regular geometric fibres.

Theorem 2.37. *Let $f : X \to S$ be a morphism of schemes. Then, f is smooth if and only if f is flat, locally of finite presentation, and with regular geometric fibers.*

Proof. By definition, smooth maps are locally of finite presentation. By Lemma 2.32, we also know that its geometric fibers are smooth, and we already know that this can occur if and only if they are regular (note that the base field is *algebraically closed*). In the previous lemma, we saw that smooth maps are also flat.

Conversely, suppose f is flat, locally of finite presentation and with regular geometric fibers. In particular, the fibers of f are smooth and after restricting to appropriate affine opens, we may assume that they are standard smooth, hence gci by the above lemma. We claim that the associated map $A \to B$ has gci fibers given by a single unique sequence $f_1, \dots, f_c$ after inverting an element in B. Indeed, we can write $B = A[x_1, \dots, x_n]/I$ and a more refined version of the argument in the previous lemma shows that, after inverting an element of B, the B-module I/I^2 is free of rank c. Choose elements $f_1, \dots, f_c \in I$ defining a B-basis modulo I^2 and consider the finitely presented A-algebra $C = A[x_1, \dots, x_n]/(f_1, \dots, f_c)$. We first claim that C is standard smooth under the given presentation. Indeed, after base change, we may let $A = k$ be an algebraically closed field, so by smoothness of geometric fibers, we know $\Omega_{B/k}$ is free of rank $d = n - c$ and the right exact sequence of cotangent sheaves shows that the image of $I/I^2 \to \oplus_{1 \leq i \leq n} B dx_i$ is free of rank c. But $f_1, \dots, f_c$ give a B-basis of I/I^2 and the natural map is given by $\mathrm{Jac}(f_1, \dots, f_c)$, so we derive the required condition for standard smoothness of $A \to C$ (because invertibility of a minor lifts from the $\kappa(\mathfrak{p})$-fiber to the local ring). Next, we claim that the surjection $C \to B$ becomes an isomorphism on fibers, equivalently by A-flatness of B and C, that there is an equality $I \otimes_A \kappa(\mathfrak{p}) = (f_1, \dots, f_c) \otimes_A \kappa(\mathfrak{p})$ for all primes $\mathfrak{p} \subset A$. This is implied by the fact that $C \otimes_A \kappa(\mathfrak{p}) \to B \otimes_A \kappa(\mathfrak{p})$ is a surjection of regular rings of the same dimension, so by Krull's Hauptidealsatz it must be an isomorphism. As a consequence, we can now see that the C-fibers of $J = I/(f_1, \dots, f_c)$ vanish as well, so by Nakayama applied to every local ring of C we deduce that $J = 0$, and hence $B = C$ is a standard smooth A-algebra. □

Here is a differential criterion of smoothness at a point.

Lemma 2.38. *Let $f : X \to S$ be a morphism of schemes, locally of finite presentation. Let $x \in X$ and set $s = f(x)$. The following are equivalent:*

(1) *The morphism f is smooth at x.*
(2) *The local ring map $\mathcal{O}_{S,s} \to \mathcal{O}_{X,x}$ is flat and $X_s \to \mathrm{Spec}(\kappa(s))$ is smooth at x.*
(3) *The local ring map $\mathcal{O}_{S,s} \to \mathcal{O}_{X,x}$ is flat and the $\kappa(x)$-vector space $\Omega_{X/S,x} \otimes_{\mathcal{O}_{X,x}} \kappa(x)$ has dimension equal to $\dim_x(X_{f(x)})$.*

Proof. We already saw the equivalence of the first two properties, and that the second one implies the third. Note that by stability under base change, we can compute the $\kappa(x)$-fiber of the stalk $\Omega_{X/S,x}$ by assuming $S = \mathrm{Spec}(\kappa(s))$ and $X = X_s$. But then, we see by the Jacobian presentation that this vector space is the dual of $T_x X$ and the condition means the $\kappa(s)$-scheme X is smooth, so we have proved equivalence with the second property. □

In the next few lemmas, we prove that some of the right exact sequences we encountered for differential sheaves actually turn out to be short exact under smoothness assumptions.

For this, we rather need more sophisticated machinery than differential sheaves, which yield another characterization of smoothness.

Definition 2.39. Let B be a A-algebra equipped with a finite presentation $\alpha\colon B \simeq A[x_1, \dots, x_n]/I$ with $I = (f_1, \dots, f_c)$. We associate to α its *truncated cotangent complex*

$$\tau_{\geq -1} L_{B/A} := [I/I^2 \to \Omega_{A[x_1,\dots,x_n]} \otimes_A B] \tag{2.15}$$

sitting in cohomological degrees $[-1, 0]$.

Remark 2.40. We stress that as a complex, $\tau_{\geq -1} L_{B/A}$ is *not* independent of the choice of a presentation α. Instead, it is true that any two such presentations induce homotopic, thus quasi-isomorphic, complexes. Therefore, the truncated cotangent complex $\tau_{\geq -1} L_{X/S}$ for a finitely presented map $f\colon X \to S$ of schemes exists only in the derived category $D_{\mathrm{qc}}(X)$ of quasi-coherent sheaves on X. We will try to avoid using the derived category in this course, but you should be aware of its key important in algebraic geometry.

We see immediately from Lemma 2.11 that $H^0(L_{X/S}) = \Omega_{X/S}$. As for the -1-th cohomology group, we have the following:

Proposition 2.41. *Let $f\colon X \to S$ be a finitely presented map of schemes. Then, f is smooth if and only if $H^{-1}(L_{X/S}) = 0$ and $\Omega_{X/S}$ is locally free.*

Proof. Note that if f is smooth, then $I/I^2 \to \oplus_{1\leq i\leq n} B dx_i$ is split injective, because the left side is free of rank c and the image is a direct summand of rank c by our assumptions on $\mathrm{Jac}(f_1, \dots, f_c)$. This implies that $H^{-1}(L_{B/A}) = 0$. Conversely, the vanishing of H^{-1} implies that I/I^2 injects into the middle term. On the other hand, $\Omega_{X/S}$ is locally free, so the short exact sequence locally splits, and we deduce that I/I^2 is locally free and the corresponding Jacobian matrices have maximal rank. □

Remark 2.42. Actually f is smooth if and only $L_{X/S}$ is concentrated in degree 0 and $\Omega_{X/S}$ is locally free, but constructing that object and proving this goes way beyond the scope of these notes.

Lemma 2.43. *Let $f : X \to Y$, $g : Y \to S$ be morphisms of schemes. Assume f is smooth. Then, the sequence*

$$0 \to f^*\Omega_{Y/S} \to \Omega_{X/S} \to \Omega_{X/Y} \to 0 \tag{2.16}$$

(see Lemma 2.10) is short exact.

Proof. Since the assertion is local, we can pass to affine opens and consider the maps of rings $A \to B \to C$ with $B \to C$ standard smooth. We have to prove that the map $C \otimes_B \Omega_{B/A} \to \Omega_{C/A}$ is injective. Using compatible presentations, compare with Lemma 2.32, one shows that there is a distinguished triangle of cotangent complexes

$$L_{B/A} \otimes^{\mathbb{L}}_B C \to L_{C/A} \to L_{C/B}, \tag{2.17}$$

where the tensor product on the left is derived: this means there is some Tor correction going on in strictly negative degrees, but nothing changes in degree 0 which we care about. (By the way, this is an example of the general principle that many anomalies, singularities or obstructions can be partially fixed at the *derived* level.) Taking the long exact sequence of cohomology, it suffices to know that $H^{-1}(L_{C/B}) = 0$ and this was done in the previous lemma. □

Lemma 2.44. *Let Z be a smooth S-scheme and $i : Z \to X$ be an immersion of locally finitely presented S-schemes. Then, the exact sequence*

$$0 \to \mathcal{C}_{Z/X} \to i^*\Omega_{X/S} \to \Omega_{Z/S} \to 0 \tag{2.18}$$

of Lemma 2.11 is short exact.

Proof. We pass to corresponding affine opens, so that $A \to B \to C$ are the corresponding ring maps with $A \to C$ smooth and $B \to C$ surjective with kernel I. By Lemma 2.11, we have to show that $I/I^2 \to C \otimes_B \Omega_{B/A}$ is injective. Again, we can deduce this from the long exact sequence of cohomology attached to the short exact sequence of cotangent complexes thanks to the vanishing of $\Omega_{C/B}$ and $H^{-1}(L_{C/A})$. □

Lemma 2.45. *Let $i\colon Z \to X$ and $j\colon Z \to Y$ be immersions and $f\colon X \to Y$ a smooth map with $f \circ i = j$. Then, the sequence*

$$0 \to \mathcal{C}_{Z/Y} \to \mathcal{C}_{Z/X} \to i^*\Omega_{X/Y} \to 0 \tag{2.19}$$

of Lemma 2.16 is exact.

Proof. Again, we may pass to affine opens and consider the ring maps $A \to B \to C$ with $A \to C$ surjective and $A \to B$ standard smooth. We must show that $I/I^2 \to J/J^2$ is injective where $I = \ker(A \to C)$ and $J = \ker(B \to C)$. This is implied by the vanishing of $H^{-1}(L_{B/A})$. □

Lemma 2.46. *Let $p\colon X \to S$ and $q\colon Y \to S$ be two S-schemes locally of finite presentation and $f\colon X \to Y$ be an S-map. If p is smooth and f is a smooth cover, then q is also smooth.*

Proof. We will see later that flat is a local property in the fpqc topology, which implies that q is flat. In particular, it suffices to consider the geometric fibers of q, p, and f, so we take S as the spectrum of an algebraically closed field k. Let $x \in X$ and $y = f(x) \in Y$. Suppose f has relative dimension a at x, i.e., the fiber X_y has dimension a, and X has dimension b at x: flatness tells us that Y has dimension $b - a$ at y. By Lemma 2.33, we know that $\Omega_{X/S,x}$ is free of rank b and $\Omega_{X/Y,x}$ is free of rank a. Now, Lemma 2.43 implies that $(f^*\Omega_{Y/S})_x$ is free of rank $b - a$. Hence we conclude that $Y \to S$ is smooth at y by Lemma 2.38. □

We mentioned in the beginning that smoothness for real manifolds means that it has local charts isomorphic to opens of $\mathbb{R}^d$. While local charts for schemes do not exist in the Zariski topology, we can say something about complete local rings.

Lemma 2.47. *Let $f : X \to S$ be a morphism of schemes admitting a section σ. Let $s \in S$ be a point such that f is smooth at $x = \sigma(s)$. Then, there exists an isomorphism $\hat{\mathcal{O}}_{X,x} \simeq \hat{\mathcal{O}}_{S,s}[[x_1, \ldots, x_n]]$.*

Proof. Passing to appropriate affines, we may assume that f is induced by a standard smooth map of rings $A \to B$ and σ by an epimorphism $B \to A$ with kernel I. We can identify $I/I^2 \simeq \Omega_{B/A} \otimes_B A$, so it is a free A-module of rank n. Let $x_1, \dots, x_n \in I$ be lifts of an A-basis of I/I^2 and define the resulting map

$$\hat{\mathcal{O}}_{S,s}[[x_1, \dots, x_n]] \to \hat{\mathcal{O}}_{X,x}. \tag{2.20}$$

This is a surjection by construction. On the other hand, it is also faithfully flat, because it arises as the local completion of an étale map, and thus in particular injective. □

2.5. **Unramified morphisms.** We briefly discuss unramified morphisms before the much more interesting class of étale morphisms.

Definition 2.48. Let $f : X \to S$ be a morphism of schemes. We say that f is unramified if it is locally of finite presentation and $\Omega_{X/S} = 0$.

Remark 2.49. Some divergence in the literature persists on whether unramified maps have to be locally of finite type or locally of finite presentation, but here we stick with the latter for consistency, knowing that in the comfy noetherian world nothing bad happens anyway.

Lemma 2.50. *Unramified maps are stable under composition and base change.*

Proof. This is true for locally finitely presented maps, so stability under composition follows from Lemma 2.10 and under base change from Lemma 2.9. □

Lemma 2.51. *A finitely presented immersion $i : Z \to X$ is unramified.*

Proof. Obvious by definition, as $\Omega_{Z/X}$ vanishes. □

Lemma 2.52. *Let $f : X \to S$ be morphism of schemes. If f is unramified, then it is locally quasi-finite.*

Proof. By stability under base change, see Lemma 2.9, we know that $\Omega_{X/S} \otimes_{\mathcal{O}_S} k = \Omega_{X_k,k}$ for any field k and any k-valued point of S. Therefore, the geometric fibers of f are smooth of dimension 0 by our criterion, as every scheme is flat over a field. Therefore, the geometric fibers are discrete, and this implies local quasi-finiteness of f. □

The following lemma characterizes an unramified morphisms as morphisms locally of finite type with unramified fibres.

Lemma 2.53. *Let $f : X \to S$ be a morphism of schemes locally of finite presentation. Then, f is unramified if and only if the geometric fibers are reduced and discrete.*

Proof. We saw during the previous lemma that an unramified f has smooth fibers of dimension 0, so they are discrete and reduced. Conversely, if f has reduced discrete fibers, then we see that $\Omega_{X/S} \otimes \kappa(s) = 0$ for any $s \in S$, so the stalk at any point x vanishes by Nakayama's lemma and f is unramified. □

Here is a characterization of unramified morphisms in terms of their diagonals.

Lemma 2.54. *Let $f : X \to S$ be a morphism locally of finite presentation. Then, f is unramified if and only if its diagonal $\Delta_f \colon X \to X \times_S X$ is an open immersion.*

Proof. If the diagonal is an open immersion, then $\mathcal{C}_{X/X\times_S X}$ vanishes and hence so does $\Omega_{X/S}$ by Lemma 2.12. Conversely, we may pass to affine opens and the map f is induced by an unramified map $A \to B$ of rings. Thus, if we set $I = \ker(B \otimes_A B \to B)$, then $I = I^2$ again by Lemma 2.12. By hypothesis, I is finitely generated and we can use the I-variant of Nakayama's lemma to find $f \in 1 + I$ annihilating I. Then $f^2 = f + if = f$ is an idempotent and so is $1 - f \in I$. This shows that Δ_f is a clopen immersion. ☐

Lemma 2.55. *Let $f : X \to S$ be a morphism of schemes, locally of finite presentation. Let $x \in X$ and set $s = f(x)$. The following are equivalent:*

(1) *The morphism f is unramified at x.*
(2) *The $\kappa(x)$-vector space $\Omega_{X/S,x} \otimes_{\mathcal{O}_{X,x}} \kappa(x)$ vanishes.*
(3) *We have $\mathfrak{m}_s\mathcal{O}_{X,x} = \mathfrak{m}_x$ and the field extension $\kappa(x)/\kappa(s)$ is finite separable.*

Proof. We have basically already seen that all of these are equivalent. For the third property, just note that if the fibers are smooth, then they must be given by disjoint unions of spectra of finite separable extensions. ☐

Lemma 2.56. *Let $f : X \to Y$ be a locally finitely presented morphism of locally finitely presented S-schemes. If $p\colon X \to S$ is unramified, then so is f.*

Proof. By assumption we have $\Omega_{X/S} = 0$. Hence $\Omega_{X/Y} = 0$ by Lemma 2.10 and thus f is unramified. ☐

2.6. Étale morphisms.

The Zariski topology is very coarse, as made clear by considering varieties over $\mathbb{C}$. Grothendieck's solution to this issue was to algebraize the notion of a local isomorphism in the complex-analytic topology, into what is now known as an étale map. In this section we will be handling these in full generality.

Definition 2.57. Let $f : X \to S$ be a morphism of schemes. We say that f is *étale* if it is smooth and unramified. We say that a map of rings $A \to B$ is *standard étale* if there is a presentation $B = A[x, y]/(f, gy - 1)$ with $f, g \in A[x]$, f monic and $f' \in B^\times$.

Remark 2.58. A standard étale map is standard smooth as $\mathrm{Jac}(f, gy - 1)$ is upper triangular with diagonal entries f', g being units in B. However, if $A \to B$ is standard smooth with of relative dimension 0, then it is étale, but not necessarily standard étale. Standard étale maps are also not stable under composition.

A morphism is étale if and only if it is smooth of relative dimension 0, because unramified fibers are discrete. Again, there is an open (possibly empty) étale locus for any $f\colon X \to S$.

Lemma 2.59. *Étale maps are stable under composition and base change.*

Proof. Both follow from the corresponding statements for unramified and smooth maps, see Lemmas 2.32 and 2.50. ☐

Note that unramified schemes over fields are automatically étale, so we understand already how fibers of étale maps look like due to the case of unramified maps. Next, we give the characterization of étaleness in terms of flatness and geometric fibers.

Lemma 2.60. *Let $f : X \to S$ be a morphism of schemes, locally of finite presentation. Then, f is étale if and only if it is flat and with discrete reduced geometric fibers.*

Proof. Again, this follows from the corresponding statements for smooth and unramified maps. □

Note that open immersions are étale, as these are local isomorphisms. In general, there are many étale maps that are not open immersions.

The following lemma says locally any étale morphism is standard étale. The proof is a bit convoluted and included only for self-containment. We encourage the reader to skip it without regrets.

Lemma 2.61. *Let $f : X \to S$ be an étale morphism of schemes. Then, for any $x \in X$, there exist affine open neighborhoods $x \in U \subset X$ and $s := f(x) \in f(U) \subset V \subset S$ such that $U \to V$ is induced by a standard étale map of rings.*

Proof. We can assume that f is a map of affines induced by an étale map $A \to B$ of rings. Let $\mathfrak{q} \subset B$ be any prime ideal and $\mathfrak{p} \subset A$ be its pullback. We assume that A is excellent and omit the trickier arguments in the non-excellent case. Let $C \subset B$ be the normalization of $A \to B$, so that $A \to C$ is finite because A is excellent. After localizing B, we may assume it is a principal localization of C. Note that $C \otimes_A \kappa(\mathfrak{p})$ is a product of finite $\kappa(\mathfrak{p})$-algebras indexed by the fibers of the map $A \to C$ on spectra, with the $\mathfrak{q}$-factor being isomorphic to $\kappa(\mathfrak{q})$ by étaleness of $A \to B$. Let c be an element that generates the $\mathfrak{q}$-factor as a $\kappa(\mathfrak{p})$-algebra and consider the corresponding map $A[x] \to C$ with kernel I. Its image D is still finite over A and is locally isomorphic to B around $\mathfrak{q}$. The ideal I is generated modulo $\mathfrak{p}$ by a polynomial $h \in I \subset A[x]$. After localizing A, we may assume the image $\bar{h} \in \kappa(\mathfrak{p})[x]$ is monic, but it is not clear that h is monic yet (choosing it so could a priori make us leave ideal I). By étaleness of $A \to B$, we check that $\bar{h}$ factors as a product $\bar{h}_{\mathfrak{q}}\bar{h}^{\mathfrak{q}}$ with the first terms monic irreducible vanishing at $\mathfrak{q}$ and the other term prime to $\mathfrak{q}$. Summing a large monic polynomial in I to h, we can find a monic $f \in I$ whose image in $\kappa(\mathfrak{p})[x]$ shares a similar factorization, i.e., it has multiplicity 1 at $\mathfrak{q}$. This means its derivative $g := f'$ is invertible at $\mathfrak{q}$, and hence $A[x, y]/(f, gy - 1)$ is standard étale. It is not too difficult to check now that it is locally isomorphic to B around $\mathfrak{q}$. □

Here is a differential criterion of étaleness.

Lemma 2.62. *Let $f : X \to S$ be a morphism of schemes, locally of finite presentation. Let $x \in X$ and set $s = f(x)$. The following are equivalent:*

1. *The morphism f is étale at x.*
2. *The local ring map $\mathcal{O}_{S,s} \to \mathcal{O}_{X,x}$ is flat and the $\kappa(x)$-vector space $\Omega_{X/S,x} \otimes_{\mathcal{O}_{X,x}} \kappa(x)$ is zero.*
3. *The local ring map $\mathcal{O}_{S,s} \to \mathcal{O}_{X,x}$ is flat, we have $\mathfrak{m}_s\mathcal{O}_{X,x} = \mathfrak{m}_x$ and the field extension $\kappa(x)/\kappa(s)$ is finite separable.*
4. *There exist affine opens $U \subset X$, and $V \subset S$ such that $x \in U$, $f(U) \subset V$ and the induced morphism $f|_U : U \to V$ is standard smooth of relative dimension 0.*
5. *There exist affine opens $U \subset X$, and $V \subset S$ such that $x \in U$, $f(U) \subset V$ and the induced morphism $f|_U : U \to V$ is standard étale.*

Proof. Left as an exercise, as it follows rapidly from our previous work. □

Our proof of the following lemma uses the critère de platitude par fibres to see that a morphism $X \to Y$ of S-étale schemes is automatically flat.

Lemma 2.63. *Let $f : X \to Y$ be a morphism of schemes over S, locally of finite presentation. If X and Y are étale over S, then f is étale.*

Proof. The geometric fibers of f are closed subschemes of those of $p\colon X \to S$, so we conclude that they are discrete and reduced as well. Besides, we know that X and Y are flat S-schemes, and the geometric fibers of f with respect to S are maps of discrete and reduced varieties over an algebraically closed field, hence trivially flat. This means we can apply the critère de platitude par fibres to get flatness of f itself. □

The next permanence property is analogous to the one for smooth maps.

Lemma 2.64. *Let $p\colon X \to S$ and $q\colon Y \to S$ be maps of schemes, locally of finite presentation. If p is étale and $f\colon X \to Y$ is an étale cover of S-schemes, then q is étale.*

Proof. We already know by Lemma 2.46 that q is smooth. It is rather easy to check that q has relative dimension 0, by additivity of dimensions for flat maps, and the fact that f and p have relative dimension 0. □

Finally, we can describe smooth maps in terms of étale maps towards affine spaces.

Lemma 2.65. *Let $f : X \to Y$ be a smooth morphism of schemes. Let $x \in X$ and set $y = f(x)$. Then, there exist affine open neighborhoods $x \in U \subset X$ and $y \in f(U) \subset V \subset Y$ such that the restriction of f factors through an étale map $U \to \mathbb{A}^d_V$ over V.*

Proof. We may assume that f is induced by a standard smooth map of rings $A \to B$. Then, we can write $B = A[x_1, \dots, x_n]/(f_1, \dots, f_c)$, with $\mathrm{Jac}(f_1, \dots, f_c)$ having maximal rank in B. Say the invertible minor is given by the first c columns, i.e., involving the coordinates $x_1, \dots, x_c$. Then, it is clear that $A[x_{c+1}, \dots, x_n] \to B$ is standard smooth of relative dimension 0, thus étale. □

3. Homological algebra

3.1. **Complexes and homotopies.** In this section, we discuss the notions of complexes and homotopies in an additive category. This will allow us to define the cohomology of a complex and the associated long exact sequence starting from a short exact sequence of complexes. Before doing this, we need to explain what is meant by additive and abelian categories.

Definition 3.1. Let $\mathcal{A}$ be a category.

(1) The category $\mathcal{A}$ is *pointed* if it possesses an object that is both initial and final, denoted as 0. We call it a zero object in $\mathcal{A}$.
(2) A pointed category $\mathcal{A}$ is *additive* if it admits finite biproducts, (i.e., for any $X, Y \in \mathcal{A}$, there exists an object $X \oplus Y$ which is a product and coproduct), and the natural monoid $\mathrm{Hom}_{\mathcal{A}}(X, Y)$ is an abelian group.
(3) An additive category $\mathcal{A}$ is *abelian* if for all $f : X \to Y$ in $\mathcal{A}$, both the kernel $\ker(f) := 0 \times_{Y,f} X$ and the cokernel $\mathrm{coker}(f) := 0 \sqcup_{X,f} Y$ exist, and furthermore the natural map $\mathrm{coim}(f) := \mathrm{coker}(\ker(f) \to X) \to \ker(Y \to \mathrm{coker}(f)) =: \mathrm{im}(f)$ is an isomorphism.

Remark 3.2. Notice that the definition is intrinsic, i.e. given in terms of properties that ought to be satisfied. Often one finds an extrinsic definition in the literature, i.e., given in terms of some extra structure, but it turns out that this is not really necessary.

Let us look at some examples, which serve as some sanity check and for getting a better feel of the notion.

Example 3.3. The category Ab of abelian groups is an example of an abelian category. However, the full subcategory TorFr of torsion-free abelian group is only additive, but not abelian despite admitting kernels, as it lacks cokernels.

Example 3.4. More generally, we can consider the abelian category Mod_R of R-modules over a commutative ring R. It contains the full subcategory Proj_R of projective R-modules which is additive, but not abelian.

Definition 3.5. An additive functor is a functor $F : \mathcal{A} \to \mathcal{B}$ between additive categories that preserves direct sums and zero objects.

Now, we can define the notion of cochain complexes. The term cochain only means that indices increase along the maps, whereas in a chain complex they decrease. Since we deal mostly with cohomology in this course, we will drop the adjective cochain.

Definition 3.6. Let $\mathcal{A}$ be an additive category. A complex $A^\bullet$ in $\mathcal{A}$ consists of objects A^i of $\mathcal{A}$ for all integers i and morphisms $d^i : A^i \to A^{i+1}$ such that $d^{i+1} \circ d^i = 0$ for all i. A morphism of complexes $f^\bullet : A^\bullet \to B^\bullet$ is given by maps $f^i : A^i \to B^i$ such that $f^{i+1} \circ d^i = d^i \circ f^i$ for all i. We denote by $\mathrm{C}(\mathcal{A})$ the category of complexes of $\mathcal{A}$.

This is an additive category, and even abelian if so is $\mathcal{A}$. The full subcategory consisting of complexes $A^\bullet$ such that $A^i = 0$ for $i < 0$ is denoted $\mathrm{C}^{\geq 0}(\mathcal{A})$. Similarly, we define $\mathrm{C}^{\leq 0}(\mathcal{A})$ as the full subcategory of complexes $A^\bullet$ such that $A^i = 0$ for $i > 0$. One may also define analogous versions $\mathrm{C}^{[a,b]}(\mathcal{A})$ for any integers $a \leq b$. Finally, we let $\mathrm{C}^b(\mathcal{A})$ be the full subcategory of bounded complexes $A^\bullet$, meaning only finitely many of the A^i are non-zero. There is also the notion of left (resp. right) bounded complexes giving rise to the full subcategory $\mathrm{C}^+(\mathcal{A})$ (resp. $\mathrm{C}^-(\mathcal{A})$).

Given an additive category $\mathcal{A}$, we can identify $\mathcal{A}$ with the full subcategory of $\mathrm{C}(\mathcal{A})$ consisting of complexes $A^\bullet$ such that $A^i = 0$ when $i \neq 0$. We denote the complex attached to A by $A[0]$. Without further ado, let us discuss shift functors.

Definition 3.7. Let $\mathcal{A}$ be an additive category and $A^\bullet$ be a complex in $\mathcal{A}$. For any $k \in \mathbb{Z}$ we define the k-shifted chain complex $A[k]^\bullet$ as follows: we set $A[k]^n = A^{n+k}$, with transition maps $d^n \colon A^{n+k} \to A^{n+k+1}$ twisted by the sign $(-1)^k$. If $f : A^\bullet \to B^\bullet$ is a morphism of chain complexes, then we let $f[k] : A[k]^\bullet \to B[k]^\bullet$ be the morphism of complexes with $f[k]^n = f^{n+k}$.

In particular, we get endofunctors $[k]$ of $\mathrm{C}(\mathcal{A})$ such that $A[k][l]^\bullet = A[k+l]^\bullet$ and with $[0] = \mathrm{id}_{\mathrm{C}(\mathcal{A})}$. In accordance with our embedding $\mathcal{A} \to \mathrm{C}(\mathcal{A})$, the notation $A[k]$ for any object A of $\mathcal{A}$ indicates the complex obtained by placing A in degree $-k$ with vanishing terms otherwise. Please beware the mean sign!

Next, we concern ourselves with the crucial notion of homotopies.

Definition 3.8. A *homotopy* $h^\bullet$ between a pair of morphisms of complexes $f^\bullet, g^\bullet : A^\bullet \to B^\bullet$ is a collection of maps $h^i : A^i \to B^{i-1}$ such that $f^i - g^i = d^{i-1} \circ h^i + h^{i+1} \circ d^i$ holds for all i. In that case, we say that $f, g : A^\bullet \to B^\bullet$ are *homotopic*.

When $g = 0$, we simply say that f is nullhomotopic. This leads us to the definition of homotopy equivalences.

Definition 3.9. Let $\mathcal{A}$ be an additive category. A morphism $f : A^\bullet \to B^\bullet$ of complexes in $\mathcal{A}$ is a *homotopy equivalence* if there exists a map $g : B^\bullet \to A^\bullet$ such that $f \circ g - \mathrm{id}_B$ and $g \circ f - \mathrm{id}_A$ are nullhomotopic. In that case, $A^\bullet$ and $B^\bullet$ are said to be *homotopy equivalent* and g is a *homotopy inverse* to f.

Lemma 3.10. *Let $\mathcal{A}$ be an additive category. The set of homotopies between $f, g\colon A^\bullet \to B^\bullet$ two maps of complexes in $\mathcal{A}$ is either empty or in bijection with the* $\mathrm{Hom}_{\mathrm{C}(A)}(A^\bullet, B[1]^\bullet)$.

Proof. Let $h^i_j : A^i \to B^{i-1}$ define two nullhomotopies of $f - g$ for $j = 1, 2$. We see that the differences $h^i_{12} := h^i_1 - h^i_2$ define a nullhomotopy of 0. By the homotopy equation, this means $-d^{i-1} \circ h^i_{12} = h^{i+1}_{12} \circ d^i$. In particular, the h^i_{12} define a map of complexes $A^\bullet \to B[1]^\bullet$. Conversely, if there exists a nullhomotopy of $f - g$, we can modify it with such a map $A^\bullet \to B[1]^\bullet$ of complexes (regarded as a nullhomotopy of 0), and get another nullhomotopy of $f - g$. □

A special feature of derived categories that we will see later on is that they possess a certain rotation symmetry. For now, we can prove this at the level of complexes:

Lemma 3.11. *Let $0 \to A^\bullet \to B^\bullet \to C^\bullet \to 0$ be a short exact sequence of complexes with splittings $B^i = A^i \oplus C^i$ for all i (not necessarily compatible with the d^i). The resulting maps $C^i \to B^i \to B^{i+1} \to A^{i+1}$ define a map $C^\bullet \to A^\bullet[1]$ of complexes.*

Proof. Let $j^i\colon A^i \to B^i$ and $q^i\colon B^i \to C^i$ be the natural complex maps. Denote by $s^i\colon C^i \to B^i$ the individual sections, and by $\pi^i\colon B^i \to A^i$ the corresponding projections. The map $\theta^i\colon C^i \to A^{i+1}$ in the statement equals $\pi^{i+1} \circ d^i_B \circ s^i$. Note that $j^{i+1} \circ \pi^{i+1} = \mathrm{id}^{i+1}_B - s^{i+1} \circ q^i$, so it follows that $j^{i+1} \circ \theta^i = d^i_B \circ s^i - s^{i+1} \circ d^i_C$. Now, we compute $d^{i+1}_A \circ \theta^i$ by post composing with j^{i+2} which coincides with $d^{i+1}_B \circ (d^i_B \circ s^i - s^{i+1} \circ d^i_C) = d^{i+1}_B \circ s^{i+1} \circ d^i_C$. On the other hand, we have that $\theta^{i+1} \circ d^i_C$ is the only map which postcomposed with j^{i+2} yields $(d^{i+1}_B \circ s^{i+1} - s^{i+2} \circ d^{i+1}_C) \circ d^i_C = d^{i+1}_B \circ s^{i+1} \circ d^{i+1}_C$ as well. □

One can also show that if two splittings differ by maps $h^i\colon C^i \to A^i$, then these define a homotopy between the two different maps of complexes $C^\bullet \to A^\bullet[1]$.

Definition 3.12. Let $\mathcal{A}$ be an abelian category. The i-th *cohomology group* of a complex $A^\bullet$ is given by

$$H^i(A^\bullet) = \ker(d^i)/\mathrm{im}(d^{i-1}). \tag{3.1}$$

We denote by $H^i : \mathrm{C}(\mathcal{A}) \longrightarrow \mathcal{A}$ the resulting additive functor of cohomology.

The last sentence is reasonable, because a map $f : A^\bullet \to B^\bullet$ of complexes of $\mathcal{A}$ satisfies the inclusion $f^i(\ker(d^i_A)) \subset \ker(d^i_B)$ (resp. $f^i(\mathrm{im}(d^{i-1}_A)) \subset \mathrm{im}(d^{i-1}_B)$), so we get a corresponding map $H^i(f) : H^i(A^\bullet) \to H^i(B^\bullet)$ in cohomology.

Definition 3.13. Let $\mathcal{A}$ be an abelian category. We say that a map of complexes $f : A^\bullet \to B^\bullet$ in $\mathcal{A}$ is a *quasi-isomorphism* if $H^i(f) : H^i(A^\bullet) \to H^i(B^\bullet)$ are isomorphisms for all i. If $A^\bullet$ has vanishing cohomologies, then we say it is *acyclic*.

Lemma 3.14. *Let $\mathcal{A}$ be an abelian category. Given homotopic maps of complexes $f, g : A^\bullet \to B^\bullet$, we get an equality $H^i(f) = H^i(g)$. In particular, homotopy equivalences are quasi-isomorphisms.*

Proof. Subtracting g, we are reduced to showing that a nullhomotopic map $f : A^\bullet \to B^\bullet$ induces trivial maps upon taking cohomology. If we restrict $f^i = d^{i-1} \circ h^i + h^{i+1} \circ d^i$ to $\ker(d^i)$ then the second term in the sum on the right vanishes, so we see that $f^i(\ker(d^i)) \subset \mathrm{im}(d^{i-1})$. In particular, $H^i(f) = 0$. For the last claim, let $f : A^\bullet \to B^\bullet$ be a homotopy equivalence with a homotopy inverse g. We observe that $H^i(f \circ g) = 1 = H^i(g \circ f)$ and since we are in the presence of a functor H^i, this implies $H^i(f)$ and $H^i(g)$ are inverses to one another and thus f is a quasi-isomorphism. □

Theorem 3.15. *Let $\mathcal{A}$ be an abelian category. Suppose that $0 \to A^\bullet \to B^\bullet \to C^\bullet \to 0$ is a short exact sequence of complexes in $\mathcal{A}$. Then there is a long exact cohomology sequence*

$$\dots \longrightarrow H^i(A^\bullet) \longrightarrow H^i(B^\bullet) \longrightarrow H^i(C^\bullet) \longrightarrow H^{i+1}(A^\bullet) \longrightarrow \dots \tag{3.2}$$

functorial in short exact sequences.

Proof. Consider the commutative diagram

$$\begin{array}{ccccccc} & & \mathrm{coker}(d_A^{i-1}) & \longrightarrow & \mathrm{coker}(d_B^{i-1}) & \longrightarrow & \mathrm{coker}(d_C^{i-1}) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \ker(d_A^{i+1}) & \longrightarrow & \ker(d_B^{i+1}) & \longrightarrow & \ker(d_C^{i+1}) & & \end{array} \tag{3.3}$$

with the obvious maps induced by the differentials d^i thanks to the complex equations $d^i \circ d^{i-1} = 0 = d^{i+1} \circ d^i$. Note that both rows are exact by an application of the snake lemma to the relevant portions of the short exact sequence $0 \to A^\bullet \to B^\bullet \to C^\bullet$ of complexes. Next, we want to apply the snake lemma again to the displayed commutative diagram instead. The kernels of the vertical maps are the cohomology groups $H^i(A^\bullet)$, $H^i(B^\bullet)$, $H^i(C^\bullet)$, whereas their cokernels equal the cohomology groups $H^{i+1}(A^\bullet)$, $H^{i+1}(B^\bullet)$, $H^{i+1}(C^\bullet)$. In particular, we get the desired long exact sequence from the snake lemma.

For functoriality under short exact sequences, we can extend the diagram above to a parallelepiped. A simple diagram chase shows that even the connecting homomorphism coming from the snake lemma are natural in $0 \to A^\bullet \to B^\bullet \to C^\bullet \to 0$. □

Lemma 3.16. *Let $\mathcal{A}$ be an abelian category. Consider a commutative diagram*

$$\begin{array}{ccccccc} & & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle c} & & \\ 0 & \longrightarrow & A' & \longrightarrow & B' & \longrightarrow & C' & & \end{array} \tag{3.4}$$

with exact rows. Then we get an exact sequence

$$\ker(a) \to \ker(b) \to \ker(c) \to \mathrm{coker}(a) \to \mathrm{coker}(b) \to \mathrm{coker}(c) \tag{3.5}$$

If both rows are furthermore short exact, then the sequence of kernels and cokernels remains exact upon adding two 0*s to the left and to the right. The long exact sequence is natural in diagrams of the given form.*

Proof. Let us consider the first half of the sequence $\ker(a) \to \ker(b) \to \ker(c)$ whose maps are induced by those of the sequence $A \to B \to C \to 0$. It is clearly a complex because that is true for the original first row. We need to prove exactness at $\ker(b)$. The kernel of $\ker(b) \to \ker(c)$ lies in the image of $A \to B$. Since $A' \to B'$ is injective, we deduce that the kernel of the map of kernels is necessarily contained in the image of $\ker(a) \to \ker(b)$. Dual arguments give exactness of $\operatorname{coker}(a) \to \operatorname{coker}(b) \to \operatorname{coker}(c)$. Also note that if $A \to B$ is injective, then so is the map $\ker(a) \to \ker(b)$ between their subobjects. Similarly, surjectivity of $B' \to C'$ yields that of $\operatorname{coker}(b) \to \operatorname{coker}(c)$.

We are left with producing the *connecting homomorphism* $\delta : \ker(c) \to \operatorname{coker}(a)$ and checking exactness of the resulting sequence at those two places. The preimage of $\ker(c)$ in B contains the image of A and maps to A' under b by exactness of the diagram rows. If we project this preimage further to $\operatorname{coker}(a)$, then we kill the subobject A, and deduce the connecting map δ. Next, we check exactness of the long sequence at $\ker(c)$. Notice that $\ker(b)$ dies under b meaning its image under δ must also vanish. This shows that we have a complex. If we look at $\ker(\delta)$, it lifts to a submodule of B containing the image of A and also $\ker(b)$. The restriction of b to that submodule has image in A' and it is clear that the map towards $\operatorname{coker}(a)$ has kernel equal to the image of $A \oplus \ker(b)$. This means that $\ker(b) \to \ker(\delta)$ is surjective, as desired. Passing to the dual abelian category or repeating the dual arguments, we deduce that our final sequence is also exact at $\operatorname{coker}(a)$. Naturality of the long exact sequence in the diagrams can be checked by drawing the corresponding parallelepiped and a diagram chase reveals that the relevant maps commute. □

3.2. **Derived functors.** In the realm of homological algebra, derived functors are fundamental tools used to correct exactness deficiencies of our preferred additive functors of abelian categories. They provide a means to compute homology and cohomology groups in various categories.

Definition 3.17. If $\mathcal{A}$ and $\mathcal{B}$ are abelian, an additive functor $F : A \to B$ is called left (resp. right) exact if, for all $f : X \to Y$, the canonical map $F(\ker(f)) \to \ker(F(f))$ (resp. $F(\operatorname{coker}(f)) \to \operatorname{coker}(F(f))$) is an isomorphism.

In particular, given a left exact functor $F : \mathcal{A} \to \mathcal{B}$ of abelian categories, and a short exact sequence

$$0 \to A \to B \to C \to 0, \tag{3.6}$$

in $\mathcal{A}$, we obtain the following exact sequence in $\mathcal{B}$:

$$0 \to F(A) \to F(B) \to F(C) \tag{3.7}$$

The corresponding dual statement holds for right exact functors.

Example 3.18. Let $\mathcal{A}$ be an abelian category and A an object thereof. The additive endofunctor $B \mapsto \operatorname{Hom}_{\mathcal{A}}(A, B)$ is always left exact but not right exact in general (take non-injective A), whereas the additive endofunctor $B \mapsto \operatorname{Hom}_{\mathcal{A}}(B, A)$ is always right exact but not left exact in general (take non-projective A).

Example 3.19. Let R be a commutative ring and M be an R-module. The additive endofunctor $N \mapsto N \otimes_R M$ of Mod_R is always right exact, but not left exact in general (take M to be non-flat).

Example 3.20. Let X be a scheme and consider the additive functor $\Gamma(X, -)$ from the category Shv_X of abelian sheaves on X to the category Ab of abelian groups. This is a left exact functor.

Next, we try to measure the exactness failure of a left exact functor, which will lead us straight to the notion of cohomology groups.

Definition 3.21. Let $F : \mathcal{A} \to \mathcal{B}$ be a left exact functor of abelian categories. A cohomological δ-functor $F^\bullet$ extending F is a sequence $F^i : \mathcal{A} \to \mathcal{B}$ of additive functors such that $F^0 = F$, together with natural boundary maps $\delta : F^i(Z) \to F^{i+1}(X)$ for all short exact sequences $0 \to X \to Y \to Z \to 0$ in $\mathcal{A}$, inducing an exact complex:

$$0 \to F^0(X) \to F^0(Y) \to F^0(Z) \xrightarrow{\delta} F^1(X) \to F^1(Y) \to F^1(Z) \xrightarrow{\delta} F^2(X) \to \cdots . \quad (3.8)$$

We say that δ is universal if it is initial in the category of cohomological δ-functors extending F.

In general, it is difficult to check whether a certain δ-functor is universal, but there is a helpful criterion due to Grothendieck.

Definition 3.22. A cohomological δ-functor $F^\bullet$ is effaceable if for each $X \in \mathcal{A}$, there exists an injection $X \to X_i$ such that $F^i(X) \to F^i(X_i)$ vanishes for all $i > 0$.

Recall the notions from commutative algebra of injective and projective objects we alluded to in the examples above. We say that an abelian category $\mathcal{A}$ has enough injectives (resp. projectives) if every object embeds in some injective (resp. is covered by a projective).

Lemma 3.23. *If $\mathcal{A}$ has enough injectives, then a δ-functor $F^\bullet$ is effaceable if and only if F^i vanishes on injectives for all $i > 0$.*

Proof. The reverse implication is obvious. For the direct implication, note that the injection $X \to X_i$ splits by injectivity of X, so $F^i(X)$ is a direct summand of $F^i(X_i)$. It also maps to zero by effacibility, so it vanishes. □

The following theorem explains the interest behind effaceable δ-functors.

Theorem 3.24 (Grothendieck)**.** *Effaceable cohomological δ-functors are universal.*

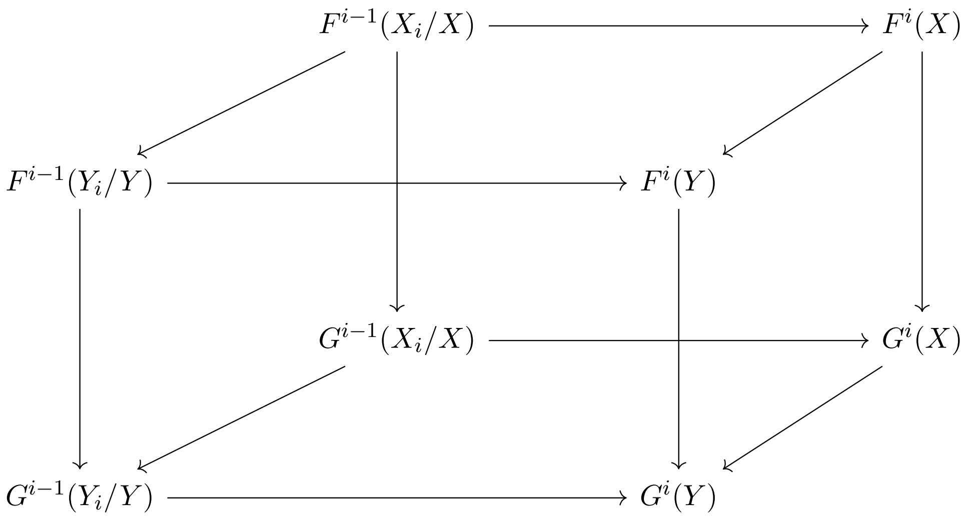
Proof. Let $G^\bullet$ be any cohomological δ-functor extending F. We aim at constructing a sequence of natural transformations $F^i \to G^i$ for all $i \geq 0$ respecting the boundary maps. Since $F^0 = G^0 = F$, this is clear for $i = 0$. Assume that we can produce these natural transformations for all $j < i$ for a given $i > 0$. For any object X of $\mathcal{A}$, choose an effacement X_i for X, thanks to our hypothesis on $F^\bullet$. From this we obtain the following diagram with exact rows:

$$\begin{array}{ccccccc} F^{i-1}(X_i) & \xrightarrow{a} & F^{i-1}(X/X_i) & \xrightarrow{b} & F^i(X) & \to & F^i(X_i) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ G^{i-1}(X_i) & \to & G^{i-1}(X/X_i) & \to & G^i(X) & \to & G^i(X_i) \end{array} , \quad (3.9)$$

where the two vertical arrows on the right still require an explanation. Note that $F^i(X) = \operatorname{coker}(a)$ by exactness of the top row and our effaceability hypothesis on $F^\bullet$. By functoriality of cokernels, we deduce the desired map $\varphi^i : F^i(X) \to G^i(X)$. In particular, φ^i factors through the image of $G^{i-1}(X/X_i) \to G^i(X)$ so it vanishes after post-composing with $G^i(X) \to G^i(X_i)$. Therefore, the last vertical arrow can be an arbitrary morphism, and the diagram will commute.

We still need to verify independent of φ^i from our choice of X_i. First of all, it is not too hard to see that the category of effacements of X is filtered (use the existence of finite coproducts). But if we have a map $X_i \to X_i'$ respecting X, the independence becomes clear by functoriality and our induction hypothesis.

Finally, we have to see that the construction of φ^i is natural in X. Again, we can choose the effacements $X \to X_i$ and $Y \to Y_i$ in a functorial manner (take Y_i to be an effacement of $X_i \sqcup_X Y$). Then, we consider the following cube

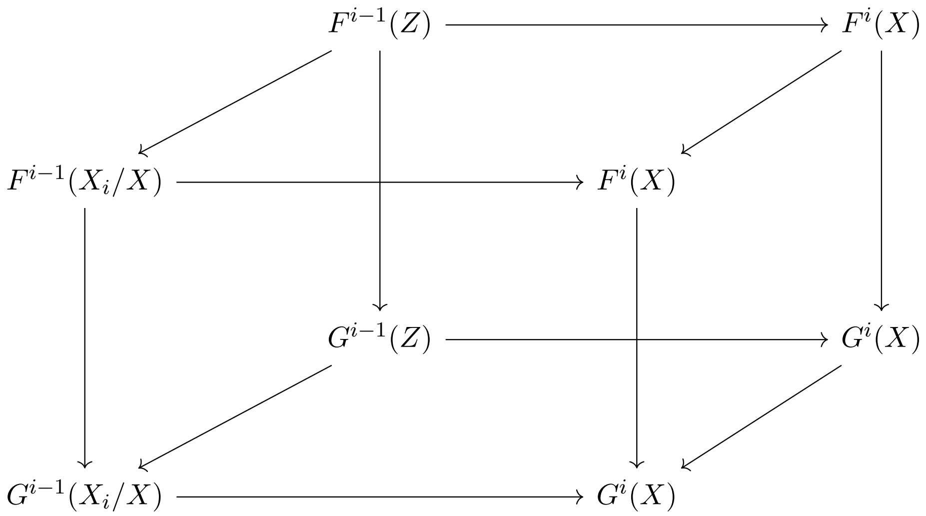


(3.10)

and claim that it commutes. This is true for the top and bottom faces by definition of cohomological δ-functors. It holds for the front and back faces by construction of φ. By induction on i (the $i = 0$ case being trivial), we may and do assume that the left face commutes. We assert that so does the right face and by surjectivity of $F^{i-1}(X_i/X) \to F^i(X)$, it is enough to check the claim after pre-composing with that quotient map. Then, a simple diagram chase reveals the desired commutativity, and thus functoriality of φ^i in X.

The last requirement is commutativity with respect to boundary maps. In order to do this, we consider an effacement X_i of X for $i > 0$ carrying a map from Y. Then, the short exact sequence $0 \to X \to Y \to Z \to 0$ admits a natural map to $0 \to X \to X_i \to$

$X_i/X \to 0$. This leads us to consider the following cube with the obvious maps



(3.11)

and we claim again that it commutes. Notice again that the top and bottom faces commute by definition of cohomological δ-functors. The right face trivially commutes, the front face commutes basically by construction of $\varphi_\bullet$, and the left face can be assumed to commute by induction on i (the initial case $i = 1$ being trivial). Now, a diagram chase along the cube reveals the desired commutativity of the back face, so $\varphi^\bullet$ respects boundaries. $\square$

In the presence of enough injectives, we get an existence statement:

Theorem 3.25. *Let $F : \mathcal{A} \to \mathcal{B}$ be a left-exact functor of abelian categories. If $\mathcal{A}$ has enough injectives, then F extends to an effaceable (and in particular universal) cohomological δ-functor $F_\bullet$*

Proof. Given an object X in $\mathcal{A}$, we choose an injective resolution $X \to I^\bullet$, i.e., a sequence I^i of injective modules for $i \geq 0$ with maps $I^i \to I^{i+1}$ which is exact at every $i > 0$ and such that $0 \to X \to I^0 \to I^1$ is left exact. Such a resolution can be constructed inductively thanks to the existence of enough injectives. We define $F^i(X)$ as the i-th cohomology group of the complex $F(I^\bullet)$. By left exactness of F, we do indeed recover $F^0(X) = X$. When applied to an injective object X, it yields vanishing of $F^i(X)$ by taking X as its own injective resolution. However, we still have to verify that this group $F^i(X)$ for $i > 0$ is independent from the choice of an injective resolution $X \to I^\bullet$.

We claim more generally that given a couple of injective resolutions $X \to I^\bullet$ and $Y \to J^\bullet$, any map $f\colon X \to Y$ extends to a map $f^\bullet\colon I^\bullet \to J^\bullet$ uniquely up to homotopy. In particular, we get the desired independence of $F^i(X)$ from $I_\bullet$ by applying this result to $f = \mathrm{id}_X$ and noticing that homotopic maps of complexes induce the same map at the level of cohomology. We construct f^i by induction on i assuming all the f^j have been defined for $j < i$. When $i = 0$, we have a map $X \to Y \to J^0$ which extends to $f^0 : I^0 \to J^0$ by injectivity. Similarly, when $i = 1$, we have a map $I^0/X \to J^0/Y \to J^1$ which extends to $f^1\colon I^1 \to J^1$ by injectivity. Suppose now that $i > 1$. Then, we look at the composition $\mathrm{coker}(I^{i-2} \to I^{i-1}) \to \mathrm{coker}(J^{i-2} \to J^{i-1}) \to J^i$ induced by f^j with $j = i-2, i-1$, and notice that it extends to $f^i : I^i \to J^i$ by injectivity of I^i.

Next we need to ensure that any other map $g^\bullet \colon I^\bullet \to J^\bullet$ of complexes extending f is homotopic to our favorite $f^\bullet$. Again, we are going to construct maps $h^i : I^i \to J^{i-1}$ for $i > 0$ by induction on i such that $f^0 - g^0 = h^1 \circ d^0$ when and $f^{i-1} - g^{i-1} = d^{i-2} \circ h^{i-1} + h^i \circ d^{i-1}$ for $i > 1$. The initial case $i = 1$ is done as follows. Observe that the difference $f^0 - g^0 : I^0 \to J^0$ necessarily kills X, so it factors through a map $I^0/X \to J_0$, meaning it extends to $h^1 \colon I^1 \to J^0$ so that the homotopy equation holds. The next case that we consider separately is when $i = 2$. We look at $f^1 - g^1 : I^1 \to J^1$ and notice that its restriction to the image of I^0 coincides with h^1 followed by $J^0 \to J^1$. Hence, we get a map $\mathrm{coker}(I^0 \to I^1) \to J^1$ given by the difference $f^1 - g^1 - d^0 \circ h^1$. This extends by injectivity to a map $h^2 : I^2 \to J^1$, yielding the claimed homotopy equation by construction. Next, we assume that the h^j have been constructed for all $j < i$, where $i > 2$. Again, the map $f^{i-1} - g^{i-1} : I^{i-1} \to J^{i-1}$ restricts to $d^{i-2} \circ h^{i-1}$ on the image of I^{i-2}. This means that we get a map $\mathrm{coker}(I^{i-2} \to I^{i-1}) \to J^{i-1}$ given by $f^{i-1} - g^{i-1} - d^{i-2} \circ h^{i-1}$ and it extends to $h^i : I^i \to J^{i-1}$ by injectivity, as desired.

Unfortunately, we are not yet done with the proof. While we have shown that F^i for $i > 0$ is a well-defined functor vanishing on injectives, we still have to prove $F^\bullet$ forms a cohomological δ-functor, i.e., that we get long exact sequences attached to short exact sequences $0 \to X \to Y \to Z \to 0$ in $\mathcal{A}$. We claim that there exists a short exact sequence

$$0 \to I^\bullet \to J^\bullet \to K^\bullet \to 0 \tag{3.12}$$

of injective resolutions extending the given short exact sequence in $\mathcal{A}$. We let $X \to I^\bullet$ and $Z \to K^\bullet$ be arbitrary injective resolutions and will now construct a resolution $Y \to J^\bullet$ filling the short exact sequence above. We define J^i and the maps either from X^i and J^{i-1} or to Z^i by induction on i. The case $i = 0$ follows by extending $X \to I^0$ to a map $Y \to I^0$ and then defining the corresponding map $Y \to J^0 = I^0 \oplus K^0$. The case $i = 1$ will be subsumed into the general $i > 1$ case by extending the injective resolutions to degree -1 via the original short exact sequence. Then, we have a short exact sequence

$$0 \to \mathrm{coker}(I^{i-2} \to I^{i-1}) \to \mathrm{coker}(J^{i-2} \to J^{i-1}) \to \mathrm{coker}(K^{i-2} \to K^{i-1}) \to 0 \tag{3.13}$$

by the snake lemma. The left and right terms embed respectively into the injectives I^i and K^i and we can find J^i fitting in the desired exact sequence by invoking the initial case. Now, we apply the long exact sequence of cohomology to the complex $0 \to F(I^\bullet) \to F(J^\bullet) \to F(K^\bullet) \to 0$. For mental health reasons, we omit the verification that the long exact sequence obtained in this manner is independent of the choice of injective resolutions, and functorial in short exact sequences. In conclusion, we have constructed an effaceable cohomological δ-functor $F^\bullet$, hence universal. □

Definition 3.26. Given a left exact functor $F \colon \mathcal{A} \to \mathcal{B}$ of abelian categories with enough injectives, we denote by $R^iF : \mathcal{A} \to \mathcal{B}$ for $i \geq 0$ the i-th component of the universal δ-cohomological extension of F and call it the *i-th right derived functor* of $F : \mathcal{A} \to \mathcal{B}$.

When F equals the global sections functor $\mathcal{M} \mapsto \Gamma(X, \mathcal{M})$ on abelian sheaves on a scheme X, it is common to write $\mathcal{M} \mapsto H^i(X, \mathcal{M})$ for R^iF. We will verify later the category Shv_X contains enough injectives.

3.3. **Spectral sequences.** Spectral sequences are a highly technical machinery meant to understand what composing derived functors does on cohomology groups. All our spectral sequences will lie in the first quadrant.

Definition 3.27. Let $\mathcal{A}$ be an abelian category.

(1) A *spectral sequence in* $\mathcal{A}$ is a system of objects $E_r^{p,q}$ in $\mathcal{A}$ for all non-negative integers $p, q \geq 0$, and maps $d_r^{p,q} : E_r^{p,q} \to E_r^{p+r,q+1-r}$ such that $d_r^{p,q} \circ d_r^{p-r,q-1+r} = 0$, and $E_{r+1}^{p,q} = \ker(d_r^{p,q})/\mathrm{im}(d_r^{p-r,q-1+r})$ for all $r \geq 0$.
(2) A *morphism of spectral sequences* a family of morphisms $f_r^{p,q} : E_r^{p,q} \to E_r'^{p,q}$ such that $f_r^{p+r,q+1-r} \circ d_r^{p,q} = d_r'^{p,q} \circ f_r^{p,q}$ and such that $f_{r+1}^{p,q}$ is induced by $f_r^{p,q}$.

Note that, for the above definition to make sense, we implicitly extend $E_r^{p,q}$ and $d_r^{p,q}$ to all integers by declaring them as 0 if either p or q is negative. Given a spectral sequence, we define an increasing sequence of subobjects $B_r^{p,q} \subset B_{r+1}^{p,q} \subset E_0^{p,q}$ and a decreasing sequence $Z_{r+1}^{p,q} \subset Z_r^{p,q} \subset E_0^{p,q}$ such that $B_r^{p,q} \subset Z_r^{p,q}$ with cokernel isomorphic to $E_r^{p,q}$. Indeed, by induction we let $Z_{r+1}^{p,q} \subset Z_r^{p,q}$ be the lift of the kernel of $d_r^{p,q}$ on $Z_r^{p,q}/B_r^{p,q}$, and $B_{r+1}^{p,q} \subset Z_r^{p,q}$ be the lift of the image of $d_r^{p-r,q-1+r}$. Note that for any fixed pair of integers (p, q), eventually none of $(p + r, q + 1 - r)$ and $(p - r, q - 1 + r)$ are pairs of positive integers for $r \gg 0$. In other words, $E_r^{p,q}$ stabilizes for $r \gg 0$.

Definition 3.28. Let $\mathcal{A}$ be an abelian category and $(E_r^{p,q}, d_r^{p,q})$ be a spectral sequence in $\mathcal{A}$. We define the *limit* $E_\infty^{p,q}$ of the spectral sequence as the stabilizing value of $E_r^{p,q}$ for $r \gg 0$ (depending on p, q). We say that the spectral sequence *degenerates at* E_r if the differentials $d_s^{p,q}$ are zero for all p, q and $s \geq r$.

Similaely, one defines subobjects $B_\infty^{p,q} \subset Z_\infty^{p,q} \subset E_0^{p,q}$ whose quotient recovers $E_\infty^{p,q}$. Next, we wish to produce actual examples of spectral sequences. As the notation itself suggests, they arise from the notion of double complexes.

Definition 3.29. Let $\mathcal{A}$ be an additive category. A *double complex* in $\mathcal{A}$ is given by objects $A^{p,q}$ in $\mathcal{A}$ and maps $d_1^{p,q} : A^{p,q} \to A^{p+1,q}$ and $d_2^{p,q} : A^{p,q} \to A^{p,q+1}$ such that $d_1^{p+1,q} \circ d_1^{p,q} = 0$, $d_2^{p,q+1} \circ d_2^{p,q} = 0$, and $d_1^{p,q+1} \circ d_2^{p,q} = d_2^{p+1,q} \circ d_1^{p,q}$ for all $p, q \geq 0$.

Note that $A^{p,q}$ is a complex as p is fixed and q varies, and also as p varies and q is fixed. Beware that in the literature the squares of the double complex may be anti-commutative.

Example 3.30. Let $M^\bullet$ and and $N^\bullet$ be complexes of R-modules over a ring. We obtain a double complex $K^{\bullet,\bullet} = M^\bullet \otimes N^\bullet$ with the obvious differentials.

Definition 3.31. Let $\mathcal{A}$ be an additive category and $A^{\bullet,\bullet}$ be a double complex. The *associated total complex* $\mathrm{Tot}(A^{\bullet,\bullet})$ is given by

$$\mathrm{Tot}^n(A^{\bullet,\bullet}) = \bigoplus_{n=p+q} A^{p,q} \tag{3.14}$$

with differential $d^n := \sum_{n=p+q}(d_1^{p,q} + (-1)^p d_2^{p,q})$

We want to produce a spectral sequence out of a double complex, and for this we start by guessing what $B_r^{p,q} \subset Z_r^{p,q} \subset E_0^{p,q} = A^{p,q}$ ought to be. Recall that the total complex K^n of $A^{p,q}$ carries a natural filtration

$$F^p K^n = \oplus_{p' \geq p} A^{p',n-p'} \tag{3.15}$$

indexed by p and compatible with the differentials d^n. This leads us to define

$$Z_r^{p,q} := \mathrm{gr}^p(F^pK^n \cap (d^n)^{-1}(F^{p+r}K^{n+1})) \tag{3.16}$$

and also

$$B_r^{p,q} := \mathrm{gr}^p(F^pK^n \cap d^{n-1}(F^{p+1-r}K^{n-1})) \tag{3.17}$$

where we set $n = p + q$. Intuitively, you should picture the cycles (resp. the boundaries) as the (p, q)-graded of certain anti-diagonals with first non-vanishing term at $A^{p,q}$ whose total differential vanishes in the first r terms (resp. which arise as the first non-vanishing term in the differential of an anti-diagonal after r steps).

Theorem 3.32. *Let $(A^{p,q}, d_1^{p,q}, d_2^{p,q})$ be a double complex with associated total complex (K^n, d^n). Then, $Z_r^{p,q}$ and $B_r^{p,q}$ given by the above formulae are the cycles and boundaries of a unique spectral sequence $E_r^{p,q}$ with differentials $d_r^{p,q}$ induced by d^n. Furthermore, $E_\infty^{p,q}$ equals $\mathrm{gr}^pH^n(K^\bullet)$.*

Proof. Let $B^n \subset Z^n \subset K^n$ be the cycles and boudaries of the total complex. Note that $d^{n-1}(F^{p+1-r}K^{n-1})$ lies in Z^n, so it follows that $B_r^{p,q} \subset Z_r^{p,q}$, and we may define $E_r^{p,q} = Z_r^{p,q}/B_r^{p,q}$. We want to show that the differentials d^n of the total complex induce a spectral sequence structure on the $E_r^{p,q}$. During the proof, we assume that $\mathcal{A}$ is the category of modules over a ring R, so that we can choose elements inside our objects. This makes it easier to understand what is happening in the proof. The sceptics should either rewrite the whole thing without taking objects or invoke the Freyd–Mitchell embedding theorem.

First, we define the maps $d_r^{p,q} \colon E_r^{p,q} \to E_r^{p+r,q+1-r}$. Note that d^n carries $F^pK^n \cap (d^n)^{-1}(F^{p+r}K^{n+1})$ to $F^{p+r}Z^{n+1}$ by definition and as $d^{n+1} \circ d^n = 0$. Now suppose that $x, y \in F^pK^n \cap (d^n)^{-1}(F^{p+r}K^{n+1})$ have the same p-graded component. We see that $d(x - y) \in d^n(F^{p+1}K^n) \cap F^{p+r}K^{n+1}$, so it has $p + r$-graded contained in $B_r^{p+r,q+1-r}$ by unraveling the definition of the latter. In particular, passing to the respective gradeds yields a well-defined map $Z_r^{p,q} \to E_r^{p+r,q+1-r}$. Clearly, if we want to understand the image of the boundaries $B_r^{p,q}$, we may represent it by an element of F^pB^n, whose image under the constructed map vanishes as $d^n \circ d^{n-1} = 0$. In particular, d^n induces a map $d_r^{p,q} \colon E_r^{p,q} \to E_r^{p+r,q+1-r}$ as desired. It is clear by functoriality that $d_r^{p,q} \circ d_r^{p-r,q-1+r} = 0$.

Next, we have to prove an equality $\ker(d_r^{p,q}) = Z_{r+1}^{p,q}/B_r^{p,q}$. The right term is trivially included in the left one. Conversely, if an element is in the kernel of $d_r^{p,q}$, then it lifts to some $x \in F^pK^n$ and there must be some $y \in F^{p+1}K^n$ such that $d^n(x-y) \in F^{p+r+1}K^{n+1}$. But $x - y$ has the same p-graded and witnesses its inclusion in $Z_{r+1}^{p,q}$.

The final equality that we have to prove is $\mathrm{im}(d_r^{p-r,q-1+r}) = B_{r+1}^{p,q}/B_r^{p,q}$. An element in the image of $d_r^{p-r,q+1-r}$ arises as the p-graded of $F^pK^n \cap d^{n-1}(F^{p-r}K^{n-1})$ after unraveling the definition of $Z_r^{p-r,q-1+r}$. On the other hand, this is the same as $B_{r+1}^{p,q}$ by definition. It follows from our calculations that $\ker(d_r^{p,q})/\mathrm{im}(d_r^{p-r,q-1+r}) = Z_{r+1}^{p,q}/B_{r+1}^{p,q} = E_{r+1}^{p,q}$, and thus $(E_r^{p,q}, d_r^{p,q})$ is indeed a spectral sequence.

For the final assertion, we start by noticing that for $r \gg 0$ we get $F^{p+r}K^{n+1} = 0$ and $F^{p+1-r}K^{n-1} = K^{n-1}$, so we deduce that $\mathrm{gr}^pZ^n = Z_\infty^{p,q}$ and $\mathrm{gr}^pB^n = B_\infty^{p,q}$. In particular, $E_\infty^{p,q} = \mathrm{gr}^pH^n(K^\bullet)$, as promised. ☐

After having seen how to produce spectral sequences out of double complexes, we turn our attention to the composition of right derived functors.

Definition 3.33. Let $\mathcal{A}$ be an abelian category and $K^\bullet \in \mathrm{C}^{\geq 0}(\mathcal{A})$ be a non-negative complex. A *Cartan-Eilenberg resolution* of $K^\bullet$ consists of a double complex $I^{\bullet,\bullet}$ in non-negative degrees and a map $\epsilon : K^\bullet \to I^{\bullet,0}$ such that $I^{p,\bullet}$ (resp. $\ker(d_1^{p,\bullet})$, resp. $\mathrm{im}(d_1^{p,\bullet})$, resp. $H_1^p(I^{\bullet,\bullet})$) is an injective resolution of K^p (resp. $\ker(d_K^p)$, resp. $\mathrm{im}(d_K^p)$, resp. $H^p(K^\bullet)$).

Lemma 3.34. *Let $\mathcal{A}$ be an abelian category with enough injectives and $K^\bullet$ be a non-negative complex. There exists a Cartan-Eilenberg resolution of $K^\bullet$.*

Proof. Define $B^p \subset Z^p \subset K^p$ be the cycles and boundaries in degree p, and set $H^p = Z^p/B^p$. We inductively construct a Cartan–Eilenberg resolution as follows. Use the horse shoe lemma to find extend the short exact sequence $0 \to Z^p \to K^p \to B^{p+1}$ to a short exact sequence of injective resolutions

$$0 \to J_Z^{p,\bullet} \to I^{p,\bullet} \to J_B^{p+1,\bullet} \to 0. \tag{3.18}$$

Similarly, we find a short exact sequence of injective resolutions

$$0 \to J_B^{p+1,\bullet} \to J_Z^{p+1,\bullet} \to J_H^{p+1,\bullet} \to 0 \tag{3.19}$$

extending the short exact sequence $0 \to B^{p+1} \to Z^{p+1} \to H^{p+1} \to 0$. Taking as maps $d_1^\bullet : I^{p,\bullet} \to I^{p+1,\bullet}$ the obvious composition, it is easy to check that we obtain the desired double complex. □

Finally, we prove the Grothendieck spectral sequence that describes the composition of right derived functors under a mild assumption.

Lemma 3.35. *Let $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ be abelian categories with enough injectives. Let $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{C}$ be left exact functors. If F sends injectives to G-acyclics, there is a spectral sequence $(E_r^{p,q}, d_r^{p,q})_{r\geq 0}$ such that $E_2^{p,q} = R^pG \circ R^qF(X)$ converging to $\mathrm{gr}^p R^n(G\circ F)(X)$ for a natural filtration.*

Proof. Let $X \to I^\bullet$ be an injective resolution and choose a Cartan-Eilenberg resolution $F(I^\bullet) \to J^{\bullet,\bullet}$ using Lemma 3.34. We now consider the double complex $G(J^{\bullet,\bullet})$ and study the associated spectral sequence $(E_r^{p,q}, d_r^{p,q})$. Since the cycles and boundaries of the complex $J^{p,\bullet}$ are injective, we see that the first page $E_1^{p,q}$ coincides with $G(H^q(J^{p,\bullet}))$. Then, we use that $H^q(J^{p,\bullet})$ is an injective resolution of $R^qF(X)$ to deduce that $E_2^{p,q} = R^pG \circ R^qF(X)$. Finally, we look at the spectral sequence $({}^tE_r^{p,q}, d_r^{p,q})$ associated to the transpose of $I^{\bullet,\bullet}$ (i.e., the double complex obtained by switching rows and columns). The first page ${}^tE_1^{p,q}$ is given by $R^qG \circ F(I^\bullet)$ because $F(I^p) \to J^{p,\bullet}$ is an injective resolution. Since $F(I^p)$ is G-acyclic, we have $R^q \circ F(I^\bullet) = 0$ if $q > 0$ and otherwise equal to $G(F(I^\bullet))$, so in the second page we get a degeneration ${}^tE_2^{p,q} = R^n(G\circ F)(X)$. Since the total complex of $I^{\bullet,\bullet}$ is invariant under transposition, we deduce that $E_\infty^{p,q}$ are the graded pieces of a certain filtration of ${}^tE_\infty^{p,q} = R^n(G\circ F)(X)$. □

4. Coherent cohomology

4.1. **Quasi-coherent sheaves.** In this section, we introduce the abelian category of quasi-coherent sheaves on a scheme X. Before doing this, we start with the abelian category of $\mathcal{O}_X$-modules.

Definition 4.1. Given a scheme X, let Shv_X be the abelian category of abelian sheaves on X. We define the abelian category Mod_X as the (non-full) subcategory of Shv_X whose objects are equipped with an $\mathcal{O}_X$-module structure and whose morphisms respect the latter.

By an $\mathcal{O}_X$-module structure on an abelian sheaf $\mathcal{F}$, we mean a map $\mathcal{O}_X \oplus \mathcal{F} \to \mathcal{F}$ making $\mathcal{F}(U)$ into an $\mathcal{O}_X(U)$-module for every open set $U \subset X$. The category Mod_X inherits a zero object from Shv_X given by $0(U) = 0$ on every open subset $U \subset X$. Similarly, the biproduct in Mod_X is given by the direct sum of abelian sheaves and this yields an additive category. Kernels of maps $\varphi : \mathcal{F} \to \mathcal{G}$ of $\mathcal{O}_X$-modules exist and are computed in the underlying category of presheaves, i.e., $\ker(\varphi)(U) := \ker(\varphi(U))$. The same is not true for cokernels, because the cokernel presheaf does not have to be sheaf, but we let $\mathrm{coker}(\varphi)$ be instead the sheafification, and this is still an $\mathcal{O}_X$-module. One can see, e.g. by computing stalks, that Mod_X is an abelian category.

Let $f\colon X \to Y$ be a map of schemes. We get a pair of adjoint functors (f^{-1}, f_*) between Shv_X and Shv_Y. Concretely, $f_*\mathcal{F}(V) = \mathcal{F}(f^{-1}(V))$ for all open $V \subset Y$, and $f^{-1}\mathcal{G}(U) = \mathrm{colim}_{V \supset f(U)}\mathcal{G}(V)$ for all open $U \subset X$. Note that f_* is left exact, calculating sections explicitly and using that the kernel is computed on presheaves, while f^{-1} is exact, because it preserves stalks. However, f^{-1} does not preserve $\mathcal{O}$-modules, so we correct this by defining the pullback $f^* := f^{-1} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X$, where $\otimes$ means sheafifying the termwise tensor product at the presheaf level. We get a pair (f^*, f_*) of adjoint functors between Mod_X and Mod_Y, but now f^* is at most right exact.

Note that global sections $s \in \mathcal{F}(X)$ of an $\mathcal{O}_X$-module $\mathcal{F}$ correspond bijectively to maps $\mathcal{O}_X \to \mathcal{F}$ by evaluating at 1.

Definition 4.2. An $\mathcal{O}_X$-module $\mathcal{F}$ is generated by global sections if there exists a surjection $\oplus_I \mathcal{O}_X \to \mathcal{F}$ for some set I.

Given a global section $s \in \mathcal{F}(X)$, we define its support $\mathrm{supp}(s)$ as the set of points $x \in X$ such that the stalk s_x is non-zero. The above definition signifies that we can find global sections $s_i \in \mathcal{F}(X)$ whose supports form an open cover of X. The usefulness of this notion will become clear once we work with ample line bundles. These represent some notion of positivity for invertible $\mathcal{O}_X$-modules (i.e., those that are locally isomorphic to $\mathcal{O}_X$) on proper schemes and a theorem of Serre asserts that every $\mathcal{O}_X$-module becomes globally generated upon tensoring with sufficiently large powers of ample line bundles.

Now, we can finally define quasi-coherent $\mathcal{O}_X$-modules.

Definition 4.3. Let X be a scheme. An $\mathcal{O}_X$-module $\mathcal{F}$ is a *quasi-coherent sheaf of $\mathcal{O}_X$-modules* if every point $x \in X$ admits an open neighbourhood $x \in U \subset X$ such that $\mathcal{F}|_U$ is the cokernel of a map

$$\bigoplus_J \mathcal{O}_U \to \bigoplus_I \mathcal{O}_U \tag{4.1}$$

The full subcategory of Mod_X consisting of quasi-coherent $\mathcal{O}_X$-modules is denoted QCoh_X.

In particular, X is covered by open sets U such that $\mathcal{F}|_U$ has a *presentation* by free modules, i.e. a right exact sequence

$$\bigoplus_J \mathcal{O}_U \to \bigoplus_I \mathcal{O}_U \longrightarrow \mathcal{F}|_U \longrightarrow 0. \tag{4.2}$$

One can show from this definition that QCoh_X is stable under finite direct sums, but not clearly under infinite direct sums, because U might depend on $\mathcal{F}$ a priori. Fortunately, this category has all the desired properties, thanks to the following observation in the affine case.

Theorem 4.4. *Let X be an affine scheme and $R = \Gamma(X, \mathcal{O}_X)$. The assignment $M \mapsto M \otimes_R \mathcal{O}_X$ defines an equivalence between Mod_R and QCoh_X with inverse given by $\mathcal{F} \mapsto \mathcal{F}(X)$.*

Proof. Let us first show that $M \otimes_R \mathcal{O}_X$ is quasi-coherent. We can choose a presentation $R^J \to R^I \to M \to 0$ for M by choosing generators for M and then generators for the relations ideal. Since tensoring is a right exact operation, we conclude that our induced $\mathcal{O}_X$-module admits a global presentation. Descent theory actually implies that no sheafification is needed in defining $M \otimes_R \mathcal{O}_X$, i.e., the tensor presheaf is already a sheaf. In a more elementary fashion, we have an exact complex

$$0 \to R \to \oplus_{1 \le i \le n} R[f_i^{-1}] \to \oplus_{1 \le i \le j \le n} R[(f_i f_j)^{-1}] \to \cdots \to R[(f_1 \dots f_n)^{-1}] \to 0 \quad (4.3)$$

of flat modules (hence acyclic for $\otimes_R$) by the sheaf property of $\mathcal{O}_X$, so it remains exact upon tensoring with M.

Let $\mathcal{F}$ be a globally presented $\mathcal{O}_X$-module. Using the fact that X is quasi-compact, one can show that the map $\alpha : \oplus_J \mathcal{O}_X \to \oplus_I \mathcal{O}_X$ is defined by global sections, i.e., it comes from $\alpha(X) : \oplus_J R \to \oplus_I R$ by tensoring with $\mathcal{O}_X$. By right exactness of the tensor product, we see that $\mathcal{F}$ equals $\mathrm{coker}(\alpha(X)) \otimes \mathcal{O}_X$. Note that the global sections of $\mathcal{F}$ equal $\mathrm{coker}(\alpha(X))$ by the previous paragraph.

Suppose now that $\mathcal{F}$ is an arbitrary $\mathcal{O}_X$-module. Let $r_i \in R$ be a collection of elements in R spanning the unit ideal and such that the restriction of $\mathcal{F}$ to the principal open set $D(r_i^{-1})$ admits a presentation. We get $R[r_i^{-1}]$-modules $M_i = \mathcal{F}(D(r_i))$ equipped with isomorphisms $M_i[r_j^{-1}] \simeq M_j[r_i^{-1}]$ of $R[(r_i r_j)^{-1}]$-modules satisfying a cocycle condition for varying i, j, k. This defines a descent datum in a sense to be seen later and it is effective, so it arises from a unique R-module M up to isomorphism. In particular, we get a map $M \otimes_R \mathcal{O}_X \to \mathcal{F}$ that is an isomorphism on an open cover, and thus itself an isomorphism by the glueing property of sheaves. □

Next, we examine preservation of quasi-coherence under pullback and pushforwards.

Lemma 4.5. *Pullback f^* along a map $f : X \to Y$ of schemes preserves quasi-coherence.*

Proof. Shrinking Y, we may assume $\mathcal{G}$ is globally presented. Since f^* is right exact, so it also preserves arbitrary direct sums, and $f^*\mathcal{O}_Y = \mathcal{O}_X$, we see that $f^*\mathcal{G}$ is also globally presented. □

Pushforwards do not necessarily preserve quasi-coherent modules, at least not until imposing some finiteness conditions on our morphisms. Recall that a map of schemes is qcqs if it is quasi-compact (i.e., it preserves quasi-compactness under fiber products) and quasi-separated (i.e., the diagonal is quasi-compact).

Lemma 4.6. *Let $f \colon X \to Y$ be a qcqs map of schemes. Then, f_* preserves quasi-coherence.*

Proof. We may assume that Y is affine and thus X is a qcqs scheme by assumption on f. First, let us treat the case where X is also affine. Then, we know that $f_*\mathcal{F} = \mathcal{F}(X) \otimes_{\mathcal{O}_X(X)} \mathcal{O}_X$, so it suffices to show quasi-coherence of $f_*\mathcal{O}_X$. By definition of the fiber product, $f_*\mathcal{O}_X(V) = \mathcal{O}_X(f^{-1}(V)) = \mathcal{O}_X(X) \otimes_{\mathcal{O}_Y(Y)} \mathcal{O}_Y(V)$, which implies quasi-coherence.

In the general case, let U_i be a finite affine open cover of X and V_{ijk} be a finite affine open cover of the intersections $U_i \cap U_j$. We have a left exact sequence

$$0 \to f_*\mathcal{F} \to \oplus_i f_*\mathcal{F}_{|U_i} \to \oplus_{ijk} f_*\mathcal{F}_{|V_{ijk}} \tag{4.4}$$

as one sees by checking on stalks. The morphism f becomes affine when restricted to the U_i or the V_{ijk}, and hence the two left terms of the complex are quasi-coherent. Since quasi-coherent sheaves form an abelian category, we deduce our claim. □

Next, we impose a finiteness condition on our quasi-coherent sheaves

Definition 4.7. Let X be locally Noetherian. A coherent $\mathcal{O}_X$-module is a quasi-coherent sheaf on X which is locally finitely presented, i.e., the local presentations can be chosen so that I and J are finite. The full subcategory of QCoh_X whose objects are coherent is denoted by Coh_X.

Lemma 4.8. *The category* Coh_X *is abelian.*

Proof. It is clearly additive, as it is stable under finite direct sums. For the existence of kernels and cokernels, we can construct these locally. If X is an affine Noetherian scheme, then QCoh_X is equivalent to Mod_R and it is easy to see, because finitely presentedness descends, that Coh_X is equivalent to the subcategory of finitely presented R-modules, which is abelian by the noetherianness assumption. □

Lemma 4.9. *Pullback f^* along maps $f : X \to Y$ of locally noetherian schemes preserves coherence.*

Proof. Going back to our proof of stability of quasi-coherence, we see that the finiteness of the presentations is clearly also preserved under pullback. □

Note that f_* for maps of affine schemes corresponds to restricting a module along a homomorphism of rings, which does not preserve finiteness in general. For proper maps, this becomes salvageable. Unfortunately, the method of proof is inductive and one has to show simultaneously that all the derived functors $R^i f_*$ preserve coherence, which we have not yet defined.

4.2. **Higher direct images and Čech cohomology.** The first thing we need to do is verify the existence of enough injectives in Shv_X.

Lemma 4.10. *Let X be a scheme. Then* Shv_X *has enough injectives.*

Proof. The category Ab of abelian groups has enough injectives and we will not review nor prove this fact. Let $\mathcal{F}$ be an abelian sheaf on X. The stalk $\mathcal{F}_x$ is an honest abelian group, so we can find a monomorphism $\mathcal{F}_x \to I_x$ with I_x being an injective abelian group. Define the product $\mathcal{I} := \prod_{x\in X} i_{x,*}I_x$ of skyscraper sheaves, where $i_{x,*} \colon \mathrm{Spec}(\kappa(x)) \to X$ is the natural map. As $\mathcal{F}_x = i_x^{-1}\mathcal{F}$, we get a map $\mathcal{F} \to i_{x,*}\mathcal{I}_x$ by adjunction. These assembles into a natural map $\mathcal{F} \to \mathcal{I}$ because products are limits. To check that it is a monomorphism, we can do it on stalks, and it follows already from injectivity of $\mathcal{F}_x \to \mathcal{I}_x$. It suffices to check that $\mathcal{I}$ is injective. As products preserve injectivity, we are reduced to considering the skyscraper sheaves $i_{x,*}I_x$. But here, we notice that by adjunction, maps $\mathcal{G} \to i_{x,*}I_x$ correspond bijectively to maps $\mathcal{G}_x \to I_x$, so we can verify injectivity inside abelian groups, where it is a given. □

Definition 4.11. Let X be a scheme. We define $H^i(X, -)\colon \mathrm{Shv}_X \to \mathrm{Ab}$ to be the i-th right derived functor of $\Gamma(X, -)$. If $f\colon X \to Y$ is a morphism of schemes, we let $R^i f_*\colon \mathrm{Shv}_X \to \mathrm{Shv}_Y$ be the i-th right derived functor of $f_*\colon \mathrm{Shv}_X \to \mathrm{Shv}_Y$.

Because both functors $\Gamma(X-)$ and f_* agree whether they are taken on Shv_X or Mod_X or even QCoh_X, we will often restrict them to the previous smaller categories of modules. One shows that $R^i f_*$ lands in Mod_Y when restricted to Mod_X by choosing the injective abelian groups I_x to have an $\mathcal{O}_{X,x}$-module structure and the maps $\mathcal{F}_x \to I_x$ to be $\mathcal{O}_{X,x}$-linear.

Next, we want to understand how local the definition of Rf_* is, i.e., what happens when we replace Y by an open cover.

Lemma 4.12. *Let $f : X \to Y$ be a morphism of schemes and $\mathcal{M}$ be an $\mathcal{O}_X$-module. Then, $R^i f_*\mathcal{M}$ is the sheafification of the presheaf $V \mapsto H^i(f^{-1}(V), \mathcal{M})$.*

Proof. Let $\mathcal{M} \to \mathcal{I}^\bullet$ be an injective resolution in Mod_X. First, note that if $j\colon U \to X$ is an open immersion of schemes, then j^* is exact and preserves injective abelian sheaves. In particular, $j^*\mathcal{M} \to j^*\mathcal{I}^\bullet$ is an injective resolution of $\mathcal{O}_U$-modules. This allows us to construct natural transition maps making the assignment $V \mapsto H^i(f^{-1}(V), \mathcal{M})$ into an actual presheaf.

On the other hand, we know $R^i f_*\mathcal{M} = H^i(f_*\mathcal{I}^\bullet)$. We can take this cohomology by sheafifying $V \mapsto H^i(\mathcal{I}^\bullet(f^{-1}V))$. But the last term here is by definition equal to $H^i(f^{-1}(V), \mathcal{M})$. □

Corollary 4.13. *Let $f : X \to Y$ be a morphism of schemes, $i\colon V \to Y$ an open immersion, and $g\colon f^{-1}(V) \to V$, $j\colon f^{-1}(V) \to X$ be the base changed morphisms. We have an isomorphism*

$$R^p g_* j^* \simeq i^* R^p f_* \tag{4.5}$$

of functors.

Proof. This follows immediately from Lemma 4.12. □

Let X be a scheme and $\mathcal{U} = \{U_i : i \in I\}$ be an open cover of X. To an abelian sheaf $\mathcal{F}$ on X, we associate its Čech complex

$$\check{\mathcal{C}}^p(\mathcal{U}, \mathcal{F}) = \prod\nolimits_{(i_0,\dots,i_p)\in I^{p+1}} \mathcal{F}(U_{i_0\dots i_p}). \tag{4.6}$$

where we set $U_{i_0\dots i_p} = U_{i_0} \cap \dots \cap U_{i_p}$. For $s \in \check{\mathcal{C}}^p(\mathcal{U}, \mathcal{F})$ we denote by $s_{i_0\dots i_p}$ its value in $\mathcal{F}(U_{i_0\dots i_p})$. We define the differentials by the formula

$$d^p(s)_{i_0\dots i_{p+1}} = \sum\nolimits_{j=0}^{p+1} (-1)^j s_{i_0\dots\hat{i}_j\dots i_{p+1}}|_{U_{i_0\cdots i_{p+1}}} \tag{4.7}$$

It is straightforward to see that $d \circ d = 0$, so $\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F})$ is a complex.

Definition 4.14. Let X be a scheme with an open cover $\mathcal{U}$ and $\mathcal{F}$ be an abelian presheaf on X. The complex $\check{\mathcal{C}}^\bullet(\mathcal{U},\mathcal{F})$ is the *Čech complex* associated to $\mathcal{F}$ and the open cover $\mathcal{U}$. Its cohomology groups $\check{H}^i(\mathcal{U},\mathcal{F}) := H^i(\check{\mathcal{C}}^\bullet(\mathcal{U},\mathcal{F}))$ are called the *Čech cohomology groups* associated to $\mathcal{F}$ and the cover $\mathcal{U}$.

Lemma 4.15. *An abelian presheaf $\mathcal{F}$ on a scheme is a sheaf if and only if $\mathcal{F}(U) \to \check{H}^0(\mathcal{U},\mathcal{F})$ is an isomorphism for every open cover $\mathcal{U}$ of every open $U \subset X$.*

Proof. This is obvious after unwinding the definitions of the Čech complex. □

Lemma 4.16. *Let X be a scheme with an open cover $\mathcal{U}$ and let $\mathcal{F}$ be an abelian presheaf on X. If $U_i = X$ for some $i \in I$, then the natural map $\mathcal{F}(X) \to \check{\mathcal{C}}^\bullet(\mathcal{U},\mathcal{F})$ is a homotopy equivalence.*

Proof. Write i_{-1} for the specific element of I such that $X = U_{-1}$. Observe that $U_{i_0\dots i_p} = U_{i_0\dots\hat{i}_j\dots i_p}$ if $i_j = i_{-1}$. Also the claim is equivalent to proving that the augmented complex $[\mathcal{F}(X) \to \check{\mathcal{C}}(\mathcal{U},\mathcal{F})]$ is nullhomotopic. Let us define a nullhomotopy

$$h : \prod\nolimits_{i_0\dots i_{p+1}} \mathcal{F}(U_{i_0\dots i_{p+1}}) \longrightarrow \prod\nolimits_{i_0\dots i_p} \mathcal{F}(U_{i_0\dots i_p}) \tag{4.8}$$

by the rule $h(s)_{i_0\dots i_p} = s_{i_{-1}i_0\dots i_p}$. We get

$$dh(s)_{i_0\dots i_p} = s_{i_0\dots i_p} + \sum\nolimits_{j=0}^{p} (-1)^{j+1} s_{ii_0\dots\hat{i}_j\dots i_p} \tag{4.9}$$

and also

$$hd(s)_{i_0\dots i_p} = \sum\nolimits_{j=0}^{p} (-1)^{j} s_{ii_0\dots\hat{i}_j\dots i_p} \tag{4.10}$$

and the sum of these maps equals the identity. □

We can moreover regard the Čech complex as a functor $\mathrm{PMod}_X \to \mathrm{C}(\mathrm{Mod}_{\mathcal{O}_X(X)})$ which is exact because we are working with presheaves. It also follows from the associated long exact sequence that $\mathcal{F} \mapsto \check{H}^n(\mathcal{U},\mathcal{F})$ define a δ-functor for varying n. To prove effaceability, we need to show that the higher Čech cohomology of injectives vanish, so we need to realize the Čech complex in terms of Hom.

First, notice that for any open immersion $j\colon U \to X$, we have a left adjoint $j_!$ to the exact functor j^* on the categories of pre-modules PMod, so it preserves projectives. Because $\mathcal{O}_U$ corepresents the functor $\mathcal{F} \mapsto \mathcal{F}(U)$, which is exact at the presheaf level, we get the projective object $j_!\mathcal{O}_U$ in PMod_X.

Lemma 4.17. *Given an open cover $\mathcal{U}$ of a scheme, we consider the complex $K(\mathcal{U})_\bullet$ of pre-modules such that $K^{-p}(\mathcal{U}) = \oplus_{i_0\dots i_p} j_{i_0\dots i_p}\mathcal{O}_{U_{i_0\dots i_p}}$ and d^{-p} given by the obvious maps times $(-1)^j$ when deleting the j-th index. Then, we have functorial isomorphisms $\mathrm{Hom}_{\mathcal{O}_X}(K^\bullet(\mathcal{U}),\mathcal{F}) = \check{\mathcal{C}}^\bullet(\mathcal{U},\mathcal{F})$ in PMod_X.*

Proof. This is a simple exercise in using the adjoint pair $(j_!, j^*)$. □

Lemma 4.18. *Let X be a scheme with an open cover $\mathcal{U}$. The complex $K^\bullet(\mathcal{U})$ has vanishing cohomology in negative degrees and its 0-th cohomology equals $\mathcal{O}_{\mathcal{U}} := \mathrm{im}(\oplus j_!\mathcal{O}_{U_i} \to \mathcal{O}_X)$ as a map of presheaves.*

Proof. Consider the augmentation $[K^\bullet(\mathcal{U}) \to \mathcal{O}_\mathcal{U}]$. We claim that this is nullhomotopic. It can be done after evaluating at any open $V \subset X$. This leads to a partition of I according to whether $V \subset U_i$ holds or not, and the non-vanishing terms of K^{-p} are indexed by $(p+1)$ many indexes of the first kind.

Pick i_{-1} such that $V \subset U_{-1}$ (if it does not exist, then the augmented complex vanishes at V), and define a nullhomotopy h via the following rule: $h(s)_{i_0 \dots i_{p+1}}$ vanishes unless $i_0 = i_{-1}$ in which case it equals $s_{i_1 \dots i_{p+1}}$. We can verify that this is a nullhomotopy via a straightforward calculation, as for the acyclicity of the augmented Čech complex in the presence of a degeneracy. □

Lemma 4.19. *Let X be a scheme with an open cover $\mathcal{U}$. The Čech δ-functor $\check{H}^p(\mathcal{U}, -)$ is universal.*

Proof. By the same argument as for Shv_X and Mod_X, we can show that both PShv_X and PMod_X have enough injectives. Note that $\check{H}^0(\mathcal{U}, -)$ is a left exact functor $\mathrm{PMod}_X \to \mathrm{Mod}_{\mathcal{O}_X(X)}$, so it admits right derived functors.

Let $\mathcal{I}$ be an injective pre-module on X. By Lemma 4.17 we havee $\mathrm{Hom}_{\mathcal{O}_X}(K^\bullet(\mathcal{U}), \mathcal{I}) \simeq \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{I})$. On the other hand, $K(\mathcal{U})_\bullet$ is quasi-isomorphic to $\mathcal{O}_\mathcal{U}[0]$ by Lemma 4.18. Using injectivity of $\mathcal{I}$, we see that $\check{H}^i(\mathcal{U}, \mathcal{I}) = 0$ for all $i > 0$. and our δ-functor is universal. □

Let us note that Čech cohomology vanishes on injective *modules*.

Lemma 4.20. *Let $\mathcal{U}$ be an open cover of a scheme X and $\mathcal{I}$ be an injective $\mathcal{O}_X$-module. Then $\check{H}^p(\mathcal{U}, \mathcal{I})$ vanishes for all $p > 0$.*

Proof. Because sheafification is exact, its right adjoint forgetful functor preserves injectives, so this reduces to Lemma 4.19. □

Lemma 4.21. *Let X be a scheme and consider the forgetful functor $i\colon \mathrm{Mod}_X \to \mathrm{PMod}_X$. This is left exact with right derived functor given by the cohomology presheaves $U \longmapsto H^p(U, \mathcal{F})$.*

Proof. Left exactness is well known at this point. Choose an injective resolution $\mathcal{F} \to \mathcal{I}^\bullet$. By definition $R^p i$ is the p-th cohomology *presheaf* of the complex $\mathcal{I}^\bullet$. But that coincides exactly with $H^p(U, \mathcal{F})$. □

Lemma 4.22. *Let $\mathcal{U}$ be an open cover of a scheme. For any $\mathcal{O}_X$-module $\mathcal{F}$, there is a spectral sequence $(E_r, d_r)_{r \geq 0}$ with*

$$E_2^{p,q} = \check{H}^p(\mathcal{U}, R^q i(\mathcal{F})) \implies H^{p+q}(X, \mathcal{F}) \tag{4.11}$$

functorial in $\mathcal{F}$.

Proof. This is a Grothendieck spectral sequence applied to the composable functors i and $\check{H}^0(\mathcal{U}, -)$ (recall that the second one was defined indeed on PMod_X). Because $\mathcal{F}$ is a sheaf, we know that composing them yields the global sections functor $\mathcal{F} \to H^0(X, \mathcal{F})$, see Lemma 4.15. Also, we know that $i(\mathcal{I})$ is Čech acyclic by Lemma 4.20 and that $\check{H}^p(\mathcal{U}, -)$ equal the right derived functors of $\check{H}^0(\mathcal{U}, -)$ by Lemma 4.19. In total, we may in fact invoke the Grothendieck spectral sequence. □

Corollary 4.23. *Let $\mathcal{U}$ be an open cover of a scheme X. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module and assume that $H^i(U_{i_0\dots i_p}, \mathcal{F}) = 0$ for all $i > 0$, all $p \geq 0$ and all $i_0, \dots, i_p \in I$. Then $\check{H}^p(\mathcal{U}, \mathcal{F}) = H^p(X, \mathcal{F})$.*

Proof. This is an immediate consequence of Lemma 4.22. Indeed, the term $E_2^{p,q} = 0$ vanishes as soon as $q > 0$. Hence the spectral sequence degenerates at E_2 and the result follows. □

Lemma 4.24. *Let X be a scheme and*

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0 \tag{4.12}$$

be a short exact sequence of $\mathcal{O}_X$-modules. If there exists a cofinal system of open covers $\mathcal{U}$ of X such that $\check{H}^1(\mathcal{U}, \mathcal{F}) = 0$, then the map $\mathcal{G}(X) \to \mathcal{H}(X)$ is surjective.

Proof. Take a global section $s \in \mathcal{H}(X)$. By assumption, we can find an open cover $\mathcal{U}$ of X such that the first Čech cohomology $\check{H}^1(\mathcal{U}, \mathcal{F})$ vanishes the restriction of s to U_i comes from some $s_i \in \mathcal{G}(U_i)$. Taking the difference of the restrictions $s_{i_0 i_1} = s_{i_1}|_{U_{i_0 i_1}} - s_{i_0}|_{U_{i_0 i_1}}$ to $U_{i_0 i_1}$ gives rise to an element of $\mathcal{F}(U_{i_0 i_1})$ by hypothesis. Since $\check{H}^1(\mathcal{U}, \mathcal{F})$ vanishes, we can find sections $t_i \in \mathcal{F}(U_i)$ such that $s_{i_0 i_1} = t_{i_1}|_{U_{i_0 i_1}} - t_{i_0}|_{U_{i_0 i_1}}$. Now, if we modify the s_i by subtracting the t_i, they now glue to a global section of $\mathcal{G}$ lifting s. □

Lemma 4.25. *Let X be a scheme and $\mathcal{F}$ be an $\mathcal{O}_X$-module such that $\check{H}^p(\mathcal{U}, \mathcal{F})$ vanishes for all $p > 0$ and all open covers $\mathcal{U}$ of open subschemes $U \subset X$. Then, $H^p(U, \mathcal{F}) = 0$ for all $p > 0$.*

Proof. Note that $\mathcal{F}$ is Čech-acyclic for any open cover of an open subscheme of X. Choose an injection $\mathcal{F} \to \mathcal{I}$ into an injective $\mathcal{O}_X$-module. By Lemma 4.20, we also know that $\mathcal{I}$ is Čech-acyclic for all open covers. We denote by $\mathcal{Q}$ the quotient of the previous injection. Notice that forming the quotient commutes with taking sections over any open $U \subset X$, because of the Čech acyclicity of $\mathcal{F}$. In particular, we get a short exact sequence of *presheaves*, and an associated long exact sequence of Čech cohomology for any open cover $\mathcal{U}$. This implies that $\mathcal{Q}$ is also Čech-acyclic.

Next, we look at the long exact sequence of cohomology at any open $U \subset X$. Since $\mathcal{I}$ is injective, we have $H^n(U, \mathcal{I}) = 0$ for $n > 0$. In particular, $H^n(U, \mathcal{F}) \simeq H^{n-1}(U, \mathcal{G})$ for all $n > 1$. We also know that $H^0(U, \mathcal{I}) \to H^0(U, \mathcal{Q})$ is surjective by a preceding lemma, so also $H^1(U, \mathcal{F}) = 0$. Note that also $H^1(U, \mathcal{Q}) = 0$ by repeating the same argument for $\mathcal{Q}$ instead, so we get vanishing of higher cohomology by induction. □

Actually, if we inspect the proof, we notice that it would be enough to have Čech acyclicity for a cofinal set of open covers of elements in a basis of the Zariski topology of X. This will be important to reduce most of our tasks to the affine case.

Corollary 4.26. *Let $f : X \to Y$ be a morphism of schemes and $\mathcal{I}$ be an injective $\mathcal{O}_X$-module. Then, both $H^p(Y, f_*\mathcal{I})$ and $R^p f_*\mathcal{I}$ vanish for all $p > 0$.*

Proof. Let $V \subset Y$ be an open and set $U = f^{-1}(V)$. For any open covering $\mathcal{V}$ of V, we define similarly its pullback $\mathcal{U} = f^{-1}(\mathcal{V})$. It is clear by construction that Čech complexes commute with pushforward. Thus, we get vanishing of $\check{H}^p(\mathcal{V}, f_*\mathcal{I})$ for all $p > 0$ by Lemma 4.20. Then, Lemma 4.25 implies vanishing of $H^p(V, f_*\mathcal{I})$ for all $p > 0$, and finally also of $R^p f_*\mathcal{I}$ for all $p > 0$ as this is the sheaf associated to the cohomology presheaf $V \mapsto H^p(V, f_*\mathcal{I})$. □

Flat pushforwards actually preserve injectivity, because the left adjoint is exact. We are now ready to define the Leray spectral sequence.

Lemma 4.27. *Let $f : X \to Y$ be a morphism of schemes and $\mathcal{F}$ be an $\mathcal{O}_X$-module. There is a spectral sequence*

$$E_2^{p,q} = H^p(Y, R^q f_*(\mathcal{F})) \implies H^{p+q}(X, \mathcal{F}) \tag{4.13}$$

functorial in $\mathcal{F}$.

Proof. This is just the Grothendieck spectral sequence applied to the composition $\Gamma(X, -) = \Gamma(Y, -) \circ f_*$. This is valid, because f_* maps injectives to $\Gamma(Y, -)$-acyclics. □

Corollary 4.28. *Let $f : X \to Y$ be a morphism of ringed spaces. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module.*

(1) *If $R^q f_*\mathcal{F} = 0$ for $q > 0$, then $H^p(X, \mathcal{F}) = H^p(Y, f_*\mathcal{F})$.*
(2) *If $H^p(Y, R^q f_*\mathcal{F}) = 0$ for $p > 0$, then $H^q(X, \mathcal{F}) = H^0(Y, R^q f_*\mathcal{F})$.*

Proof. These conditions force the Leray spectral sequence to degenerate at E_2. □

There is also a relative version of the Leray spectral sequence.

Lemma 4.29. *Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of schemes, and $\mathcal{F}$ be an $\mathcal{O}_X$-module. There is a spectral sequence*

$$E_2^{p,q} = R^p g_*(R^q f_*\mathcal{F}) \implies R^{p+q}(g \circ f)_*\mathcal{F} \tag{4.14}$$

functorial in $\mathcal{F}$.

Proof. This is a Grothendieck spectral sequence for the composition $(g \circ f)_* = g_* \circ f_*$. We just need to verify that f_* maps injectives to g_*-acyclics. But we say that $f_*\mathcal{I}$ is $\Gamma(V, -)$-acyclic for all opens $V \subset Y$, so by sheafifying we get also that $R^p g_*(f_*\mathcal{I})$ vanishes for $p > 0$. □

The next order to business is to show vanishing of Čech cohomology on standard affine covers. This is at the heart of comparing abstract cohomology to Čech cohomology of quasi-coherent sheaves.

Lemma 4.30. *Let $\mathcal{U}$ be a standard open cover of an affine scheme X and $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Then, $\check{H}^p(\mathcal{U}, \mathcal{F}) = 0$ for all $p > 0$.*

Proof. We set $X = \mathrm{Spec}(A)$ and $U_i = D(f_i)$ where the finitely many $f_i \in A$ generate the unit ideal. If $M = \mathcal{F}(X)$, we also have $\mathcal{F} = \mathcal{M} \otimes_A \mathcal{O}_X$. Here, the tensor product can be taken inside presheaves, and there's no need to sheafify. Now, the Čech complex $\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F})$ is given by

$$\prod\nolimits_{i_0} M_{f_{i_0}} \to \prod\nolimits_{i_0 i_1} M_{f_{i_0} f_{i_1}} \to \prod\nolimits_{i_0 i_1 i_2} M_{f_{i_0} f_{i_1} f_{i_2}} \to \dots \tag{4.15}$$

We now show exactness of the augmented complex by adding M on the left. It is enough to show that this holds after localizing at each of the f_i, as they generate the unit ideal. But we get as a result the augmented complex $[M[f_i^{-1}] \to \check{\mathcal{C}}^\bullet(\mathcal{U}[f_i^{-1}], \mathcal{F}[f_i^{-1}])]$ which carries an extra degeneracy, so we already know it is nullhomotopic. □

Now we get vanishing of higher cohomology on affines.

Lemma 4.31. *Let X be an affine scheme and $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Then, $H^p(X,\mathcal{F}) = 0$ for all $p > 0$.*

Proof. We are going to apply a variant of Lemma 4.25 for a given basis of the topology consisting of standard opens and their corresponding covers. Since these are cofinal within the set of all covers, we are reduced to verifying $\check{H}(\mathcal{U},\mathcal{F})$ for a standard open cover of a standard affine open of X. But this was the content of the previous lemma. □

We also get a relative analogue of affine vanishing.

Lemma 4.32. *Let $f : X \to S$ be an affine morphism of schemes, and $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Then, $R^i f_*\mathcal{F} = 0$ for all $i > 0$.*

Proof. We know already that $R^i f_*\mathcal{F}$ is the sheafification of the presheaf $V \mapsto H^i(f^{-1}(V), \mathcal{F}|_{f^{-1}(V)})$. We may therefore assume $S = V$ is affine, so X is also affine, and the previous lemma gives vanishing of $H^i(X,\mathcal{F})$. □

The following two lemmas explain when Čech cohomology can be used to compute cohomology of quasi-coherent modules.

Proposition 4.33. *Let $\mathcal{U}$ be an affine open cover of a separated scheme X. For any quasi-coherent sheaf $\mathcal{F}$, we have $\check{H}^p(\mathcal{U},\mathcal{F}) = H^p(X,\mathcal{F})$ as $\Gamma(X,\mathcal{O}_X)$-modules for all p.*

Proof. Note that the intersections $U_{i_0\dots i_p}$ appearing in the Čech complex are affine, because X is separated. In view of Lemma 4.31, we deduce that the Čech spectral sequence degrenerates, yielding the desired equality. □

As an application of this comparison result between Čech cohomology and coherent cohomology, we prove finiteness of cohomological dimension.

Corollary 4.34. *Let X be a separated quasi-compact scheme. Then, $H^n(X,\mathcal{F})$ vanishes for all $n > C$ and all quasi-coherent $\mathcal{F}$, where C is a constant depending only on X.*

Proof. Let $\mathcal{U}$ be a finite affine open covering. By separatedness, the intersections $U_{i_0\dots i_p}$ are all affine. The alternating Čech complex $\check{\mathcal{C}}^\bullet_{\mathrm{alt}}(\mathcal{U},\mathcal{F})$ obtained by supressing repeated indexes, vanishes beyond a fixed degree that depends only on $\mathcal{U}$, so the same holds for Čech cohomology. Now, we simply apply the comparison result. □

4.3. **Coherent cohomology of projective space.** In this section, we are going to discuss the cohomology of coherent sheaves on projective schemes. In order to start, let us recall how to construct these sheaves via graded modules.

Let $S = \oplus_{d\geq 0} S_d$ be a graded ring such that S_d are finite S_0-modules and S is generated by S_1 as an S_0-algebra. We know how to construct a proper scheme $\mathrm{Proj}(S)$ by glueing the standard opens $D_+(f)$. These are the affine spectra of the rings $S[f^{-1}]_0$ given as the degree 0 part of the localization of S at an homogeneous element $f \in S_d$ for some $d \geq 1$.

Let $M = \oplus_d M_d$ be a graded S-module. We consider the sheaf $\mathcal{M}$ on $\mathrm{Proj}(S)$ whose values on a standard open $D_+(f)$ are given by $M[f^{-1}]_0$. One can show that these glue on standard open covers as in the case of the structure sheaf $\mathcal{O}_{\mathrm{Proj}(S)}$. As this definition is compatible with localization, we see that $\mathcal{M}$ is quasi-coherent.

Note that we have a twisting functor (n) on graded modules that is given as follows $M(n)_d = M_{n+d}$ with the obvious graded action of S. This yields a natural map

$$M \longrightarrow \bigoplus_n \Gamma(X, \widetilde{M(n)}). \tag{4.16}$$

of graded S-modules. More generally, we have the following definition.

Definition 4.35. Let S be a graded ring and $X = \mathrm{Proj}(S)$. We define the n-th twist $\mathcal{O}_X(n)$ of the structure sheaf as the quasi-coherent sheaf associated with $S(n)$. For any quasi-coherent sheaf $\mathcal{F}$ on X, we set $\mathcal{F}(n) = \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(n)$.

Since $S(n) \otimes_S S(m) = S(n+m)$, we can construct an isomorphism $\mathcal{O}_X(n) \otimes_{\mathcal{O}_X} \mathcal{O}_X(m) \simeq \mathcal{O}_X(n+m)$. It follows that $\mathcal{O}_X(n)$ is an invertible sheaf on X, with inverse given by $\mathcal{O}_X(-n)$. Another way of seeing this is by noticing that the map $S \to S(n)$ for $n > 0$ given by multiplication by $f \in S_n$ is an isomorphism over $D_+(f)$.

Lemma 4.36. *Let S be a finitely S_1-generated graded S_0-algebra and $X = Proj(S)$. Let $\mathcal{F}$ be a quasi-coherent sheaf on X and set $M = \bigoplus_{n\in\mathbf{Z}} \Gamma(X, \mathcal{F}(n))$. Then there is a canonical isomorphism $\mathcal{M} \simeq \mathcal{F}$ functorial in $\mathcal{F}$.*

Proof. We send an element $mf^{-n} \in \mathcal{M}(D_+(f))$ with $d = \deg(f)$ to the corresponding section of $\mathcal{F}(D_+(f))$ obtained by reversing the map $\mathcal{F} \to \mathcal{F}(nd)$ over $D_+(f)$. One checks easily that this recipe glues. Let us check that this map is injective. We consider $g_1, \ldots, g_n \in S_1$ generating S as an S_0-algebra, so that X is covered by $D_+(g_i)$. If $mf^{-n} \in \mathcal{M}(D_+(f))$ vanishes after mapping it to $\mathcal{F}$, we conclude that m as a global section of $\mathcal{F}(nd)$ vanishes on the distinguished affine open $D(fg_i^{-1})$ of the affine scheme $D_+(g_i)$: this implies $f^e m g_i^{-(n+e)d}$ is the zero element in $\mathcal{F}((n+e)d)(D_+(g_i))$ for $e \gg 0$. Since the g_i cover X, this means that the global section $f^e m$ of $\mathcal{F}((n+e)d)$ simply vanishes, so the same holds true for $mf^{-n} = f^e m f^{-e-n}$ over $D_+(f)$.

Now, we check surjectivity. Let $t' \in \Gamma(D_+(f), \mathcal{F})$ and observe that $f^e t$ is the image of some $t_i \in \Gamma(D_+(g_i), \mathcal{F}(ed))$ for $e \gg 0$. Note that $t_i = t_j$ inside $\Gamma(D_+(fg_ig_j), \mathcal{F}(ed))$, so by the preceding injectivity, we deduce that $t_i = t_j$ as elements of $\Gamma(D_+(fg_ig_j), \mathcal{M}(ed))$. This translates into an equality $f^{e'} t_i = f^{e'} t_j$ in $\Gamma(D_+(g_ig_j), \mathcal{M}((e+e')d))$, so they glue to a global section t of $\mathcal{M}((e+e')d)$ such that $tf^{-e-e'} \in \Gamma(D+(f), \mathcal{M})$ lifts t'. ☐

Now, we are going to specialize to the case where $S = R[T_0, \ldots, T_n]$ and thus X is the n-dimensional projective space $\mathbb{P}^n_R := \mathrm{Proj}(R[T_0, \ldots, T_n])$ over an arbitrary ring R. We dispose of natural line bundles $\mathcal{O}_{\mathbb{P}^n_R}(d)$ on $\mathbb{P}^n_R$. The degree d summand $R[T_0, \ldots, T_n]_d$ is finite free over R of rank $\binom{n+d}{d}$, and has an obvious basis consisting of monomials $T_0^{e_0} \ldots T_n^{e_n}$ with $e_i \geq 0$ and $\sum e_i = d$. Similarly, we consider the graded R-algebra $R[T_0^{-1}, \ldots, T_n^{-1}]$ with T_i^{-1} in degree -1. In particular, its graded module $T_0^{-1} \ldots T_n^{-1} R[T_0^{-1}, \ldots, T_n^{-1}]$ vanishes in degrees $\geq -n$, and is free over R of rank $\binom{-d-1}{-d-n-1}$ for $d \leq -n-1$, with basis consisting of monomials $T_0^{e_0} \ldots T_n^{e_n}$ with $e_i < 0$ and $\sum e_i = d$.

Lemma 4.37. *Let R be a ring and $n \geq 0$ a non-negative integer. We have*

$$H^q(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n_R}(d)) = \begin{cases} R[T_0, \ldots, T_n]_d & \text{if} \quad q = 0 \\ 0 & \text{if} \quad 0 < q < n \\ T_0^{-1} \ldots T_n^{-1} R[T_0^{-1}, \ldots, T_n^{-1}]_d & \text{if} \quad q = n \end{cases}$$

as R-modules.

Proof. We will use the standard affine open covering $\mathcal{U} = D_+(T_i)_{i=0,\dots,n}$ of $\mathbf{P}^n_R$ to compute the cohomology using the Čech complex

$$\check{\mathcal{C}}^p(\mathcal{U}, \mathcal{O}_{\mathbb{P}^n_R}(d)) = \bigoplus\nolimits_{i_0<\ldots<i_p} (R[T_0, \ldots, T_n, T_{i_0}^{-1}, \ldots T_{i_p}^{-1}])_d$$

with differentials given by $d(s)_{i_0\ldots i_{p+1}} = \sum_{j=0}^{p+1}(-1)^j s_{i_0\ldots\hat{i}_j\ldots i_{p+1}}$. Note that the complex is naturally graded by $\mathbb{Z}^{n+1}$ via the exponents of the T_i, so it suffices to compute the graded pieces $\check{\mathcal{C}}^p_e$ for any vector $e \in \mathbb{Z}^{n+1}$. For such an e, assume $\sum e_i = d$ and let $\mathrm{Neg}_e \subset [n]$ be the set of negative exponents $e_i < 0$. Then, it follows that

$$\check{\mathcal{C}}^p_e = \bigoplus\nolimits_{\mathrm{Neg}_e \subset \{i_0<\ldots<i_p\}} R \cdot T_0^{e_0} \ldots T_n^{e_n} \tag{4.17}$$

If $\mathrm{Neg}_e = [n]$, then the e-graded complex is concentrated in degree n and satisfies $\check{\mathcal{C}}^n_e = R \cdot T_0^{e_0} \ldots T_n^{e_n}$. This matches our claim in degree e. If instead $\mathrm{Neg}_e = \emptyset$, then $\check{\mathcal{C}}^p_e$ is the sum of several copies of $R \cdot T_0^{e_0} \ldots T_n^{e_n}$ indexed by $i_0 < \ldots < i_p$. But this is clearly the Čech complex of the trivial cover of $\mathrm{Spec}(R)$ by itself repeated n times, so its cohomology vanishes for $q > 0$ and is given by a single copy of $R \cdot T_0^{e_0} \ldots T_n^{e_n}$ when $q = 0$. This also matches our claim.

To finish the proof of the lemma we have to show that the complexes $\check{\mathcal{C}}^\bullet_e$ are acyclic when $\emptyset \subsetneq \mathrm{Neg}_e \subsetneq [n]$. Pick an index $i_{-1} \notin \mathrm{Neg}_e$ which exists by hypothesis and define the homotopy map $h\colon \check{\mathcal{C}}^{p+1}_e \to \check{\mathcal{C}}^p_e$ given by the rule

$$h(s)_{i_0\ldots i_p} = \begin{cases} s_{i_{-1}i_0\ldots i_p} & \text{if} \quad i_{-1} < i_0 \\ (-1)^a s_{i_0\ldots i_{a-1}i_{-1}i_a\ldots i_p} & \text{if} \quad i_{a-1} < i_{-1} < i_a \\ (-1)^p s_{i_0\ldots i_p} & \text{if} \quad i_p < i_{-1} < n \end{cases} \tag{4.18}$$

Note that this is well defined as the negative exponents are contained in $\{i_0, \ldots, i_p\}$ if and only if they also are after adjoining i_{-1}. We claim that $hd + dh = \mathrm{id}$, so h defines a nullhomotopy of the identity and the complex $\check{\mathcal{C}}^\bullet(\vec{e})$ is nullhomotopic, thus acyclic.

To check this claim, suppose first that $i_{a-1} < i_{-1} < i_a$ for some $1 \leq a \leq p$. Then we have on the one hand

$$dh(s)_{i_0\ldots i_p} = \sum\nolimits_{j=0}^{a-1}(-1)^{a+j} s_{i_0\ldots\hat{i}_j\ldots i_{-1}\ldots i_p} + s_{i_0\ldots i_p} + \sum\nolimits_{j=a}^{p}(-1)^{a+j+1} s_{i_0\ldots i_{-1}\ldots\hat{i}_j\ldots i_p} \tag{4.19}$$

and on the other hand we get

$$hd(s)_{i_0\ldots i_p} = \sum\nolimits_{j=0}^{a-1}(-1)^{j+a-1} s_{i_0\ldots\hat{i}_j\ldots i_{-1}\ldots i_p} + \sum\nolimits_{j=a}^{p}(-1)^{j+a} s_{i_0\ldots i_{-1}\ldots\hat{i}_j\ldots i_p} \tag{4.20}$$

the two of which sum to $s_{i_0\ldots i_p}$ as desired. The other cases are similar and left to the reader □

We can now verify Serre duality essentially by hand for line bundles on projective space. We have a pairing of free R-modules

$$R[T_0, \ldots, T_n] \times T_0^{-1} \ldots T_n^{-1} R[T_0^{-1}, \ldots, T_n^{-1}] \longrightarrow R \tag{4.21}$$

which is defined by sending (f, g) to the $T_0^{-1} \ldots T_n^{-1}$-coefficient of fg. In other words, the basis $T_0^{e_0} \ldots T_n^{e_n}$ is dual to $T_0^{-1-e_0} \ldots T_n^{-1-e_n}$. Using this pairing, we can deduce from

Lemma 4.37 that

$$H^n(\mathbb{P}^n_R, \mathcal{O}_{\mathbb{P}^n_R}(d)) \simeq \mathrm{Hom}_R(H^0(\mathbb{P}^n_R, \mathcal{O}_{\mathbb{P}^n_R}(-d-n-1)), R) \tag{4.22}$$

4.4. **Ample line bundles.** In this section, we discuss the crucial notion of ampleness in algebraic geometry. This is a positivity condition on line bundles that controls the existence of many global sections and vanishing of higher cohomology. This matches our calculations of cohomology of line bundles in $\mathbb{P}^n_R$, as we saw that this is holds for positive d (and a few negative d), but we want to formulate it abstractly. Note that a notion of positivity, whatever it may be, must be stable under rescaling. In our case, this means taking powers of a line bundle should not change its ampleness. Recall that a *line bundle* $\mathcal{L}$ on a scheme X is a locally free sheaf of rank 1. In the literature, this is also called an invertible sheaf.

Lemma 4.38. *Given a line bundle $\mathcal{L}$ on a scheme X, and a global section s of $\mathcal{L}$, the set $X_s = \{x \in X \mid s \notin \mathfrak{m}_x\mathcal{L}_x\}$ is open.*

Proof. Note that the global section corresponds to a map $s\colon \mathcal{O}_X \to \mathcal{L}$. If its fiber at x does not vanish, then the stalk s_x is surjective, and hence necessarily bijective by torsion-freeness. In particular, s is an isomorphism in an open neighborhood of x. □

Note that we have also $X_s \cap X_{s'} = X_{ss'}$, where ss' denotes the section $s \otimes s' \in \Gamma(X, \mathcal{L} \otimes \mathcal{L}')$ (check this).

Definition 4.39. Let X be a qcqs scheme and $\mathcal{L}$ be a line bundle on X. We say $\mathcal{L}$ is *ample* (resp. *semi-ample*) if for every $x \in X$ there exists an $n \geq 1$ and $s \in \Gamma(X, \mathcal{L}^{\otimes n})$ such that X_s is an affine (resp. not necessarily affine) open neighborhood of x.

It is immediate that $\mathcal{L}$ is ample (resp. semi-ample) if and only if $\mathcal{L}^{\otimes n}$ is ample (resp. semi-ample), and that ampleness is stable under affine pullback, whereas semi-ampleness is stable under arbitrary pullback. Also note that $\mathcal{L}$ is semi-ample if and only if a power of it is globally generated.

Lemma 4.40. *Let s be a global section of a line bundle $\mathcal{L}$ on a scheme X. Then, the open immersion $X_s \to X$ is affine.*

Proof. We may and do assume that X is the affine spectrum of a ring R and that $\mathcal{L} = \mathcal{O}_X$ is the structure sheaf. Then, s corresponds to an element of R, and its non-vanishing locus is a distinguished affine open of X. □

Lemma 4.41. *Let $\mathcal{L}$ and $\mathcal{M}$ be line bundles on a qcqs scheme X. If $\mathcal{L}$ is ample and $\mathcal{M}$ is semi-ample, then $\mathcal{L} \otimes \mathcal{M}$ is ample.*

Proof. Let $x \in X$. Choose $n \geq 1$, $m \geq 1$, $s \in \Gamma(X, \mathcal{L}^{\otimes n})$, and $t \in \Gamma(X, \mathcal{M}^{\otimes m})$ such that $x \in X_s \cap X_t$. This intersection and X_s is affine. Since $X_t \to X$ is affine by Lemma 4.40 and X_s is affine by ampleness of $\mathcal{L}$, their intersection is also affine. On the other hand, we have $s^m t^n \in \Gamma(X, (\mathcal{L} \otimes \mathcal{M})^{\otimes nm})$, and we can write $X_{s^m t^n} = X_s \cap X_t$, so we have found our affine open neighborhood. □

Lemma 4.42. *Let $\mathcal{L}$ be an ample line bundle on a qcqs scheme X. Then, X is separated.*

Proof. We are going to use the valuative criterion. Thus, let V be a valuation ring with fraction field K and consider two morphisms $f, g : \mathrm{Spec}(V) \to X$ that agree on $\mathrm{Spec}(K)$. As V is local, there exists (after perhaps raising $\mathcal{L}$ to a sufficiently divisible power) sections $s \in \Gamma(X, \mathcal{L})$, and $t \in \Gamma(X, \mathcal{L})$ such that X_s and X_t are affine, and f (resp. g) factors through X_s (resp. X_t).

The quasi-coherent module $f^*\mathcal{L}$ (resp. $g^*\mathcal{L}$) corresponds to a free V-module M (resp. N) of rank 1, because V is a discrete valuation ring. The associated K-vector spaces are isomorphic via some $\varphi \colon M \otimes_V K \to N \otimes_V K$ as f and g agree when restricted to $\mathrm{Spec}(K)$. Let $x \in M$ and $y \in N$ be the elements corresponding to the pullback of s along f and g, respectively. We get $\phi(x \otimes 1) = y \otimes 1$ and we know f factors through X_s, so x generates M and ϕ identifies M with $yN \subset N$. By symmetry, we conclude $M \simeq N$ and y generates N, so g also factors through X_s. But now we can use the fact that X_s is affine to see that f and g must agree on the whole of $\mathrm{Spec}(V)$. □

Semi-ampleness is intimately linked with morphisms towards projective schemes.

Lemma 4.43. *Let $\mathcal{L}$ be a line bundle on a qcqs scheme X and $S := \bigoplus_{n \geq 0} H^0(X, \mathcal{L}^{\otimes n})$ be the associated graded ring. If $\mathcal{L}$ is semi-ample, there is a canonical morphism of schemes $f \colon X \to Y := \mathrm{Proj}(S)$ equipped with maps $f^*\mathcal{O}_Y(n) \to \mathcal{L}^{\otimes n}$ that are isomorphisms for sufficiently divisible n.*

Proof. We define maps $S[s^{-1}]_0 \to \mathcal{O}(X_s)$ by performing the usual trick of dividing by a section of a line bundle on its non-vanishing locus: this is allowed because $s \colon \mathcal{O}_X \to \mathcal{L}$ is an isomorphism on X_s. One checks straightforwardly that all these maps glue, and hence we get a map $f \colon X \to Y$ because the X_s cover X by semi-ampleness of $\mathcal{L}$. Similarly, one obtains maps $f^*\mathcal{O}_Y(n) \to \mathcal{L}^{\otimes n}$. To check that they are isomorphisms for suitably divisible degrees, take n such that $\mathcal{L}^{\otimes n}$ becomes globally generated. Then, we can see that f factors through the open locus of Y where $\mathcal{O}_Y(n)$ is invertible and so we get a surjection $f^*\mathcal{O}_Y(n) \to \mathcal{L}^{\otimes n}$ of line bundles, and hence an isomorphism. □

The proof generalizes to the case where we have a map $\psi \colon S \to \bigoplus_{n \geq 0} H^0(X, \mathcal{L}^{\otimes n})$ of graded rings such that X is covered by $X_{\psi(s)}$ for $s \in S_d$. This allows us to define morphisms from X towards $\mathbb{P}^n_R$ in terms of generating global sections of line bundles $\mathcal{L}$.

Lemma 4.44. *Let $\mathcal{L}$ be a semi-ample line bundle on a qcqs scheme X with associated graded ring $S := \bigoplus_{n \geq 0} H^0(X, \mathcal{L}^{\otimes n})$. Then, the canonical morphism of schemes $f \colon X \to \mathrm{Proj}(S)$ has dense image.*

Proof. Assume that the image of f is not dense. Then, since the opens $D_+(s)$ with $s \in S_+$ homogeneous form a basis for the topology on $\mathrm{Proj}(S)$, we can find an s such that $f(X)$ does not meet the non-empty $D_+(s)$. In other words, by Lemma 4.43 this means X_s is empty. In turn, this implies that a power of s is the zero section in $\mathcal{L}^{\otimes n \deg(s)}$, so actually $D_+(s)$ is empty. □

Lemma 4.45. *Let $\mathcal{L}$ be an ample line bundle on a qcqs scheme X with associated graded ring $S := \bigoplus_{n \geq 0} H^0(X, \mathcal{L}^{\otimes n})$. Then, the canonical morphism of schemes $f \colon X \longrightarrow \mathrm{Proj}(S)$ is an open immersion with dense image. If X is proper over $H^0(X, \mathcal{O}_X)$, then it is an isomorphism.*

Proof. Choose $s_1, \dots, s_n \in S_+$ homogeneous such that X_{s_i} are affine, and $X = \bigcup X_{s_i}$. Say s_i has degree d_i. The inverse image of $D_+(s_i)$ under f is X_{s_i}, see Lemma 4.43. One checks easily by hand that $\Gamma(D_+(s_i), \mathcal{O}_{\mathrm{Proj}(S)})$ identifies with $\Gamma(X_{s_i}, \mathcal{O}_X)$, so f induces an isomorphism $X_{s_i} \to D_+(s_i)$. Thus f is an isomorphism of X onto the open $\bigcup_{i=1,\dots,n} D_+(s_i)$ of $\mathrm{Proj}(S)$. The image is dense by Lemma 4.44. If X is proper, then so is the map f, and we just have to notice that dense clopen immersions are isomorphisms. □

Now, we give a different criterion for ampleness in terms of twisted sheaves $\mathcal{O}(1)$.

Corollary 4.46. *Let X be a proper R-scheme with $H^0(X, \mathcal{O}_X) = R$. A line bundle $\mathcal{L}$ on X is ample if and only if there is a closed immersion $\iota \colon X \to \mathbb{P}^n_R$ for some n such that $\iota^*\mathcal{O}_{\mathbb{P}^n_R}(1)$ is isomorphic to $\mathcal{L}^{\otimes m}$ for some $m > 0$.*

Proof. First, note that $\mathcal{O}_{\mathbb{P}^n_R}(1)$ is ample, because its standard global sections $s_i := T_i$ have affine distinguished opens $D_+(T_i)$ covering $\mathbb{P}^n_R$. This shows the converse direction, because pullback along affine maps preserves ampleness, which is also invariant under positive powers.

For the forward direction, we can identify X with the projective spectrum of the graded ring $S = \bigoplus_{n \geq 0} H^0(X, \mathcal{L}^{\otimes n})$, and $\mathcal{L}$ with $\mathcal{O}_S(1)$. For some large m, we may find finitely many global sections s_i with $i = 1, \dots, n$ of $\mathcal{O}_S(m)$ such that X_{s_i} cover X. After replacing $\mathcal{L}$ by its m-th power, we may even assume that $m = 1$. This means that the irrelevant ideal $S_{>0} \subset S$ has the same radical as the R-algebra generated by the finitely many $s_i \in S_1$. Note that X_{s_i} is the affine spectrum of $S[s_i^{-1}]_0$, so the latter is a finitely generated R-algebra. After picking generators $t_{ij} s_i^{-m}$ for some $m \gg 0$, we may again replace the original line bundle to assume $m = 1$. But then we have a graded map $R[T_0, \dots, T_n] \to S$ taking T_k to the global sections s_i and t_{ij} of $\mathcal{L}$. One checks again easily by construction that this defines a closed immersion $\iota \colon X \to \mathbb{P}^n_R$ along which $\mathcal{O}_{\mathbb{P}^n_R}(1)$ pulls back to $\mathcal{L} = \mathcal{O}_S(1)$. □

Line bundles appearing as pullbacks of $\mathcal{O}(1)$ on projective space are called very ample. Now, we give an even stronger characterization of ampleness.

Proposition 4.47. *Let X be proper over $H^0(X, \mathcal{O}_X)$ and $\mathcal{L}$ be a line bundle on X. Then $\mathcal{L}$ is ample if and only if, for every coherent sheaf $\mathcal{F}$ on X, there exists an integer n_0 such that $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n}$ is globally generated for all $n \geq n_0$,*

Proof. For the converse direction, we let $x \in X$ be any point and $U \subset X$ be an affine open neighborhood of x. If $\mathcal{I}$ is the coherent ideal sheaf defining some closed subscheme $Z \subset X$ with U as open complement, then we can pick some global section s of $\mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n}$ generating it at x. But when considered as a global section of $\mathcal{L}^{\otimes n}$, we see that it also vanishes along Z, so we get $X_s \subset U$. This means we can write $X_s = X_s \cap U$, and it must be affine because $X_s \to X$ is an affine map and U is affine.

Now, we handle the forward direction. By hypothesis, there exists some $d > 0$ such that $\mathcal{L}^{\otimes d}$ is very ample, i.e., arising as $\mathcal{O}_X(1)$ with respect to a closed immersion $\iota \colon X \to \mathbb{P}^n_R$ with $R = H^0(X, \mathcal{O}_X)$. In order to prove our claim, we may and do replace $\mathcal{L}$ by its very ample d-th power. Indeed, if we apply the result to all $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes j}$ and the very ample line bundle $\mathcal{L}^{\otimes d}$ for any $j < d$, then we get the original result anyway. We therefore refer to $\mathcal{O}_X(1)$ instead of $\mathcal{L}$ and to the twists $\mathcal{F}(n)$ instead of $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n}$.

Let $\mathcal{F}_m \subset \mathcal{F}$ be the subsheaf generated by the global sections of $\mathcal{F}(m)$, in other words, the image of the canonical map $\Gamma(X, \mathcal{F}(m)) \otimes \mathcal{O}_X(-m) \to \mathcal{F}$. By construction, the twist $\mathcal{F}_m(n)$ is globally generated as soon as $n \geq m$. On the other hand, X identifies with an open subset of $\mathrm{Proj}(S)$ with $S = \bigoplus_{n \geq 0} H^0(X, \mathcal{O}_X(n))$ and $\mathcal{F}$ is the restriction to X of the quasi-coherent sheaf on $\mathrm{Proj}(S)$ associated to the graded S-module $M = \bigoplus_n H^0(X, \mathcal{F}(n))$, so it coincides with the sum of the subsheaves $\mathcal{F}_m$. By coherence, we have $\mathcal{F} = \sum_{m=1,\dots,N} \mathcal{F}_m$ for some $N \geq 1$. It follows that $\mathcal{F}(n)$ is globally generated whenever $n \geq N + 1$. □

4.5. **Cohomology of coherent sheaves on projective schemes.** Next, we discuss coherent sheaves on $\mathrm{Proj}(A)$ where A is a Noetherian graded ring generated by A_1 over A_0. We start by handling the case where $A = R[T_0, \dots, T_n]$ is a graded polynomial R-algebra. We are going to prove finiteness of cohomology and Serre vanishing at the same time.

Proposition 4.48. *Let R be a Noetherian ring. For every coherent sheaf $\mathcal{F}$ on $\mathbb{P}^n_R$, we have the following:*

1. *There exists a surjection $\mathcal{O}(-m)^{\oplus r} \to \mathcal{F}$ for $d \gg 0$ and some $r \geq 0$.*
2. *For any i, the R-module $H^i(\mathbb{P}^n_R, \mathcal{F})$ is finite, and vanishes unless $0 \leq i \leq n$.*
3. *If $i > 0$, then $H^i(\mathbb{P}^n_R, \mathcal{F}(d)) = 0$ for all sufficiently large $d \gg 0$.*
4. *For any k, the graded $R[T_0, \dots, T_n]$-module $\bigoplus_{d \geq k} H^0(\mathbb{P}^n_R, \mathcal{F}(d))$ is finite.*

Proof. We will use that $\mathcal{O}_{\mathbb{P}^n_R}(1)$ is an ample line bundle on $\mathbb{P}^n_R$. This follows directly from the definition since $\mathbb{P}^n_R$ covered by the standard affine opens $D_+(T_i)$. Hence, by our characterization of ample sheaves, we know that $\mathcal{F}(m)$ is globally generated for $m \gg 0$. In other words, there is a surjection $\mathcal{O}_{\mathbb{P}^n_R}(-m)^{\oplus r} \to \mathcal{F}$.

Also, $\mathbb{P}^n_R$ is covered by $n+1$ affines, namely the standard opens $D_+(T_i)$, $i = 0, \dots, n$, so we get $H^i(\mathbb{P}^n_R, \mathcal{F}) = 0$ for $i \geq n+1$ for any quasi-coherent sheaf $\mathcal{F}$ on $\mathbb{P}^n_R$, by Corollary 4.34. Now, we prove finiteness of cohomology for all coherent sheaves on $\mathbb{P}^n_R$ by descending induction on i. Clearly the result holds for $i \geq n+1$. Suppose we know the result for $i+1$ and we want to show the result for i. Choose a surjection $\mathcal{O}_{\mathbb{P}^n_R}(-m)^{\oplus r} \to \mathcal{F}$ and let $\mathcal{G}$ be the kernel, which is also a coherent sheaf on $\mathbb{P}^n_R$. The long exact cohomology sequence gives an exact sequence

$$H^i(\mathbb{P}^n_R, \mathcal{O}_{\mathbb{P}^n_R}(-m))^{\otimes r} \to H^i(\mathbb{P}^n_R, \mathcal{F}) \to H^{i+1}(\mathbb{P}^n_R, \mathcal{G}). \tag{4.23}$$

By induction assumption the right R-module is finite. We also calculated the cohomology of line bundles on $\mathbb{P}^n_R$ in Lemma 4.37, and saw that the left R-module is finite. Since R is Noetherian, it follows immediately that $H^i(\mathbb{P}^n_R, \mathcal{F})$ is a finite R-module.

Next, we handle Serre vanishing by descending induction on i. Notice that twisting on $\mathbb{P}^n_R$ is an exact functor, since it is given by tensoring with a locally free, thus flat sheaf. Again, the long exact sequence twisted by d gives us

$$H^i(\mathbb{P}^n_R, \mathcal{O}_{\mathbb{P}^n_R}(d-m))^{\oplus r} \to H^i(\mathbb{P}^n_R, \mathcal{F}(d)) \to H^{i+1}(\mathbb{P}^n_R, \mathcal{G}(d)). \tag{4.24}$$

By induction assumption, we see the module on the right is zero for $d \gg 0$ and by the computation in Lemma 4.37 the same holds for the left side if $i > 0$, so we get the desired assertion.

Finally, we note that, by Serre vanishing applied to $\mathcal{G}(d)$, the R-module $M_{\geq k} := \oplus_{d\geq k} H^0(\mathbb{P}^n_R, \mathcal{F}(d))$ is a quotient of $N_{\geq k} := \oplus_{d\geq k} H^0(\mathbb{P}^n_R, \mathcal{O}(-m))^{\oplus r}$ for $k \gg 0$. Also if $l < k$, we have that $M_{\geq l}$ is an extension of $M_{\geq k}$ by a finite R-module due to finiteness of cohomology. Hence, to get finiteness of $M_{\geq k}$ as a $R[T_0, \dots, T_n]$-module, it is enough to prove the same for $N_{\geq k}$. If $k < m$, then it follows from our calculations of cohomology of $\mathcal{O}(-m)$, that $N_{\geq k} = R[T_0, \dots, T_n](-m)^{\oplus r}$ which is finite. If $k \geq m$, then $N_{\geq k}$ is a submodule of $N_{\geq m-1}$, so it is also finite as an $R[T_0, \dots, T_n]$-module. □

Now, we may deduce the general case.

Corollary 4.49. *Let A be a graded ring such that A_0 is Noetherian and A is generated by finitely many elements of A_1 over A_0. Set $X = \mathrm{Proj}(A)$ and let $\mathcal{F}$ be a coherent sheaf on X.*

1. *There exists a surjection $\mathcal{O}_X(-m)^{\oplus r} \to \mathcal{F}$ for $d \gg 0$ and some, $m, r \geq 0$.*
2. *For any i, the A_0-module $H^i(X, \mathcal{F})$ is finite.*
3. *If $i > 0$, then $H^i(X, \mathcal{F}(d)) = 0$ for all sufficiently large $d \gg 0$.*
4. *For any k, the graded A-module $\bigoplus_{d\geq k} H^0(X, \mathcal{F}(d))$ is finite.*

Proof. By assumption there exists a surjection of graded A_0-algebras $A_0[T_0, \dots, T_n] \to A$ where $\deg(T_j) = 1$ for $j = 0, \dots, n$. This induces a closed immersion $i : X \to \mathbb{P}^n_{A_0}$ such that $i^*\mathcal{O}_{\mathbb{P}^n_{A_0}}(1) = \mathcal{O}_X(1)$. Now, the claims follow directly from Proposition 4.48 applied to the coherent sheaf $i_*\mathcal{F}$ on $\mathbb{P}^n_{A_0}$. For example, the surjection $\mathcal{O}_{\mathbb{P}^n_{A_0}}(-m)^{\oplus r} \to i_*\mathcal{F}$ yields by adjunction another surjection $\mathcal{O}_X(-m)^{\oplus r} \to \mathcal{F}$. The statements on cohomology follow from stability under finite pushforward. □

We had promised to relate finite graded A-modules to coherent sheaves on X. This is related to tails of graded modules, i.e., the graded submodules of sufficiently large degree.

Corollary 4.50. *Let A be a graded ring such that A_0 is Noetherian and A is generated by finitely many elements of A_1 over A_0. Let M be a finite graded A-module. Set $X = Proj(A)$ and let $\mathcal{M}$ be the associated quasi-coherent sheaf on X. Then, $M_n \to \Gamma(X, \mathcal{M}(n))$ are isomorphisms for all sufficiently large $n \gg 0$.*

Proof. Finiteness of A as an A-module implies that $\mathcal{M}$ is a coherent sheaf on X. Set $N = \bigoplus_{n\in\mathbf{Z}} \Gamma(X, \mathcal{M}(n))$ and recall that we have an isomorphism $\mathcal{N} \simeq \mathcal{M}$. We have to show that the natural map $M \to N$ of graded A-modules identifies their tails, i.e., it is an isomorphism in sufficiently large degrees. Let $K = \ker(M \to N)$ and $Q = \mathrm{coker}(M \to N)$. Since one checks that the functor $M \mapsto \mathcal{M}$ is exact by construction itself (as localization and taking degree 0 parts are exact functors), we get vanishing of $\mathcal{K}$ and $\mathcal{Q}$. Also note that $N_{\geq k}$ is a finite A-module by Corollary 4.49, so the same holds for $K_{\geq k}$ and $Q_{\geq k}$. We know already that modules with the same tails induce the same quasi-coherent sheaves, so we are reduced to the following claim: let K be a finite graded A-module with vanishing coherent sheaf $\mathcal{K}$; then $K_n = 0$ for $n \gg 0$.

To prove tail vanishing, let $x_1, \dots, x_r \in K$ be homogeneous generators sitting in degrees $d_1, \dots, d_r$ and $f_1, \dots, f_n \in A_1$ be A_0-linear generators. For each i and j there exists an $n_{ij} \geq 0$ such that $f_i^{n_{ij}} x_j = 0$ by vanishing of $\mathcal{K}$. Then we see that K_d is zero for $d \gg 0$ as every element of K_d is a sum of monomials in the f_i times some x_j of total degree d. □

An equivalent way of formulating the condition of vanishing tails, at least for *finite* graded A-modules is to require that K is $A_{>0}$-power torsion. We get a Serre full subcategory $\mathrm{GrMod}^{\omega}_A[A_{>0}] \subset \mathrm{GrMod}^{\omega}_A$ of $A_{>0}$-power torsion modules inside the category of finite graded A-modules.

Proposition 4.51. *Let A be a graded ring such that A_0 is Noetherian and A is generated by finitely many elements of A_1 over A_0. Set $X = \mathrm{Proj}(A)$. The functor $M \mapsto \mathcal{M}$ induces an equivalence*

$$\mathrm{GrMod}^{\omega}_A/\mathrm{GrMod}^{\omega}_A[A_{>0}] \to \mathrm{Coh}_X, \tag{4.25}$$

with quasi-inverse given by $\mathcal{F} \mapsto \bigoplus_{n\geq k} \Gamma(X, \mathcal{F}(n))$ for any k.

Proof. Clearly, $A_{>0}$-power torsion modules define trivial coherent sheaves, as seen in the previous corollary. Also, we saw above that the quasi-inverse takes a coherent sheaf $\mathcal{M}$ in the essential image to a graded module N with the same tail as M, so they are isomorphic in the Serre quotient. As for essential surjectivity, we just invoke finiteness of $\oplus_{n\geq k}\Gamma(X, \mathcal{F}(n))$ and recall that its associated sheaf recovers $\mathcal{F}$. □

We conclude this section by proving coherence is preserved under proper derived pushforward.

Proposition 4.52. *Let S be a locally Noetherian scheme, $f : X \to S$ be a proper morphism, and $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Then, $R^i f_*\mathcal{F}$ is a coherent $\mathcal{O}_S$-module for all $i \geq 0$.*

Proof. We may assume that S and X are noetherian and that S is affine. Notice that the property of a coherent sheaf on X to have coherent higher direct images is stable under kernels, cokernels, and extensions by the long exact sequence in cohomology. Notice that any coherent sheaf is an extension of pushforward of ideal sheaves along closed immersions. Indeed, first we may write any coherent sheaf $\mathcal{F}$ has an extension of coherent sheaves $i_*\mathcal{G}$ pushed forward along a closed immersion $i \colon Z \to X$ with Z integral. Locally, we may write $i_*\mathcal{G}|_U \simeq i_*\mathcal{O}_Z|_U^{\oplus r}$ so we can extend this to an injection $i_*\mathcal{I}^{\oplus r} \to i_*\mathcal{G}$ for some ideal sheaf $\mathcal{I} \subset \mathcal{O}_Z$, meaning we can filter $\mathcal{F}$ by coherent sheaves of the form $i_*\mathcal{I}$, with $\mathcal{I}$ being a generic line bundle on the integral Z. By the same inductive argument, it becomes clear that for the sake of proving that coherence is preserved, we may replace the ideal sheaves $\mathcal{I}$ by our favorite generic line bundle $\mathcal{G}$ on Z. Also we may assume that $X = Z$ is integral by the Leray spectral sequence and because i is affine.

Now we apply Chow's lemma to f to get a projective birational map $\pi \colon Z' \to Z$ and a closed immersion $i \to Z' \to \mathbb{P}^n_S$ of S-schemes. Let $\mathcal{L} = \mathcal{O}_Z(1)$ be the very ample line bundle defined via the embedding i. It turns out that (π, i) embeds Z' into $\mathbb{P}^n_Z$, in such a way that $\mathcal{O}_{\mathbb{P}^n_Z}(1)$ pulls back to $\mathcal{L}$. Hence, we deduce that $R^i f'_*\mathcal{L}^{\otimes d}$ and $R^i\pi_*\mathcal{L}^{\otimes d}$ vanish for $i > 0$ and $d \gg 0$. Setting $\mathcal{G} = \pi_*\mathcal{L}^{\otimes d}$, we see that it is a generic line bundle as π is birational. On the other hand, the Leray spectral sequence for the composition $f \circ \pi$ applied to $\mathcal{L}^{\otimes d}$ degenerates at the second page, so we get that $R^p f_*\mathcal{G} = R^p(f')_*\mathcal{L}^{\otimes d}$ vanishes for $p > 0$ and is coherent for $p = 0$ as f' is projective. □

Corollary 4.53. *Let S be the spectrum of a noetherian ring A, $f : X \to S$ be a proper morphism, and $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Then $H^i(X, \mathcal{F})$ is a finite A-module for all $i \geq 0$.*

Proof. This is just the affine case of Proposition 4.52. □

4.6. **Base change.** Let $f : X \to S$ be a morphism of schemes. Suppose further that $g : S' \to S$ is any morphism of schemes. Denote by $X' = X_{S'} = S' \times_S X$ the base change of X and $f' : X' \to S'$ the base change of f. Also write $g' : X' \to X$ the projection. We will refer to this as a base change diagram and use this notation consistently.

Lemma 4.54. *Let $f : X \to S$ be a morphism of schemes and $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. If f is affine, then for every base change diagram we have an identification $g^* f_* \mathcal{F} \simeq f'_*(g')^* \mathcal{F}$.*

Proof. Note that there is always a canonical map by using adjuction twice. The statement is local on S and S'. Hence we may reduce to the situation where there are ring maps $R \to A, R'$ and an A-module M. The isomorphism boils down to the equality $(R' \otimes_R A) \otimes_A M = R' \otimes_R M$ of R'-modules. □

The most important case of base change is when g is flat.

Lemma 4.55 (Flat base change)**.** *Consider the base change diagram for $(f, g) : X \times_S S' \to S$ and let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. If g is flat and f is qcqs, then for any $i \geq 0$ we get an isomorphism*

$$g^* R^i f_* \mathcal{F} \simeq R^i f'_*(g')^* \mathcal{F}, \tag{4.26}$$

In particular, if g is a morphism of affine schemes induced by $A \to A'$, then $H^i(X, \mathcal{F}) \otimes_A A' = H^i(X', \mathcal{F} \otimes_A A')$.

Proof. To get a natural map, one has to use the adjunction pair (g^*, g_*) and then either work with injective resolutions (preserved under flat pushforward!) or use the edge maps of the Leray spectral sequence $g_* R^i f'_* \to R^i(g \circ f')_* = R^i(f \circ g') \to R^i f_*(g')^*$. Since the isomorphism claim is local, we may assume that g is induced by a flat map of rings $A \to A'$ upon taking spectra. Now, $R^i f_* \mathcal{F}$ is the quasi-coherent sheaf attached to the A-module $H^0(S, R^i f_* \mathcal{F}) = H^i(X, \mathcal{F})$, and similarly, $R^i f'_*(g')^* \mathcal{F}$ is the quasi-coherent $\mathcal{O}_{S'}$-module associated to the A'-module $H^i(X', \mathcal{F} \otimes_A A')$.

We want to show that $H^i(X, \mathcal{F}) \otimes_A A' \longrightarrow H^i(X', \mathcal{F} \otimes_A A')$ is an isomorphism and we are for simplicity going to assume X separated. Choose a finite affine open covering $\mathcal{U}$ of X and recall that $\check{H}^p(\mathcal{U}, \mathcal{F}) = H^p(X, \mathcal{F})$ by Čech comparison in Proposition 4.33. By the same token, we get $\check{H}^p(\mathcal{U}', \mathcal{F} \otimes_A A') = H^p(X', \mathcal{F} \otimes_A A')$. But it is obvious by construction (and the fact that every intersection $U_{i_0 \dots i_p}$ is affine) that Čech complexes commute with base change and now we use that flat tensoring is exact and thus preserves cohomology. □

Next, we would like to read $R^i f_* \mathcal{F}$ out of a concrete complex of quasi-coherent sheaves. Under separatedness assumptions, we do this via a relative Čech construction.

Lemma 4.56. *Let $f : X \to S$ be a morphism of separated schemes, and $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. For any finite affine open covering $\mathcal{U}$ of X, the Čech sheaf complex*

$$\check{\mathcal{C}}^p(\mathcal{U}, f, \mathcal{F}) = \bigoplus_{i_0 \dots i_p} f_{i_0 \dots i_p *} f^*_{i_0 \dots i_p} \mathcal{F} \tag{4.27}$$

with the obvious differentials has cohomology sheaves functorially isomorphic to $R^i f_* \mathcal{F}$.

Proof. We omit the proof for now as it is extremely similar to the work we did when computing cohomology via the Čech complex. □

Next, we apply this to base change.

Lemma 4.57. *Consider a base change diagram* $(f, g)\colon X \times_S S' \to S$ *of separated schemes, and let* $\mathcal{F}$ *be a quasi-coherent sheaf on* X*. Choose a finite affine open covering* $\mathcal{U} : X = \bigcup U_i$ *of* X*. Then, the cohomology of* $g^*\check{\mathcal{C}}^\bullet(\mathcal{U}, f, \mathcal{F})$ *recovers* $R^i f'_* \mathcal{F}'$*.*

Proof. We let $\mathcal{U}' := (g')^*\mathcal{U}$ regardless of whether g is affine or not. Then, we get an equality of Čech complexes $g^*\check{\mathcal{C}}^\bullet(\mathcal{U}, f, \mathcal{F}) = \check{\mathcal{C}}^\bullet(\mathcal{U}', f', \mathcal{F}')$. Moreover, exactly as in Lemma 4.56, one see that the right complex still computes $R^i f'_* \mathcal{F}'$ upon taking cohomology because $U'_i \to X'$ and $U'_i \to S'$ are still affine maps. □

We warn that this statement does not simply upgrade to the derived category, as we are using the underived pullback g^*, instead of its left derived functors $L^i g^*$. When f is proper, we can do more than the above, namely prove that the cohomology complex is perfect in grown-up language.

Lemma 4.58. *Let* A *be a Noetherian ring with spectrum* S*,* $f : X \to S$ *be a proper morphism of schemes, and* $\mathcal{F}$ *be an* S*-flat coherent sheaf on* X*. Then there exists a finite complex of finite projective* A*-modules* $M^\bullet$ *such that* $H^i(X_{A'}, \mathcal{F}_{A'}) = H^i(M^\bullet \otimes_A A')$ *functorially in* A'*.*

Proof. Choose a finite affine open covering $X = \bigcup_{i=1,\ldots,n} U_i$. By Lemmas 4.56 and 4.57, the alternating Čech complex $K^\bullet := \check{C}^\bullet_{\text{alt}}(\mathcal{U}, \mathcal{F})$ satisfies computes cohomology $H^i(X, \mathcal{F})$ after arbitrary base change $A \to A'$. Since $\mathcal{F}$ is flat over A, we see that each K^n_{alt} is flat over A, but almost never finite over A. However, $K^\bullet$ is bounded and its cohomology groups are finite over A by Corollary 4.53. Using projective resolutions, we can replace $K^\bullet$ by a bounded complex $M^\bullet$ of finite A-modules such that M^n is projective if $n > 0$, and which still computes the cohomology of $\mathcal{F}$ after any base change. Now, we observe that the mapping cone $C^\bullet$ of the map $M^\bullet \to K^\bullet$ is exact and its terms are direct sums $M^{n+1} \oplus K^n$, so we deduce that M^0 must also be flat by using Tor vanishing. □

Let us now define the Euler characteristic of a coherent sheaf on a proper k-scheme X for any field k.

Definition 4.59. Let k be a field, X a proper k-scheme, and $\mathcal{F}$ a coherent sheaf on X. Then, its Euler characteristic

$$\chi(X, \mathcal{F}) = \sum_{i \geq 0} (-1)^i \dim_k H^i(X, \mathcal{F}) \tag{4.28}$$

is the alternating sum of the dimensions of its cohomology groups.

This is well-defined because the $H^i(X, \mathcal{F})$ are finite k-modules and vanish for all $i \gg 0$. Now, we can prove that Euler characteristic remains locally constant in families, assuming flatness.

Theorem 4.60. *Let* S *be locally Noetherian,* $X \to S$ *be a proper map and* $\mathcal{F}$ *be an* S*-flat coherent sheaf on* X*. Then the Euler function* $|S| \to \mathbb{Z}$ *given by* $s \mapsto \chi(X_s, \mathcal{F}_s)$ *is locally constant.*

Proof. We have an equality $\chi(X_s, \mathcal{F}_s) = \sum (-1)^i \dim_s M^i_s$ by the previous lemma and the long exact sequence of cohomology applied inductively to the complex $M^\bullet$. It is trivial to see that $s \mapsto \dim_s M^i_s$ is locally constant, as M^i_s is finite projective over A. □

Let us recall that a function $f\colon |S| \to \mathbb{R}$ is upper semi-continuous if the strict sublevel sets $f^{-1}(\mathbb{R}_{<r})$ are open in $|S|$ for all real numbers r. This means that that the function can only jump up when passing to closeds of $|S|$. The dual notion of upper semi-continuity is called lower semi-continuity.

Theorem 4.61. *Let S be locally Noetherian, $X \to S$ be a proper map and $\mathcal{F}$ be an S-flat coherent sheaf on X. Then the function $|S| \to \mathbb{Z}$ given by $s \mapsto \dim_k H^i(X_s, \mathcal{F}_s)$ is upper semi-continuous.*

Proof. Let $B^i_{A'} \subset Z^i_{A'} \subset M^i \otimes_A A'$ be the cycles and boundaries of the complex in the key lemma *after base change.* This implies that

$$\dim_k H^i(X_s, \mathcal{F}_s) = \dim_k M^i_s - \dim_k B^i_s - \dim_k B^{i+1}_s \tag{4.29}$$

and this reduces us to prove lower semi-continuity of $s \mapsto \dim_k B^i_s$. But bounding the fiber rank of the map $M^{i-1} \to M^i$ by some d corresponds to demand vanishing of certain local minors of the corresponding matrix, defining a closed of S. □

The next result that we are going to prove is Grauert's theorem.

Theorem 4.62. *Let S be a locally noetherian reduced scheme, $X \to S$ be a proper map and $\mathcal{F}$ be an S-flat coherent sheaf on X. If the function $|S| \to \mathbb{Z}$ given by $s \mapsto \dim_k H^i(X_s, \mathcal{F}_s)$ is locally constant, then $R^i f_* \mathcal{F}$ is locally free with S-fibers equal to $H^i(X_s, \mathcal{F}_s)$.*

Proof. We may and do assume that S is the spectrum of a noetherian ring A, in order to apply our key lemma. We have that the fiber dimensions of B^i_s and B^{i+1}_s are locally constant by the formula in the previous theorem. This implies that $W^i := \mathrm{coker}(M^{i-1} \to M^i)$ and $W^{i+1} := \mathrm{coker}(M^i \to M^{i+1})$ have locally constant ranks. Because A is reduced, this implies these cokernels are locally free. Indeed, we get a surjection $A^{\oplus n} \to N$ with same local ranks by Nakayama's lemma after possibly inverting an element of A. If some vector (a_i) is in the kernel with $a_1 \neq 0$ up to reordering, then there exists some $\mathfrak{p}$ not containing a_1 (as A is reduced and the nilradical is the intersection of all primes), and hence $N_\mathfrak{p}$ has smaller rank. Now, we can see that B^{i+1} and thus also H^i are finite projective A-modules and therefore satisfy base change. □

Finally, we have the following strong predictor for the behavior of fibers of cohomology, usually called cohomology and base change.

Theorem 4.63. *Let S be a locally noetherian scheme, $X \to S$ be a proper map and $\mathcal{F}$ be an S-flat coherent sheaf on X. If $R^i f_* \mathcal{F} \to H^i(X_s, \mathcal{F}_s)$ is surjective for all s, then the base change map $g^* R^i f_* \mathcal{F} \to Rf'_* \mathcal{F}'$ is an isomorphism for any map $g\colon S' \to S$. Moreover, $R^{i-1} f_* \mathcal{F} \to H^{i-1}(X_s, \mathcal{F}_s)$ is also surjective if and only if $R^i f_* \mathcal{F}$ is locally free over S.*

Proof. We are going to assume that S is the spectrum of a noetherian ring A. Recall that we have defined the R-modules $B^i_R \subset Z^i_R \subset M^i_R$ and $H^i_R := Z^i_R / B^i_R \subset W^i_R := M^i_R / B^i_R$ after base changing, and the theorem relates to understanding when these are compatible with base change. If $H^i \to H^i_s$ is surjective, then also $Z^i \to Z^i_s$ is surjective by the snake lemma and the automatic surjectivity of $B^i \to B^i_s$. The snake lemma again tells us that B^{i+1} satisfies base change at every s. But this implies $\mathrm{Tor}_1(\kappa(s), W^{i+1})$ vanishe taking the corresponding long exact sequence. The local criterion for flatness implies that W^{i+1} is flat, so the same holds for B^{i+1} and Z^i. It now follows easily that H^i satisfies base change for every A-algebra R. If $H^{i-1} \to H_s^{i-1}$ is surjective, then we see that W^i is also finite projective, and thus so is H^i. Conversely, if H^i is finite projective, then B^i is also finite projective and hence satisfies base change at all s. Using the snake lemma once more, we get that $Z^{i-1} \to Z_s^{i-1}$ is surjective, and thus so is $H^{i-1} \to H_s^{i-1}$. □

5. Descent

5.1. **Sites.** Unfortunately, the Zariski topology is not a very fine structure in algebraic geometry, because there are just not that many open subschemes. Grothendieck's fundamental idea was that the notion of topology does not necessarily require us to think in terms of open *subsets*, but rather that we can think of the maps $U \to X$ to be themselves open, but necessarily monomorphisms. This insight led in a short amount time to the discovery of *étale cohomology* and progress on the Weil conjectures, as well as a foundation for *rigid-analytic* geometry. Armed with this, we will have a clear geometric picture of what it means to glue objects in algebraic geometry along *flat covers*, instead of just open ones.

Definition 5.1. A *site* is given by a category $\mathcal{C}$ and a set $\mathrm{Cov}(\mathcal{C})$ of families of morphisms $\{U_i \to U\}_{i\in I}$ for some set I, called *coverings of* $\mathcal{C}$, and satisfying the following axioms

(1) If $V \to U$ is an isomorphism then $\{V \to U\} \in \mathrm{Cov}(\mathcal{C})$.
(2) If $\{U_i \to U\}_{i\in I} \in \mathrm{Cov}(\mathcal{C})$ and for each i we have $\{V_{ij} \to U_i\}_{j\in J_i} \in \mathrm{Cov}(\mathcal{C})$, then $\{V_{ij} \to U\}_{i\in I, j\in J_i} \in \mathrm{Cov}(\mathcal{C})$.
(3) If $\{U_i \to U\}_{i\in I} \in \mathrm{Cov}(\mathcal{C})$ and $V \to U$ is a morphism of $\mathcal{C}$ then $U_i \times_U V$ exists for all i and $\{U_i \times_U V \to V\}_{i\in I} \in \mathrm{Cov}(\mathcal{C})$.

Usually, we consider appropriate subcategories of schemes, which are stable under fiber products and disjoint unions. It will often happen (except in the Zariski topology) that a (possibly even infinite) cover $\{U_i \to U\}$ gives rise to a singleton cover $V \to U$ with $V := \sqcup_{i\in I} U_i$, allowing us to focus most of our energy on the latter case.

Remark 5.2. If one wants to define the site of all schemes with, e.g., the Zariski topology, one quickly runs into set-theoretic difficulties. To resolve this, one should either assume the existence of universes and work inside them throughout, or fix a strong limit cardinal κ and define every category in such a way that its objects and morphisms have cardinality bounded by κ. We will not occupy ourselves with these cumbersome technicalities and will simply ignore the previous issues.

Example 5.3. Let X be a scheme and define X_{Zar} as the category of open immersions $U \to X$ with the obvious morphisms. We declare $\{U_i \to U\}_{i\in I} \in \mathrm{Cov}(X_{\mathrm{Zar}})$ to be a cover if $\bigcup U_i = U$. This is clearly a site and the our general definition of sheaves on arbitrary sites applied to X_{Zar} shall agree with the usual notion.

Example 5.4. Let G be a group. Consider the category G-Sets whose objects are sets X with a left G-action, and morphisms are G-equivariant. This category admits fiber products and we declare $\{\varphi_i : U_i \to U\}_{i\in I}$ to be a cover if $\bigcup_{i\in I} \varphi_i(U_i) = U$. One can readily check that this defines a site. For the advanced readers, we note that this site is related to an algebraic one via the theory of *stacks*, namely one can consider the classifying stack of G-torsors, denoted either BG or $[*/G]$.

Next, we can define the notion of sheaves on arbitrary sites.

Definition 5.5. Let $\mathcal{C}$ be a site, and let $\mathcal{F}$ be a presheaf of sets on $\mathcal{C}$. We say $\mathcal{F}$ is a *sheaf* if for every cover $\{U_i \to U\}_{i\in I} \in \mathrm{Cov}(\mathcal{C})$ the diagram

$$\mathcal{F}(U) \longrightarrow \prod_{i\in I} \mathcal{F}(U_i) \rightrightarrows \prod_{(i_0,i_1)\in I\times I} \mathcal{F}(U_{i_0} \times_U U_{i_1}) \tag{5.1}$$

is an equalizer. The category $\mathrm{Sh}(\mathcal{C})$ of sheaves is the full subcategory of $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Sets})$ whose objects are sheaves.

This is very similar to the usual definition, except we are not allowed to take intersections anymore (as cover maps are not monomorphisms), but instead take fiber products. More generally, given another category $\mathcal{A}$, one can define the notion of $\mathcal{A}$-valued sheaves on $\mathcal{C}$, but we do not pursue this.

To define sheafification in the site framework, we need Čech cohomology. Let $\mathcal{F}$ be a presheaf on $\mathcal{C}$, and $\mathcal{U} = \{U_i \to U\}_{i\in I}$ be a covering of $\mathcal{C}$. We use the notation $\mathcal{F}(\mathcal{U})$ to indicate the equalizer of the two pullback maps as in the previous equation – this is the 0-th Čech cohomology group $H^0(\mathcal{U}, \mathcal{F})$. There is a canonical map $\mathcal{F}(U) \to \mathcal{F}(\mathcal{U})$ compatibly with the pullback maps $\mathcal{F}(\mathcal{U}) \to \mathcal{F}(\mathcal{V})$ along maps of covers $\mathcal{V} \to \mathcal{U}$. A sufficient and necessary condition for $\mathcal{F}$ to be a sheaf is that this natural map is an isomorphism for all covers $\mathcal{U}$.

Lemma 5.6. *The assignment $\mathcal{F}^+$ via the rule $U \mapsto \mathcal{F}^+(U) := \check{H}^0(U, \mathcal{F}) := \lim_{\mathcal{U}} H^0(\mathcal{U}, \mathcal{F})$ with the limit running through all covers $\mathcal{U}$ of U defines a natural presheaf equipped with a canonical map of presheaves $\mathcal{F} \to \mathcal{F}^+$.*

Proof. This group is the zeroth Čech cohomology of $\mathcal{F}$ over U. The transition maps $\mathcal{F}^+(U) \to \mathcal{F}^+(V)$ are defined thanks to the stability under pullbacks imposed on the site $\mathcal{C}$. Their associativity is checked via stability under composition and it is routine to check that this is compatible with the obvious map $\mathcal{F} \to \mathcal{F}^+$ whose values on U is the limit of $\mathcal{F}(U) \to \mathcal{F}(\mathcal{U})$. □

More precisely, the above construction $\mathcal{F} \mapsto \mathcal{F}^+$ is an endofunctor of $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Set})$ (and the canonical map from $\mathcal{F}$ is also functorial). Because two covers can be simultaneously refined, the limit in $\mathcal{U}$ is cofiltered and hence every element of $\mathcal{F}^+(U)$ arises from $H^0(\mathcal{U}, \mathcal{F})$ for some cover $\mathcal{U}$ of U, and two elements agree if they do already on $H^0(\mathcal{V}, \mathcal{F})$ for some refinement $\mathcal{V} \to \mathcal{U}$.

Definition 5.7. We say that a presheaf $\mathcal{F}$ on a site $\mathcal{C}$ is *separated* if, for all covers $\{U_i \to U\}$, the map $\mathcal{F}(U) \to \prod \mathcal{F}(U_i)$ is injective.

Now, we can fully describe the sheafification functor.

Theorem 5.8. *The following properties hold:*

(1) *The presheaf $\mathcal{F}^+$ is separated.*
(2) *If $\mathcal{F}$ is separated, then $\mathcal{F}^+$ is a sheaf and $\mathcal{F} \to \mathcal{F}^+$ is injective.*

In particular, $\mathcal{F}^{++}$ is always a sheaf.

Proof. The first part was explained already. Indeed if two elements of $\mathcal{F}^+(U)$ coincide on a cover $\mathcal{U}$, we may after some refinement realize them as the same elements in $H^0(\mathcal{V},\mathcal{F})$, hence by definition also in $\mathcal{F}^+(U)$.

As for the second part, injectivity is clear. Now, let $\mathcal{U}$ be a cover and suppose s lies in the equalizer $H^0(\mathcal{U},\mathcal{F}^+)$. After choosing a refinement $\mathcal{V}\to\mathcal{U}$, we can use the injectivity to upgrade s to an element of $H^0(\mathcal{V},\mathcal{F})$, so it comes from $\mathcal{F}^+(U)$ as desired. □

Definition 5.9. Let $\mathcal{C}$ be a site and let $\mathcal{F}$ be a presheaf on $\mathcal{C}$. The sheaf $\mathcal{F}^\# := \mathcal{F}^{++}$ together with the canonical map $\mathcal{F}\to\mathcal{F}^\#$ is called the *sheaf associated to* $\mathcal{F}$.

The assignment above is functorial and we call it the sheafification functor. It is easy to see that $\mathcal{F}\to\mathcal{F}^\#$ is the initial map of presheaves towards a sheaf. In other words, sheafification is left adjoint to the inclusion functor $\mathrm{Sh}(\mathcal{C})\to\mathrm{Fun}(\mathcal{C},\mathrm{Set})$. It is a general fact that sheafification is an exact functor (right exactness follows from left adjointness, and the rest from commuting with finite limits, because covers are cofiltered). Just as for sheaves on topological spaces, we get the usual notion of injectivity (can be checked at the level of presheaves) and surjectivity (existence of lift up to refining covers).

Remark 5.10. There is a general definition of morphisms of sites and of topoi (i.e., categories of sheaves on sites) that we will avoid in these notes. Knowing these tools is useful for transitioning from one topology to another.

5.2. **Abstract descent data.** In this subsection, we are going to introduce the main definitions surrounding faithfully flat descent of quasi-coherent sheaves. The basic idea are that we consider quasi-coherent sheaves $\mathcal{F}_i$ on the terms of an fppf cover $S_i\to S$ that are compatible with each other in a precise sense, and then we attempt to realize all of them as arising from a quasi-coherent sheaf $\mathcal{F}$ on S, called the *descent* of the $\mathcal{F}_i$.

We are going to constantly use the following notation. If we have two S-schemes X_0 and X_1, then we write $p_i\colon X_0\times_S X_1\to X_i$ with $i=0,1$ for the canonical projections. Similarly, if we are given a further S-scheme X_2 we let $p_{ij}\colon X_0\times_S X_1\times_S X_2\to X_i\times_S X_j$ with $0\le i<j\le 2$ denote the obvious projection. At this point, you should understand how to continue these conventions, should the need ever arise (hopefully not).

Definition 5.11. Let S be a scheme and $\{f_i : S_i\to S\}_{i\in I}$ be an fpqc cover.

(1) A *descent datum* $(\mathcal{F}_i,\varphi_{ij})$ *for quasi-coherent sheaves* with respect to the given family is given by a quasi-coherent sheaf $\mathcal{F}_i$ on S_i for each $i\in I$, an isomorphism of quasi-coherent $\mathcal{O}_{S_i\times_S S_j}$-modules $\varphi_{ij} : p_0^*\mathcal{F}_i\to p_1^*\mathcal{F}_j$ for each pair (i,j) such that for every triple of indices (i,j,k), we get a commutative diagram

$$\begin{array}{ccc} p_0^*\mathcal{F}_i & \xrightarrow{\;p_{02}^*\varphi_{ik}\;} & p_2^*\mathcal{F}_k \\ & {\scriptstyle p_{01}^*\varphi_{ij}}\searrow \qquad \nearrow{\scriptstyle p_{12}^*\varphi_{jk}} & \\ & p_1^*\mathcal{F}_j & \end{array} \tag{5.2}$$

of $\mathcal{O}_{S_i\times_S S_j\times_S S_k}$-modules, called the *cocycle condition.*

(2) A *morphism* $\psi : (\mathcal{F}_i,\varphi_{ij})\to(\mathcal{F}'_i,\varphi'_{ij})$ *of descent data* is given by a family $\psi=(\psi_i)_{i\in I}$ of morphisms of $\mathcal{O}_{S_i}$-modules $\psi_i : \mathcal{F}_i\to\mathcal{F}'_i$ such that all the diagrams

$$\begin{array}{ccc} p_0^*\mathcal{F}_i & \xrightarrow{\varphi_{ij}} & p_1^*\mathcal{F}_j \\ {\scriptstyle p_0^*\psi_i}\downarrow & & \downarrow{\scriptstyle p_1^*\psi_j} \\ p_0^*\mathcal{F}'_i & \xrightarrow{\varphi'_{ij}} & p_1^*\mathcal{F}'_j \end{array} \tag{5.3}$$

commute.

Example 5.12. Let $S = \bigcup S_i$ be an open cover. Then, the category of glueing data of quasi-coherent sheaves with respect to this open cover is equivalent to QCoh_S.

For a quasi-coherent sheaf $\mathcal{F}$ on S and an fpqc cover $\{f_i \colon S_i \to S\}_{i\in I}$, we denote simply by $(f_i^*\mathcal{F}, \mathrm{can})$ the descent datum with the obvious transition maps given by composability of pullbacks.

Definition 5.13. Let S be a scheme and $\{f_i \colon S_i \to S\}_{i\in I}$ be an fpqc cover. A descent datum $(\mathcal{F}_i, \varphi_{ij})$ for quasi-coherent sheaves with respect to the given covering is said to be *effective* if there exists a quasi-coherent sheaf $\mathcal{F}$ on S such that $(\mathcal{F}_i, \varphi_{ij})$ is isomorphic to $(f_i^*\mathcal{F}, \mathrm{can})$.

We have already seen that descent data for open covers are always effective. Our goal is to prove instances of fpqc descent. This requires certain assumptions, and as always we need to start by the simplest case, i.e., that of affine fpqc covers $f \colon X := \mathrm{Spec}(B) \to S := \mathrm{Spec}(A)$.

Let $A \to B$ be a flat homomorphism of rings. This gives rise to a cosimplicial A-algebra

$$B \;\substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow}\; B \otimes_A B \;\substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow}\; B \otimes_A B \otimes_A B \tag{5.4}$$

continuing indefinitely to the right. More precisely, $(B/A)_\bullet$ is defined so that $(B/A)_n$ is the $(n+1)$-fold tensor product of B over A, and given a non-decreasing map $\varphi : [n] \to [m]$ the A-algebra map $(B/A)_\bullet(\varphi)$ is the map

$$a_0 \otimes \ldots \otimes a_n \longmapsto \prod_{\varphi(i)=0} a_i \otimes \prod_{\varphi(i)=1} a_i \otimes \ldots \otimes \prod_{\varphi(i)=m} a_i \tag{5.5}$$

where we use the convention that the empty product is 1. (Exercise: write down all the first 8 face and degeneracy maps displayed in the diagram above.) An A-module M gives rise to a cosimplicial $(B/A)_\bullet$-module $(B/A)_\bullet \otimes_R M$ by tensoring on the right. In this setting, we have an analogue of a descent datum for quasi-coherent sheaves.

Definition 5.14. Let $A \to B$ be a faithfully flat ring homomorphism.

(1) A *descent datum* (N, φ) *for modules with respect to* $A \to B$ is given by a B-module N and an isomorphism of $B \otimes_A B$-modules $\varphi : N \otimes_A B \to B \otimes_A N$ such that the *cocycle condition* holds, i.e., we get a commutative diagram

$$\begin{array}{ccccc} N \otimes_A B \otimes_A B & & \xrightarrow{\quad\varphi_{02}\quad} & & B \otimes_A B \otimes_A N \\ & {\scriptstyle \varphi_{01}}\searrow & & \nearrow{\scriptstyle \varphi_{12}} & \\ & & B \otimes_A N \otimes_A B & & \end{array} \tag{5.6}$$

of $B \otimes_A B \otimes_A B$-modules, where the maps φ_{ij} are given by scalar extending φ along the obvious cosimplicial face maps of $(B/A)_\bullet$.

(2) A *morphism* $(N, \varphi) \to (N', \varphi')$ *of descent data* is a morphism of A-modules $\psi : N \to N'$ such that the diagram

$$\begin{array}{ccc} N \otimes_A B & \xrightarrow{\varphi} & B \otimes_A N \\ {\scriptstyle \psi \otimes \mathrm{id}_B}\downarrow & & \downarrow{\scriptstyle \mathrm{id}_B \otimes \psi} \\ N' \otimes_A B & \xrightarrow{\varphi'} & B \otimes_A N' \end{array} \tag{5.7}$$

is commutative.

It is not too hard to see using the cocycle condition that the descent datum (N, φ) induces a natural cosimplicial $(B/A)_\bullet$-module $N_\bullet$, taking $[n]$ to the n-fold tensor product of B over A tensored with N on the right. Given an A-module M, we can apply this construction to the canonical descent datum $(M \otimes_A B, \mathrm{can})$, and we recover $(B/A)_\bullet \otimes_A M$. Recall that our goal is to study effectivity of (N, φ), i.e., that it arises from some A-module M. For this, we consider the complex $s(N, \varphi)$ below

$$N \to B \otimes_A N \to B \otimes_A B \otimes_A N \to \dots \tag{5.8}$$

obtained by taking the alternating sum of the face maps of $N_\bullet$. (Exercise: write down the first two maps.) If $(N, \varphi) = (M \otimes_A B, \mathrm{can})$, we see that the kernel of the first map carries a map from M, so $s(M \otimes_A B, \mathrm{can})$ extends to a complex

$$\tilde{s}(M) := [0 \to M \to s(M \otimes_A B, \mathrm{can})]. \tag{5.9}$$

The next observation is crucial for descent.

Lemma 5.15. *The complex $\tilde{s}(M)$ is exact.*

Proof. The assertion is clear stable under tensoring with faithfully flat covers $A \to A'$. Taking $A' = B$ itself, we see that the new faithfully flat homomorphism $B \to B \otimes_A B$ appearing in the descent admits a section $B \otimes_A B \to B$ induced by multiplication. In other words, it suffices to treat the case where $A \to B$ admits a section. Under this assumption, one can explicitly verify that $A \to (B/A)_\bullet$ is a homotopy equivalence of cosimplicial R-algebras. Passing to complexes, we get the desired exactness. □

In particular, we deduce that (N, φ) is effective if and only if the obvious $B \otimes_A H^0(s(N, \varphi)) \to N$ is an isomorphism. Notice that we can verify this after a faithfully flat cover $A \to A'$, and hence assume $A \to B$ has a section.

Proposition 5.16. *Every descent datum (N, φ) is effective.*

Proof. As explained before, we assume that $A \to B$ has a section $\sigma : B \to A$. Set $M := H^0(s(N, \varphi))$: we have to show that $B \otimes_A M \to N$ is an isomorphism. Take an element $n \in N$. Write $\varphi(n \otimes 1) = \sum b_i \otimes x_i$ for certain $b_i \in B$ and $x_i \in N$, so in particular $n = \sum b_i x_i$. Next, write $\varphi(x_i \otimes 1) = \sum b_{ij} \otimes y_j$ for certain $b_{ij} \in B$ and $y_j \in N$. Applying σ to the cocycle condition, we deduce on the one hand

$$\sum \sigma(b_i) \varphi(x_i \otimes 1) = \sum \sigma(b_i) \otimes x_i \tag{5.10}$$

and on the other hand

$$\sum_i b_i \otimes \sum_j \sigma(b_{ij}) y_j = \sum b_i \otimes x_i. \tag{5.11}$$

In particular, the first equation tells us $\sum \sigma(b_i)x_i \in M$, and by symmetry so does $\sum \sigma(b_{ij} \otimes y_j)$. Now, the second equation implies $\sum_i b_i(\sum_j \sigma(b_{ij})y_j) = \sum b_i x_i = n$, and thus $B \otimes_A M \to N$ is surjective. Injectivity is clear by faithful flatness of $A \to B$. □

Remark 5.17. Assume $B = \prod_i A[f_i^{-1}]$ for finitely many $f_i \in A$. Then $(B/A)_n = \prod_{i_0,\ldots,i_n} A[f_{i_0}^{-1} \ldots f_{i_n}^{-1}]$ and our previous theorem not only recovers the fact that sheafification is unnecessary for the structure presheaf, but also that quasi-coherent sheaves on affines have vanishing Čech cohomology.

Now, we can return to general schemes and prove fpqc descent for quasi-coherent sheaves.

Proposition 5.18. *Let S be a scheme and $f\colon X \to S$ be an fpqc cover. Any descent datum on quasi-coherent sheaves for X is effective and the functor* $\mathrm{QCoh}_S \to \mathrm{Desc}_{X/S}$ *is an equivalence.*

Proof. First, we show that the functor is faithful. Let $\mathcal{F}$, $\mathcal{G}$ be quasi-coherent sheaves on S and let $a, b : \mathcal{F} \to \mathcal{G}$ be homomorphisms of $\mathcal{O}_S$-modules. If $f^*(a) = f^*(b)$, then we can apply faithful flatness of $\mathcal{O}_{S,s} \to \mathcal{O}_{X,x}$ with $s = f(x)$ to the kernel of $a - b$ to deduce that $a_s = b_s : \mathcal{F}_s \to \mathcal{G}_s$. Hence $a = b$.

Before continuing, we note that, by definition of the fpqc site, there exists a fpqc cover $g\colon Y \to X$ such that $h := g \circ f$ factors as an fpqc affine cover and an open cover. Now, we prove fully faithfulness. Let $\mathcal{F}$, $\mathcal{G}$ be quasi-coherent sheaves on S and let $c : f^*\mathcal{F} \to f^*\mathcal{G}$ be a homomorphisms of $\mathcal{O}_X$-modules such that $p_0^*c = p_1^*c$ on $X \times_S X$. First, we see that g^*c equals h^*a for some $a\colon \mathcal{F} \to \mathcal{G}$ by fpqc descent in the affine case (and open covers as well, which is trivial by the sheaf property). Now we invoke faithfulness to get $c = f^*a$, proving fullness.

Finally, we treat essential surjectivity. Let $(\mathcal{G}, \varphi)$ be a descent datum for quasi-coherent sheaves relative to f. We know from fpqc affine descent that $(g^*\mathcal{G}, g^*\varphi)$ on Y assumes the form $(h^*\mathcal{F}, \mathrm{can})$ for some quasi-coherent sheaf $\mathcal{F}$ on S. Using full faithfulness, we deduce that the isomorphism above descends to an identification $(\mathcal{G}, \varphi) = (f^*\mathcal{F}, \mathrm{can})$. □

Remark 5.19. Let us consider the case of Galois descent. Finite field extensions l/k induce faithfully flat ring maps. Assume l/k is Galois, so that we have

$$l \otimes_k l = \prod_{\sigma \in \mathrm{Gal}(l/k)} l \tag{5.12}$$

via $a \otimes b \mapsto (a\sigma(b))$, and similarly for the remaining terms of the cosimplicial algebra $(l/k)_\bullet$. Given a k-scheme X, we see that descent data of quasi-coherent sheaves along $X_l := X \otimes_k l \to X$ amounts to a quasi-coherent sheaf $\mathcal{F}$ on X_l together with isomorphisms $\varphi_\sigma\colon \mathcal{F} \to \sigma^*\mathcal{F}$ giving rise to a 1-cocycle $\mathrm{Gal}(l/k) \to \mathrm{Aut}(\mathcal{F})$. Now, we know they are effective.

5.3. **Descent of properties.** In this section, we pursue the idea that many absolute or relative properties of schemes that we have encountered so far can be really seen locally in some of our favorite topologies: étale smooth, fppf, or fpqc.

Definition 5.20. Let $\mathcal{P}$ be a property of morphisms of schemes. We say that $\mathcal{P}$ is *τ-local on the source* if it respects disjoint unions and $Y \to X$ has $\mathcal{P}$ if and only if $Z \to X$ has $\mathcal{P}$ for any τ-cover $Z \to Y$.

Lemma 5.21. *Let $\mathcal{P}$ be one of the following properties: locally of finite presentation, flat, smooth, étale. Then $\mathcal{P}$ is fppf local on the source.*

Proof. Finite presentation is relatively easy (especially under noetherian assumptions), so we leave it as an exercise. Flatness reduces to an assertion about homomorphisms of rings $A \to B \to C$. If $M \to N$ is an injection such that $B \otimes_A M \to B \otimes_A N$ has non-trivial kernel K, then $C \otimes_B K \subset C \otimes_A M$ by faithful flatness, while $C \otimes_A M \to C \otimes_A N$ is injective by flatness. In the smooth/étale case, we already know that $X \to Y$ is fppf, so we only need to pass to geometric fibers to verify smoothness and or étaleness. The étale case is simple, because X is now a disjoint union of points over an algebraically closed field, and $X \to Y$ is faithfully flat, so for dimension reasons it is discrete of dimension 0, and moreover reduced as $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ with $y = f(x)$ is injective.

The smooth case is more difficult, because we need to check that regularity of local rings can be verified after a fppf cover. This is a consequence of a hard theorem of Serre in commutative algebra characterizing regular rings as the only ones whose global dimension is finite. Here, having finite global dimension is a homological notion stable under faithfully flat base change. □

Definition 5.22. Let $\mathcal{P}$ be a property of schemes. We say that $\mathcal{P}$ is τ-local if it respects disjoint unions and X has $\mathcal{P}$ if and only if Y does for any τ-cover $Y \to X$.

Proposition 5.23. *Let $\mathcal{P}$ be one of the following properties: reduced, normal, regular. Then P is local in the smooth topology.*

Proof. Let us first handle the regular case. A smooth R-algebra S is étale over some polynomial R-algebra, so ascendability reduces to the étale situation. But étale maps preserve tangent space dimensions, so this is clear. Descendability was checked in the lemma above. Next, we use Serre's criterion for reducedness and normality, namely that a ring is reduced (resp. normal) if and only if it is R_0 and S_1 (resp. R_1 and S_2). The R_k type conditions prescribe regularity in points of dimension $\leq k$, whereas the S_k condition is related to the homological notion of depth which we do not recall here. In any case, we have checked that R_k ascends and descends along smooth maps, and the same can be done for S_k, so this yields the proposition. □

The most frequently used kind of local property is the one below involving base change, because it applies to a very wide range of situations.

Definition 5.24. Let $\mathcal{P}$ be a property of morphisms of schemes. We say $\mathcal{P}$ is *τ-local on the target* if it respects disjoint unions and if $X \to Y$ has $\mathcal{P}$ if and only if $X \times_Y Z \to Z$ has $\mathcal{P}$ for any τ-cover $Z \to Y$.

Theorem 5.25. *Let $\mathcal{P}$ be one of the following properties: qcqs, finitely presented, separated, proper, flat, open immersion, closed immersion, affine, smooth, étale. Then, $\mathcal{P}$ is fpqc local on the target.*

Proof. Base change clearly preserves quasi-compactness. Let $S' \to S$ be a fpqc cover of affine schemes, and let $f : X \to S$ be a morphism. If $f' : X' := X \times_S S' \to S'$ is quasi-compact, then so is X' and thus also X by surjectivity of the projection $X' \to X$. The quasi-separated case follows from the quasi-compact one by considering the diagonal $\Delta_f : X \to X \times_S X$.

For simplicity, we will only prove locality for finite type, which is enough in the noetherian case. Being of finite type is preserved under base change, so let $S' \to S$ be a fpqc cover of affines, and let $f : X \to S$ be a morphism. By our proof for quasi-compactness, we may assume that X is affine. Then, we are reduced to the situation of a ring map $A \to B$ which becomes of finite type $A' \to B'$ after tensoring with a faithfully flat map $A \to A'$. Perhaps after some enlargement, there is an A-algebra map $C \to B$ from a finite type polynomial A-algebra C whose tensor product $C' \to B'$ with A' is surjective. Hence, $C \to B$ is also surjective by faithful flatness.

Base change preserves separatedness, so consider a fpqc cover $S' \to S$ of affines, and let $f : X \to S$ be a morphism with separated base change $f' : X' \to S'$ is separated. This means that $\Delta' : X' \to X' \times_{S'} X'$ is a closed immersion, and since Δ is an immersion in general, it suffices to show descent for universally closedness to get separatedness. This will prove descent for properness as well. So now we assume instead that $f : X \to S$ is a map with closed base change $f' : X' \to S'$. The fpqc cover $X' \to X$ is a quotient map on underlying topological spaces, so a diagram chase now reveals that f is also closed.

Now, we treat flatness. This property is stable under base change, so we consider a fpqc cover $S' \to S$ of affines. Let $f : X \to S$ be a morphism with flat base change $f' : X' \to S'$. We can reduced to the case where f is affine, and then the claim follows as for the fppf locality of flatness on the source.

Base change also preserves affineness, so let $g : S' \to S$ be an fpqc cover of affines and $f : X \to S$ be a morphism with affine base change $f' : X' \to S'$. Our previous efforts tell us that f is a separated qcqs map. We can thus form the quasi-coherent sheaf $f_*\mathcal{O}_X$ on S, whose relative spectrum defines the affine hull of f. Note that we have flat base change, i.e., $g^* f_* \mathcal{O}_X = f'_* \mathcal{O}_{X'}$, so the affine hull is fpqc local on the target. Therefore, we can replace S by the affine hull of X, thereby reducing to the case where f' is an isomorphism and need to descend this. The same argument as for universally closed maps shows that f is a universal homeomorphism. Let $x \in U \subset X$ be an affine open neighborhood and use the homeomorphism to find a global section f of S such that $x \in X[f^{-1}] \subset U$, where the notation indicates the open subscheme defined by the non-vanishing of f. Then, $X[f^{-1}] = U[f^{-1}]$ is necessarily affine and we can repeat this procedure around any point to show that f is actually an affine map. But as S was the affine hull already, we get that f is an isomorphism.

Let $S' \to S$ be an fpqc cover of affines, and let $f : X \to S$ be a morphism whose base change $f' : X' \to S'$ is an open immersion. We must show that f is an open immersion as well. As above, we can at least deduce that f is universally open and injective (look at geometric fibers). This means $f(|X|) \subset S$ is an open subset. Covering it by affine opens $U \subset S$, we reduce the problem to the case where f' is an isomorphism. We have seen above that f is at least affine, and looking at the corresponding global sections, one easily sees that also f must be an isomorphism.

Let $S' \to S$ be a fpqc cover of affines and $f : X \to S$ be a map whose base change $f' \colon X' \to S'$ is a closed immersion. Then, we know already that f must be affine, and this implies that the homomorphism on global sections is surjective as this can be checked after tensoring with a faithfully flat algebra.

As for smoothness and étaleness, we have already shown how to descend (locally) of finite presentation, and flatness. Therefore we are reduced to descending smooth and étale geometric fibers, which is obvious (because we are discussing *geometric* properties, that do not alter under change of algebraically closed field). □

Remark 5.26. There are important properties in algebraic geometry that are very sensitive to base change. Namely, being projective (or even quasi-projective) is not Zariski local.

Remark 5.27. It would also be interesting to discuss fppf descent of schemes themselves. The affine case reduces to fpqc descent of quasi-coherent sheaves via their relative spectra. In the non-affine case, it is not possible in general to descend schemes in the fppf topology, and one needs extra data, e.g., line bundles on projective schemes, etc.

6. Divisors and curves

We have already considered the category of quasi-coherent sheaves on a scheme X. A decisive example among these are invertible sheaves, i.e., those that are locally free of rank 1. They form the Picard group $\mathrm{Pic}(X)$ and one can probe them in terms of 1-codimensional cycles in X. We will define these alternative notions of Cartier and Weil divisors. We will also prove the Riemann–Roch formula for proper smooth curves over a field.

6.1. **Meromorphic functions.**

Let A be a ring. We define its ring of fractions $\mathrm{Frac}(A)$ as the localization of A at the multiplicative subset consisting of non-zero divisors of A.

Definition 6.1. Let X be a locally noetherian scheme. We denote by $\mathcal{K}_X$ the sheafification of the assignment $U \mapsto \mathrm{Frac}(\mathcal{O}_X(U))$. This is called the *sheaf of meromorphic functions* on X.

Before continuing, we should recall the notion of an associated point of a scheme.

Definition 6.2. Let X be a locally noetherian scheme. We say that $x \in X$ is an associated point if the maximal ideal of $\mathcal{O}_{X,x}$ is an associated ideal. If x is an associated but non-generic point, we call it an embedded point.

The notion of associated point on affine schemes corresponds exactly to the associated prime ideals. The collection of associated points is finite locally on X.

Example 6.3. If X is locally Noetherian and reduced, then X has no embedded points. Instead, let k be any field and X be the affine scheme given as the spectrum of $k[u, v]/(u^2, uv)$. We claim that the origin (i.e. the vanishing locus of the ideal generated by u and v) is an embedded point of X. Indeed, this maximal ideal equals the annihilator of u, so it is an associated prime ideal, and it is clearly not a minimal prime ideal.

If X has no embedded points, then $\mathcal{K}_X$ is a quasi-coherent sheaf given as the direct sum of the pushforwards of the local rings of the generic points of X.

Definition 6.4. A Cartier divisor D is a global section of $\mathcal{K}_X^*/\mathcal{O}_X^*$. We say that $D \geq 0$ is effective if it lies in the subsheaf $\mathcal{K}_X^* \cap \mathcal{O}_X/\mathcal{O}_X^*$. Finally, $D = \mathrm{div}(f)$ is a principal divisor if it is the image of a meromorphic function f, i.e., a global section of $\mathcal{K}_X^*$.

A Cartier divisor can be described as (U_i, f_i) for an open cover $\mathcal{U}$ of X and meromorphic functions $f_i \in \mathcal{K}_X(U_i)^*$ such that $f_i f_j^{-1}$ are units in $\mathcal{O}_X(U_{ij})$. Being principal means $\mathcal{U}$ can be taken to be the trivial cover, and effective means the $f_i \in \mathcal{O}_X(U_i)$ are actual functions of our scheme. The group $\mathrm{Div}(X)$ of Cartier divisors has a natural group structure inherited from the ambient sheaf and we define the class group $\mathrm{CaCl}(X)$ as the quotient of $\mathrm{Div}(X)$ by its principal divisors.

Next, we are going to associate line bundles to arbitrary Cartier divisors. It is helpful to give a cohomological description of $\mathrm{Pic}(X)$ first.

Lemma 6.5. *Assume X has embedded points. There is an isomorphism $\check{H}^1(X, \mathcal{O}_X^*) \simeq \mathrm{Pic}(X)$, canonical up to sign.*

Proof. Let $\mathcal{L}$ be a line bundle on X. We can find an open cover $\mathcal{U}$ of X with trivializing sections $s_i \colon \mathcal{O}_{U_i} \simeq \mathcal{L}_{U_i}$. Then, we obtain a unit element $f_{ij} \in \mathcal{O}_X(U_{ij})^*$ by taking the image of 1 under the automorphism $s_i^{-1}|_{U_{ij}} \circ s_j|_{U_{ij}}$ of $\mathcal{O}_{U_{ij}}$. To see that this defines a 1-cocycle of $\mathcal{O}_X^*$, we notice that $f_{ij} f_{jk} = f_{ik}$ by construction. It is easy to check that the associated cohomology class of the 1-cocycle does not depend on the choice of an open cover and trivializing sections. Conversely, if we are given a cocycle of $\mathcal{O}_X^*$, then after possibly changing the open cover, we may assume that some U_i is dense in X, and define $\mathcal{L}|_{U_j} := f_{ij}\mathcal{O}_{U_j}$. □

Corollary 6.6. *Assume X has no embedded points. Then, there is a natural isomorphism $\mathrm{CaCl}(X) \to \mathrm{Pic}(X)$ up to sign.*

Proof. The long exact sequence of Čech cohomology applied to the short exact sequence of abelian sheaves

$$1 \to \mathcal{O}_X^* \to \mathcal{K}_X^* \to \mathcal{K}_X^*/\mathcal{O}_X^* \to 1 \tag{6.1}$$

yields an injection

$$\mathrm{CaCl}(X) \to \mathrm{Pic}(X) \tag{6.2}$$

whose kernel embeds into $\check{H}^1(X, \mathcal{K}_X^*)$. But this sheaf is flasque, i.e., it induces surjections upon restrictions along opens, so it has no higher cohomology. □

We wish to explicitly compute the sheaf $\mathcal{O}_X(D)$ associated to the divisor D. If we represent it by (U_i, f_i), then the connecting homomorphism takes it to the units $f_i f_j^{-1}$ on U_{ij}. If we let $\mathcal{L}$ be the line bundle sitting inside $\mathcal{K}_X^*$ as $f_i^{-1}\mathcal{O}_{U_i}$, then we see that the 1-cocycle of $\mathcal{O}_X^*$ determined by $\mathcal{L}$ via our isomorphism above is equal to $f_i f_j^{-1}$ on U_{ij}. The reason we set up this sign convention was so that invertible ideal sheaves are anti-effective rather than effective, because it is the inverses of ideal sheaves that are usually semi-ample, so this matches our positivity intuitions.

Example 6.7. Let $X = \mathbb{P}^n_R$ and consider the line bundle $\mathcal{O}_X(d)$ for any integer d. We claim that this is associated to the divisor D_d which is locally given by the meromorphic function $T_0^d T_i^{-d}$ on $D_+(T_i)$. This makes sense at least because if $d \geq 0$, then this function is regular, so the divisor D_d is effective. The corresponding transition function on $D_+(T_iT_j)$ equals $T_j^d T_i^{-d}$, exactly the ones induced by $\mathcal{O}_X(d)$ via our isomorphism $\check{H}^1(X, \mathcal{O}_X^*) \simeq \mathrm{Pic}(X)$. Note that there is nothing special about T_0 in this situation, and we can replace it by T_1 or any other T_k for that matter. The Cartier divisors are not the same but equivalent, as they differ by the meromorphic function $T_1 T_0^{-1}$.

Now, we discuss Weil divisors. A cycle in a scheme X is a closed subscheme $Z \subset X$. Algebraic geometers study them extensively, especially via the so-called Chow groups, and there are still very important open conjectures on their basic behavior, such as the Hodge or the Tate conjectures. The simplest objects of study are cycles in codimension 1, and these lead to the notion of Weil divisors.

Definition 6.8. A Weil divisor $D = \sum_Y n_Y Y$ is a finite sum with integral coefficients of integral subschemes $Y \subset X$ of codimension 1. If $n_Y \geq 0$ for all Y, then we say D is effective.

We denote the group of Weil divisors by $Z^1(X)$. Under a mild assumption in codimension 1, one can attach Weil divisors to Cartier divisors. First, if X has no embedded points and is regular in codimension 1, i.e., it is R_1, then we define a natural group homomorphism $\mathrm{div}\colon K(X)^* \to Z^1(X)$ given as follows: it sends f to the Weil divisor $\mathrm{div}(f) = \sum v_Y(f) Y$, where $v_Y \colon \mathcal{O}_{X,\eta_Y}$ is the normalized discrete valuation of the local ring of X at the generic point η_Y of the prime divisor Y (recall that regular rings of Krull dimension 1 are DVR's!). This sum is finite, because the closed subset of zeros and poles of f is nowhere dense, and hence it has finitely many generic points and these exhaust all possible codimension 1 divisors Y for which $v_Y(f)$ might not vanish. Note that if $f \in \mathcal{O}_X(X)$ has no poles, then $\mathrm{div}(f)$ is effective, and if $f \in \mathcal{O}_X(X)^*$ is a global unit, then $\mathrm{div}(f)$ vanishes. Now that we have a notion of principal divisors, we can define the Weil class group $\mathrm{Cl}(X)$ as the quotient of $Z^1(X)$ by its principal divisors.

More generally, we can upgrade the above construction to a group homomorphism $\mathrm{Div}(X) \to Z^1(X)$ where a Cartier divisor D is sent to $\sum_Y n_Y v_Y(D)$. The valuation of D at Y can be defined by taking an open $\eta_Y \in U_i$ over which D becomes principal with meromorphic function f_i and taking the valuation of the latter (because the transition functions are units, their valuation vanishes, and the choice of U_i does not matter). It is clear by construction that we get a homomorphism $\mathrm{CaCl}(X) \to \mathrm{Cl}(X)$, and consequently also from the Picard group. We want to finish our treatment of divisors by examining the difference between these.

Proposition 6.9. *If X has no embedded points and is normal, then* $\mathrm{CaCl}(X) \to \mathrm{Cl}(X)$ *is injective. If X is moreover regular, then the same map is an isomorphism.*

Proof. If $v_Y(D)$ vanishes for all D, then the local meromorphic functions are units up to codimension 2, so they are units everywhere by Hartogs' theorem. For the bijectivity when X is regular, consider the ideal sheaf $\mathcal{I}_Y$ attached to the prime divisor Y. Because X is regular, its local rings are unique factorization domains, so we know that $\mathcal{I}_Y$ is locally principal. Then we can write $\mathcal{I}_{Y \cap U_i} = f_i \mathcal{O}_{U_i}$ for some open cover $\mathcal{U}$ of X. By construction, the quotients $f_i f_j^{-1}$ have to be units in U_{ij}, so this defines a Cartier divisor D. Moreover, it is clear that $v_Z(D) = \delta_{YZ}$ for any prime divisor Z, so we get the desired surjectivity. □

Corollary 6.10. *Let k be an algebraically closed field and $\mathbb{P}^n_k$ be the n-dimensional projective space over k. Then, the map $\mathbb{Z} \to \mathrm{Pic}(\mathbb{P}^n_k)$ given by $d \mapsto \mathcal{O}(d)$ is an isomorphism.*

Proof. Since $\mathbb{P}^n_k$ is regular, it suffices to calculate its class group. A prime divisor Y in $\mathbb{P}^n_k$ corresponds to the vanishing locus $V_+(P)$ of a homogeneous polynomial P of degree d. But then P/T_n^d is a meromorphic function on $\mathbb{P}^n_k$, which shows that Y has the same class as $dV_+(T_n)$. Now, it follows easily from our example with Cartier divisors that $V_+(T_n)$ corresponds to the line bundle $\mathcal{O}(1)$. □

From now on, we work over an algebraically closed field k and smooth connected curves C over k, i.e., k-smooth and connected of relative dimension 1. The previous corollary states that Weil divisors are Cartier, so from now on we refer to divisors only, without making a precise distinction.

Given a divisor D on the curve C, we define its *degree*, denoted $\deg(D)$, as the sum of all the coefficients $v_x(D)$ as x ranges over all closed points of C. For effective divisors, the degree has a concrete geometric meaning.

Lemma 6.11. *Let k be an algebraically closed field and C be a smooth connected k-curve. Then, we have an equality $\deg(D) = \dim_k H^0(D, \mathcal{O}_D)$ for any effective divisor D, where $\mathcal{O}_D$ is the cokernel of the inclusion $\mathcal{O}_C(-D) \to \mathcal{O}_C$.*

Proof. By hypothesis, we have $v_x(D) \geq 0$ and the ideal sheaf of $\mathcal{O}_D$ at the point x is given by the corresponding power of the maximal ideal. Since the local ring $\mathcal{O}_{C,x}$ is a discrete valuation ring with uniformizer t_x, we get that $\mathcal{O}_{D,x} \simeq k[t_x]/t_x^{v_x(D)}$ and hence its dimension over k equals precisely $v_x(D)$. As the underlying topological space of the scheme D is discrete, we get the claim by summing over all x. □

Next, we have to understand how divisors pullback and pushforward along finite dominant maps of smooth connected k-curves $\pi\colon C_1 \to C_2$. If $D_2 = \sum_{y \in C_2(k)} v_y(D_2) y$ is a divisor on C_2, then we define its pullback $D_1 := \pi^*(D_2)$ as follows: we set $v_x(D_1) = e_{x/y} v_y(D_2)$ with $y = f(x)$ and $e_{x/y} = v_x(t_y)$ being the ramification degree of $\mathcal{O}_{C_2,y} \to \mathcal{O}_{C_1,x}$. This is compatible with passing to associated line bundles:

Lemma 6.12. *Under the above assumptions, we have an isomorphism $\mathcal{O}_{C_1}(D_1) \simeq \pi^*\mathcal{O}_{C_2}(D_2)$.*

Proof. This is checked easily at the level of principal divisors, as an arbitrary divisor becomes principal after passing to an open cover. Now, the valuation at x of some meromorphic function f of C_2 after pullback to C_1 is equal to its valuation at $y = f(x)$ multiplied by the ramification index due to the renormalization. □

Just like in algebraic number theory, we can show that $\sum_{x \in f^{-1}(y)} e_{x/y}$ equals the degree $[k(C_1) : k(C_2)]$ of the finite extension $k(C_1)/k(C_2)$ of function fields. Indeed, the map π is flat by miracle flatness and thus the Euler characteristic of $\pi_*\mathcal{O}_{C_1}$ is locally constant. It is easy to see that the latter is given by the sum of the ramification degrees. Hence, π^* rescales degrees of divisors by $[k(C_1) : k(C_2)]$.

Corollary 6.13. *Let k be an algebraically closed field and C be a proper smooth connected k-curve. If $f \in H^0(C, \mathcal{K}_C^*)$ is a non-zero meromorphic function, then we have* $\deg(f)_0 = \deg(f)_\infty = [k(C) : k(f)]$.

Proof. The function f defines a finite morphism $f : C \to \mathbb{P}^1_k$ such that $k(f)$ equals the image of $k(t)$ under f^*. Then, we can write $\mathrm{div}(t) = 0 - \infty$, and use the equality $\mathrm{div}(f) = \pi^* \mathrm{div}(t) = \pi^*(0) - \pi^*(\infty)$ to calculate the degrees of the zero and pole divisors. □

Let us explain the relationship between smooth connected k-curves and their function fields.

Proposition 6.14. *The assignment $C \mapsto k(C)$ defines an equivalence between the category of proper smooth connected k-curves with non-constant maps and the category of field extensions of k of transcendence degree* 1.

Proof. We need to show that the functor is fully faithful. Given an inclusion $k(C_2) \to k(C_1)$, we can extend it uniquely to a non-constant map $C_1 \to C_2$ by identifying the left side with the normalization of C_2 in $k(C_1)$ and invoking the universal property of the normalization. For essential surjectivity, we note that any such field K/k contains $k(t)$, which is the function field of $\mathbb{P}^1_k$, and we can construct a proper smooth connected k-curve C with function field K by taking the normalization of $\mathbb{P}^1_k$ in K. □

Example 6.15. Let E be an elliptic curve, i.e., a proper smooth connected planar k-curve of degree 3 (in other words, given by a cubic homogeneous polynomial inside $\mathbb{P}^2$). The set $E(k)$ is endowed with the group law where $P + Q + R = 0$ exactly when P, Q, and R are colinear. We claim that the natural map $E(k) \to \mathbb{P}\beth^0(E)$ given by $P \mapsto P - \infty$ for the fixed point at infinity $\infty = [0 : 1 : 0]$ is a group isomorphism. Indeed, if we let f be the meromorphic function determined by the line connecting P, Q, and R, then we see that $\mathrm{div}(f) = P + Q + R - 3\infty$, so the group structure is preserved. Also the map is injective, because $P - \infty = \mathrm{div}(g)$ leads to g defining an isomorphism between E and $\mathbb{P}^1_k$, which is not possible (e.g., the genus of E is 1 instead of 0, calculate the H^1 of the structure sheaf or instead the H^0 of the canonical sheaf). It is not too hard to show surjectivity now, because the group law of $E(k)$ allows us to absorb sums of points into a single one.

We can now approach Riemann–Roch, which gives an expression for the Euler characteristic of any line bundle $\mathcal{L}$ on a proper smooth connected k-curve C. Before doing this, let us make sure we understand how to compute the global sections of $\mathcal{O}_X(D)$.

Lemma 6.16. *For a Cartier divisor on a noetherian scheme X without embedded points, there is a natural isomorphism between $H^0(X, \mathcal{O}_X(D))$ and the set of meromorphic functions f on X such that* $\mathrm{div}(f) + D \geq 0$ *is effective.*

Proof. Consider a global section $s : \mathcal{O}_X \to \mathcal{O}_X(D)$. Clearly the assertion is preserved under localizing X and then it glues along open covers thereof. Hence, we may and do assume that $D = \mathrm{div}(g)$ is principal. We get $\mathcal{O}_X(D) = g^{-1}\mathcal{O}_X$, so the image of 1 under s is a meromorphic function f such that gf is regular, i.e., $\mathrm{div}(f) + \mathrm{div}(g) \geq 0$ is effective. □

Theorem 6.17. *Let C be a proper smooth connected k-curve. Then, for every divisor D on C, we have* $\chi(C, \mathcal{O}_C(D)) = \deg(D) + \chi(C, \mathcal{O}_C)$.

Proof. Recall that χ is additive on short exact sequences. First, we take an effective divisor $D \geq 0$ and consider the short exact sequence

$$0 \to \mathcal{O}_C(-D) \to \mathcal{O}_C \to \mathcal{O}_D \to 0 \tag{6.3}$$

and this implies the assertion for $-D$ as $\chi(C, \mathcal{O}_D) = \deg(D)$ by the previous lemma, and the fact that D is 0-dimensional as a scheme. Tensoring the above exact sequence with an arbitrary divisor E, we see that the difference $\chi(C, \mathcal{O}(E)) - \chi(C, \mathcal{O}(E - D))$ equals $\deg(D)$ using the triviality of $\mathrm{Pic}(D)$ to identify $\mathcal{O}_D(E) \simeq \mathcal{O}_D$. Now, if E is also effective, then we get the corresponding assertion for E. Putting everything together, we get the claim for differences of effective divisors, which clearly covers all of them (think of Weil divisors). □

While we haven't yet seen Serre duality, it identifies the k-linear dual of $H^1(C, \mathcal{O}_C(D))$ with $H^0(C, \mathcal{O}_C(-D) \otimes \omega_{C/k})$, where $\omega_{C/k} = \Omega^1_{C/k}$ is the canonical sheaf. We define the genus $g(C)$ as the dimension of the k-vector space $H^0(C, \omega_{C/k})$ and it follows by Riemann–Roch that $\deg(K_C) = 2g - 2$ for any canonical divisor K_C, i.e., such that $\omega_{C/k} \simeq \mathcal{O}_C(K_C)$.

7. Serre duality

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